CONSTRUCTIVE UNBOUNDED OPERATORS

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Abstract: The existence of points outside the domain of an unbounded linear mapping between normed spaces is investigated constructively.

Introduction

In this paper we begin a constructive investigation of unbounded operators by studying a number of questions about their domains. We intend this to be the start of a programme to investigate systematically the constructive theory of unbounded operators, and incidentally to counter, once and for all, some common misconceptions about the viability of such a theory (see [10] and [5]).

Since constructive analysis uses intuitionistic, rather than classical, logic, even at the base level of the theory of unbounded operators there are problems that are classically trivial but constructively significant. For example, the obvious contradiction argument, based on the closed graph theorem, merely shows that the domain of a closed unbounded operator is not complete; it does not enable us to construct a point of the initial Banach space at which the operator is not defined. In the general case we have only managed a partial, weak solution to this

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problem; but in the special case of a closed unbounded operator T on a Hilbert space, such that both T and T^* are densely defined, we can find $\xi \in H$ such that $T\xi$ is not defined.

Our setting is Bishop's constructive analysis (**BISH**—see [2] or [3]), which we can regard as mathematics carried out with intuitionistic logic ([14], [15]). At one stage we add Church's thesis, to enable us to prove a stronger result in recursive constructive analysis, which is simply one model of BISH. Further information about BISH and other varieties of constructive mathematics can be found in [8], [1], and [18].

Let T be a linear mapping between normed spaces X and Y. We say that T is

- **not bounded** if it is contradictory that it be bounded;
- **unbounded** if there exists a sequence (x_n) in X such that $\lim_{n\to\infty} x_n = 0$ and $\forall n \; (\|Tx_n\| > 1)$.

There is a constructive distinction between these two concepts: whereas the second clearly implies the first, the converse implication depends on **Markov's Principle**,

If (a_n) is a binary sequence such that $\neg\forall n\,(a_n = 0)$, then $\exists n\,(a_n = 1)$,

This principle, which represents an unbounded search, cannot be derived within, but is consistent with, Heyting arithmetic—that is, Peano arithmetic with intuitionistic logic; see [8], pages 137–138. For future reference, note that Markov's Principle is equivalent to the statement

$$\forall x \in \mathbf{R}\,(\neg\,(x = 0) \Rightarrow x \neq 0)\,,$$

where, in the context of a normed space X, $x \neq 0$ means $\|x\| > 0$. We shall return to the distinction between *bounded* and *not unbounded* at the end of the paper. We say that T is **strongly extensional** if

$$\forall x, x' \in X\ \left(Tx \neq Tx' \Rightarrow x \neq x'\right).$$

Markov's Principle implies that every linear mapping between normed spaces is strongly extensional. Without Markov's Principle the best we can prove constructively is that every linear mapping of a Banach space into a normed space is strongly extensional; see [6], Corollary 2.

1. CONSTRUCTING POINTS OUTSIDE DOMAINS

In this section we examine the problem of finding points outside the domain of an unbounded linear mapping between normed spaces. Our first result is connected with the **limited principle of omniscience (LPO)**,

> *If (a_n) is a binary sequence, then either $a_n = 0$ for all n or else there exists n such that $a_n = 1$,*

a weak form of the law of excluded middle that cannot be derived in intuitionistic logic (and that is false, even with classical logic, in the recursive model of BISH—see Chapter 3 of [8]).

Recall that every normed linear space X can be embedded as a dense subset of a Banach space $\widehat{X}$, the **completion** of X.

Proposition 1 *Let T be a strongly extensional unbounded linear mapping between normed spaces X, Y. Then for each binary sequence (a_n) there exists $x \in \widehat{X}$ such that if $x \in X$, then*

$$\forall n \ (a_n = 0) \vee \exists n \ (a_n = 1) .$$

PROOF. We may assume that (a_n) has at most one term equal to 1. For each positive integer n choose a unit vector $x_n \in X$ such that $\|Tx_n\| > n^2$. Then $\sum_{n=1}^{\infty} a_n \|Tx_n\|^{-1} x_n$ converges to a sum x in the Banach space $\widehat{X}$, by comparison with $\sum_{n=1}^{\infty} n^{-2}$. Suppose that $x \in X$. Then either $Tx \neq 0$ or $\|Tx\| < 1$. In the first case, as T is strongly extensional, $x \neq 0$ and so there exists n such that $a_n = 1$. In the second, suppose that there exists N such that $a_N = 1$; then $Tx = \|Tx_N\|^{-1} Tx_N$ and so $\|Tx\| = 1$, a contradiction; whence $a_n = 0$ for all n. Q.E.D.

Ishihara [11] has shown that in recursive constructive mathematics—constructive mathematics plus Church's thesis—every linear mapping on a Banach space is sequentially continuous and therefore not unbounded; this result also holds in the intuitionistic model of constructive mathematics ([18], page 354, 2.8). Since BISH is consistent with classical (that is, traditional) mathematics, we cannot hope to prove that recursive/intuitionistic continuity result within BISH. What we can prove, however, is that the existence of unbounded linear mappings on a Banach space is essentially nonconstructive.

Corollary 2 *If there exists a strongly extensional unbounded linear mapping on a Banach space, then LPO holds.*

PROOF. This is an immediate consequence of Proposition 1. Q.E.D.

A linear mapping $T : X \to Y$ between normed spaces is said to be **closed** if its graph is a closed subset of $X \times Y$; that is, if

$$(x_n \to x \in X \text{ and } Tx_n \to y \in Y) \Rightarrow y = Tx.$$

The constructive **closed graph theorem** says that if X is complete and the graph of T is both closed and separable, then T is sequentially continuous ([12], Corollary 2). The separability hypothesis is not needed in classical analysis, where the conclusion is the stronger one that T is bounded.

Proposition 3 *If T is an unbounded closed linear mapping, with a separable graph, of a normed space X into a Banach space Y, then* $\neg\left(\widehat{X} = X\right)$.

PROOF. If $\widehat{X} = X$, then the closed graph theorem shows that T is sequentially continuous, a contradiction. Q.E.D.

Let H be a Hilbert space, and T a linear mapping of a dense subspace of H into H; then we say that T is a **densely defined operator on** H. The **adjoint** T^* of a densely defined operator T on H is defined as in classical mathematics. Thus the domain of T^* comprises those $x \in H$ for which there exists $y \in H$ (which is then uniquely defined by x) such that

$$\langle y, z\rangle = \langle x, Tz\rangle \quad (z \in H),$$

and for such an x we write $T^*x = y$. In contrast to the classical situation, T^* may not be defined even if T is a *bounded* operator defined on the whole space H; see [9].

When dealing with unbounded linear operators on linear subsets of a Hilbert space, we can obtain substantial improvements upon Corollary 2. For the first of these we need the constructive **uniform boundedness theorem** (a contrapositive of the usual classical version of that result):

> *If (A_n) is a sequence of bounded linear mappings from a Banach space X into a normed space Y, and (x_n) is a sequence of unit vectors in X such that $\|A_n x_n\| \to \infty$ as $n \to \infty$, then there exists $x \in X$ such that $\|A_n x\| \to \infty$ as $n \to \infty$* ([16], page 61).

Proposition 4 *If H is a Hilbert space, and T is an unbounded densely defined operator on H with an adjoint, then there exists an element ξ of H such that $T\xi$ is undefined.*

PROOF. Let (x_n) be a sequence converging to 0 in H such that $\|Tx_n\| \to \infty$. Applying the uniform boundedness theorem to the linear functionals $x \mapsto \langle x, Tx_n \rangle$, we obtain a unit vector $\xi \in H$ such that $|\langle \xi, Tx_n \rangle| \to \infty$ as $n \to \infty$. If $T\xi$ is defined, then $\langle \xi, Tx_n \rangle = \langle T^*\xi, x_n \rangle \to 0$, a contradiction. Hence $T\xi$ is not defined. Q.E.D.

If T is an unbounded densely defined operator on a Hilbert space H, what can we say about its adjoint? It is closed (see [13]), but need not be unbounded; in fact, it can be 0 even though T is densely defined (see [17]). Classically, T^* is unbounded: for if it were bounded, we could extend it by continuity to a bounded operator on H whose adjoint would be bounded; whence T would have a bounded extension. When H is separable, the same result holds constructively, but, of course, with a direct proof.

Proposition 5 *If T is a densely defined operator on a separable Hilbert space H such that T^* is unbounded, then T is unbounded.*

PROOF. Since it is a dense subset of a separable space, the domain of T is separable; so there exists an increasing sequence (V_n) of finite–dimensional subspaces whose union is dense in that domain and therefore in H. Let P_n be the projection of H on V_n. Given $K > 0$, choose a unit vector $\xi \in H$ such that $\|T^*\xi\| > K + 1$. Then choose n such that

$$\|T^*\xi - P_nT^*\xi\| = \rho(T^*\xi, V_n) < 1$$

and therefore $\|P_nT^*\xi\| > K$. Setting

$$z = \|P_nT^*\xi\|^{-1} P_nT^*\xi,$$

we see that $P_nz = z$, $\|z\| = 1$, and

$$K < \|P_nT^*\xi\| = \langle P_nT^*\xi, z\rangle = \langle \xi, TP_nz\rangle = \langle \xi, Tz\rangle \leq \|Tz\|. \quad \text{Q.E.D.}$$

Corollary 6 *Let T be an unbounded densely defined operator on a Hilbert space, such that T^* is densely defined. Then there exists $\xi \in H$ such that $T\xi$ is undefined.*

PROOF. Use the preceding two results. Q.E.D.

Returning to the context of Banach spaces, we show how to improve Corollary 2 under Church's thesis (that is, in the recursive model of BISH).

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