# THE POINTS OF (LOCALLY) COMPACT REGULAR FORMAL TOPOLOGIES

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**Abstract:** In a paper appeared in 1990, C.J. Mulvey established a constructive characterization of completely prime filters on a compact regular locale L; although proved by intuitionistic logic, the result relies on a notion of maximality which contains an impredicative second-order quantification. In this note we present an alternative concept of maximality, entirely phrased in first-order terms, and give a predicative characterization of the points of a compact regular formal topology (equivalently, we give a characterization of the points of a compact regular locale which can be entirely carried out within Intuitionistic Type Theory). This result is then generalized to locally compact regular formal topologies (resp. locally compact regular locale).

## Introduction

Formal Topology[1] was conceived with the aim of developing point-free topology (Locale Theory) in a constructive and predicative foundational setting, such as Martin-Löf's Intuitionistic Type Theory. Quite recently, the topological notion of regularity has been predicatively formulated in

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[1]Formal Topology was introduced in [15]; a more recent presentation is contained in [16], [17].



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this framework, and the class of compact regular formal topologies has shown to have nice and promising properties, particularly from a constructive standpoint (cf. e.g. [5], [3] and [14]). In this note we establish a constructive characterization of the points of a compact regular formal topology, in which formal points are shown to coincide with particular subsets of basic neighbourhoods, the *maximal regular* ones. The specific feature of this characterization is that regular subsets will be considered to be maximal according to an entirely first-order criterion of maximality.

This result can then be seen to improve a previous characterization appeared in the context of Locale Theory: in [9] indeed Chris Mulvey introduces a particular formulation of the notion of maximality for regular filters which allows to prove intuitionistically that the completely prime filters on a compact regular locale coincide with the maximal regular filters (cf. [9]). In such a notion of maximality, however, an impredicative second-order quantification appears which makes the result incompatible with foundational settings for constructive mathematics such as Martin-Löf's Intuitionistic Type Theory and Aczel's Constructive Set Theory. A natural relation then exists between Formal Topology and Locale Theory (cf. [15]) that allows to give the following reading of our result: a characterization of completely prime filters on a compact regular locale by means of maximal regular filters can be obtained intuitionistically *and* predicatively, and such a characterization can be entirely carried out within Intuitionistic Type Theory.

Few modifications allow to generalize this result to *locally* compact regular formal topologies (and thus to locally compact regular locales). Then, in particular, examples of topologies for which these characterizations are valid are (that giving rise to) the Continuum, Cantor space and the spaces $\mathcal{L}(\mathcal{A})$ of linear functionals of norm $\leq 1$ from a semi-normed space $A$ to the reals[2].

## 1. PRELIMINARIES

We recall the basic definitions of Formal Topology ([15]). The reader is referred to [15], [16] and [17] for a detailed account (the presentation we are to adopt appears in [16]). We use a special notation for subsets,

[2]Endowed with the *weak** topology, cf. [4].

introduced and motivated in [18], which allows to work within Intuitionistic Type Theory (henceforth simply Type Theory, cf. [8], [13]) with essentially the usual mathematical formalism: for the present purpose it will suffice to know that a *subset* $U$ of a set $S$ is a unary predicate (dependent type) on $S$, $U(x)(x \in S)$, and that a *set-indexed family of subsets* is a binary predicate $U(x,i)(x \in S, i \in I)$ on the sets $S$ and $I$, where for each $\bar{i}$, $U(x,\bar{i})(x \in S)$ is the subset of index $\bar{i}$ (for simplicity, we will also use the traditional notations $\{a \in S : U(a)\}$, to indicate the subset $U$, and $U_i (i \in I)$ for a family of subsets). Finally, we will write $a \epsilon U$ to mean $a \in S$ *and* $U(a)$ *true* (in the expression $a \epsilon U$ the symbol '$\epsilon$' is used, instead of '$\in$', to recall that we are considering a subset, i.e. a propositional function, and not a set; cf. [18]).

**1.1** A (formal) topology is a triple $\mathcal{S} \equiv (S, \lhd, \mathsf{Pos})$ where $S$ is a set, called the *base*, $\lhd$ is a relation between elements and subsets of $S$ which satisfies the following conditions:

$$\textit{(reflexivity)} \qquad \frac{a \,\epsilon\, U}{a \lhd U}$$

$$\textit{(transitivity)} \qquad \frac{a \lhd U \qquad U \lhd V}{a \lhd V}$$

$$\textit{(}\downarrow\textit{-right)} \qquad \frac{a \lhd U \qquad a \lhd V}{a \lhd U \downarrow V}$$

where

$$\begin{array}{rcl} U \lhd V & \equiv & (\forall u \,\epsilon\, U) u \lhd V \\ U \downarrow V & \equiv & \{d : S \mid (\exists u \,\epsilon\, U)\ (d \lhd \{u\})\ \&\ (\exists v \,\epsilon\, V)\ (d \lhd \{v\})\} \end{array}$$

and $\mathsf{Pos}$ is a subset of $S$ which satisfies

$$\textit{(monotonicity)} \qquad \frac{\mathsf{Pos}(a) \quad a \lhd U}{(\exists b \,\epsilon\, U)\mathsf{Pos}(b)}$$

$$\textit{(positivity)} \qquad \frac{\mathsf{Pos}(a) \to a \lhd U}{a \lhd U}.$$

We will write $\mathsf{Pos}(U)$ for $(\exists a \,\epsilon\, U)\mathsf{Pos}(a)$. The relation $\lhd$ is called *cover* and $\mathsf{Pos}$ *positivity predicate* (we pronounce $a \lhd U$ as '$U$ covers $a$', and

$\mathsf{Pos}(a)$ as '$a$ is positive' ). For simplicity, when a basic neighbourhood $a$ is covered by a singleton subset $\{b\}$ we will often write $a \lhd b$ instead of $a \lhd \{b\}$.

One can think intuitively of the elements of $S$ as of indexes for the basic neighbourhoods of a topological space; the cover relation can then be seen as a formal description of the inclusion between basic neighborhoods and subsets of $S$, and the predicate Pos as a positive way to express that a certain neighbourhood is non-empty. Then, for instance, 'monotonicity' has the following intuitive reading: if a non-empty neighbourhood is covered by a family of neighbourhoods (indexed by $U$) then there must be at least one member of the family which is non-empty.

An equivalent formulation of positivity (cf. [15]) which we will use in the following is

$$\frac{a \lhd U}{a \lhd U^+},$$

where $U^+ \equiv \{b \,\epsilon\, U : \mathsf{Pos}(b)\}$ (that is, only non-empty neighbourhoods contribute to the cover).

Finally, we recall that given two subsets $U, V$ of $S$, $U =_S V$ means exactly that $U \lhd V \;\&\; V \lhd U$, and that for $U \subseteq S$, the *(pseudo-)complement* $U^*$ of $U$ is given by $U^* \equiv \{b : \neg\mathsf{Pos}(b \downarrow U)\}$.

**1.2** In a formal topology $\mathcal{S}$ a *formal point* is a subset $\alpha \subseteq S$ such that

*i.* $(\exists a)(a \,\epsilon\, \alpha)$

*ii.* $(a \,\epsilon\, \alpha \;\&\; b \,\epsilon\, \alpha) \rightarrow (\exists c)(c \,\epsilon\, a \downarrow b \;\&\; c \,\epsilon\, \alpha)$

*iii.* $\dfrac{a \,\epsilon\, \alpha \quad a \lhd U}{(\exists b)(b \epsilon U \;\&\; b \,\epsilon\, \alpha)}$

*iv.* $a \,\epsilon\, \alpha \rightarrow \mathsf{Pos}(a)$.

We will denote by $\mathsf{Pt}(\mathcal{S})$ the collection of formal points. (Condition *iv.* is actually known to be derivable from *iii.* and positivity and could thus be skipped[3]).

**1.3** The relation with Locale Theory can be sketched as follows (a detailed discussion of this subject is contained in [15], [17]): defining, for

[3]A proof is recalled in [12]. A generalized definition of Formal Topology can however be considered in which the positivity rule is not required, cf. [16].

$U \subseteq S$, $\mathcal{S}U$ to be the subset $\{a \in S : a \lhd U\}$, we say that $U$ is *saturated* if $U = \mathcal{S}U$; denoting then by $Sat(\mathcal{S})$ the collection of saturated subset of $\mathcal{S}$, $Sat(\mathcal{S})$ endowed with the operations

$$\mathcal{S}U \wedge \mathcal{S}V \equiv \mathcal{S}U \cap \mathcal{S}V = \mathcal{S}(U \downarrow V)$$

and

$$\bigvee_{i \in I} \mathcal{S}U_i \equiv \mathcal{S}(\bigcup_{i \in I} U_i)$$

forms a *frame* (or locale, or complete Heyting algebra).

From a non-constructive point of view the converse is also valid (that is, any frame can be obtained as the frame of saturated subsets of a formal topology $\mathcal{S}$). Finally, the points of a formal topology $\mathcal{S}$ are easily shown to correspond to completely prime filters on $Sat(\mathcal{S})$.

**1.4** A formal topology $\mathcal{S} \equiv (S, \lhd, \mathsf{Pos})$ is said to be *compact* if whenever $S \lhd U$ there exists a finite[4] subset $U_0 \subseteq U$ such that $S \lhd U_0$.

The notion of regularity have been recently introduced in Formal Topology as the predicative counterpart of that given in the context of locales (see for instance [6]): for $a, b$ in $S$, we say that $b$ is *well-covered* by $a$ iff $S \lhd a \cup b^*$; defining $wc(a)$ to be the subset of neighbourhoods $b$ which are well-covered by $a$, $wc(a) \equiv \{b : S \lhd a \cup b^*\}$, a formal topology $\mathcal{S}$ is then said to be *regular* if for all $a$ in $S$, $a \lhd wc(a)$[5]. A topology $\mathcal{S}$ will be said to be *compact regular* if it is compact and regular.

The following lemmas will be used in the following, often without an explicit mention; they obtain in any formal topology $\mathcal{S}$.

**Lemma 1.5.** *Let $V, W, Z$ be subsets of $S$, and $U_i (i \in I)$ be a family of subsets of $S$. We have*

*i)* $V \cup (W \downarrow Z) =_{\mathcal{S}} (V \cup W) \downarrow (V \cup Z)$,

*ii)* $(\bigcup_i U_i) \downarrow V = \bigcup_i (U_i \downarrow V)$.

[4]Note that here and in the following a set, or a subset, is considered to be 'finite' if its elements can be listed; cf. the notions of finite and sub-finite in [2].

[5]This definition appeared in [14]; in case of compactness, it is equivalent to the one introduced in [5], [3].

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