EMBEDDING A LINEAR SUBSET OF $\mathcal{B}(H)$ IN THE DUAL OF ITS PREDUAL

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Keywords: weak-operator topology, ultraweakly continuous, predual, trace class

Abstract: The embedding of a linear set of bounded operators on a separable Hilbert space as a dense subset of the dual of its predual is explored constructively.

In this paper we continue the study of spaces of operators on a Hilbert space within constructive mathematics, as part of a programme for the systematic constructive development of the theory of operator algebras [6, 7].

The constructive framework within which we operate was erected by the late Errett Bishop [2], under the requirement that "existence" must be strictly interpreted as "computability" relative to some notion of algorithm. By not specifying formally what he meant by an algorithm, other than insisting that it must be executable by a finite number of human beings or computers in a finite time, Bishop enabled his work to have a variety of interpretations; in particular, all theorems of Bishop's constructive mathematics hold within classical (that is, traditional) mathematics and Brouwer's intuitionistic mathematics, and under a recursive interpretation.

Note, incidentally, that Bishop's algorithmic interpretation of existence makes no demands about the complexity of the algorithms used; at present, constructive mathematics addresses questions of computability in principle, rather than computability in practice. (See, however, [13].)

The main practical difference between constructive and classical mathematics is one of logic. The algorithmic interpretation of existence forces us to re–examine the interpretation of each logical connective and quantifier, and leads, it seems inevitably, to the use of intuitionistic logic, as

P. Schuster, U. Berger and H. Osswald (eds.),
Reuniting the Antipodes – Constructive and Nonstandard Views of the Continuum, 55–61.

originally codified by Heyting [11]. We also have to modify the underlying non–logical principles—for example, the axioms for the real line **R** [5] or the axioms of Zermelo–Fraenkel set theory [14]—so as to ensure that one cannot derive from them classical principles, such as the law of excluded middle, that are independent of intuitionistic logic. Once these precautions have been taken, we are, to all (non–philosophical) intents and purposes, working with intuitionistic Zermelo–Fraenkel set theory using intuitionistic logic. (For additional background material on constructive mathematics see [1, 2, 3, 5, 9, 15].)

Let H be a Hilbert space (not necessarily separable), $\mathcal{B}(H)$ the space of bounded linear operators on H, and $\mathcal{B}_1(H)$ the unit ball of $\mathcal{B}(H)$. Recall that the **weak–operator** topology τ_w on $\mathcal{B}(H)$ is the weakest topology with respect to which the mapping $T \mapsto \langle Tx, y\rangle$ is continuous for all $x, y \in H$. This topology is determined by the seminorms $T \mapsto |\langle Tx, y\rangle|$, where x, y run through any dense subset of $\mathcal{B}_1(H)$. Classically, $\mathcal{B}_1(H)$ is τ_w–compact; but constructively the most we can say, in general, is that it is τ_w–totally bounded [6].

Let $\mathcal{R}$ be a linear subset of $\mathcal{B}(H)$, let $\mathcal{R}1 = \mathcal{R} \cap \mathcal{B}_1(H)$, and let $R_\sharp$ denote the linear space of all linear functionals on R that are **ultraweakly continuous**—that is, τ_w–uniformly continuous on $\mathcal{R}_1$. If $\mathcal{R}_1$ is τ_w–totally bounded, then

$$\|f\| = \sup\{|f(T)| : T \in \mathcal{R}_1\}$$

defines a norm on $\mathcal{R}_\sharp$; taken with this norm, $\mathcal{R}_\sharp$ is called the **predual** of $\mathcal{R}$.

For convenience, we denote by $f_{x,y}$ the ultraweakly continuous functional $T \mapsto \langle Tx, y\rangle$ on $\mathcal{B}(H)$. We also recall that the weak* topology on the dual X^* of a locally convex space is the topology defined by the seminorms $f \mapsto |f(x)|$ $(x \in X)$; see [9] and [10].

Theorem 1 *Let $\mathcal{R}$ be a linear subset of $\mathcal{B}(H)$ such that $\mathcal{R}_1$ is totally bounded in the weak–operator topology τ_w, and define a mapping ϕ of $\mathcal{R}$ into the dual space $\mathcal{R}_\sharp^*$ of $\mathcal{R}_\sharp$ by*

$$\phi(T)(f) = f(T) \quad (T \in \mathcal{R}).$$

Then ϕ is one–one and linear, and is uniformly continuous on $\mathcal{R}_1$. Moreover, $\phi(\mathcal{R}_1)$ is weak–dense in the unit ball of $\mathcal{R}_\sharp^*$, and the restriction of ϕ^{-1} to $\phi(\mathcal{R}_1)$ is uniformly continuous with respect to the weak*–topology on $\mathcal{R}_\sharp^*$ and the weak–operator topology on $\mathcal{R}_1$.*

PROOF. Since ϕ is clearly linear, to prove that it is one–one we need only show that its kernel is $\{0\}$. But if $\phi(T) = 0$, then we have $\langle Tx, y\rangle = \phi(T)\,(f_{x,y}) = 0$ for all $x, y \in H$; whence $T = 0$.

For each $f \in \mathcal{R}_\sharp$ the mapping $T \mapsto \phi(T)(f)$ equals f and so is τ_w–uniformly continuous on $\mathcal{R}_1$. It follows immediately that ϕ is uniformly continuous as a mapping of $(\mathcal{R}_1, \tau_w)$ into $\mathcal{R}_\sharp^*$ (with the weak*–topology). Hence $K = \phi(\mathcal{R}_1)$ is weak*–totally bounded, and therefore located, in $\mathcal{R}_\sharp^*$ (see [7, 10]). Let u be an element of $\mathcal{R}_\sharp^*$, let $\{f_1, \dots, f_N\}$ be a finitely enumerable subset of $\mathcal{R}_\sharp$, and let ε be a positive number. To prove that $\phi(\mathcal{R}_1)$ is weak*–dense in the unit ball $(\mathcal{R}_\sharp^*)_1$ of $\mathcal{R}_\sharp^*$, it is enough to show that K intersects the neighbourhood

$$V = \left\{v \in \left(\mathcal{R}_\sharp^*\right)_1 : |(u - v)\,(f_k)| < 3\varepsilon \ \ (1 \le k \le N)\right\}$$

of u in $(\mathcal{R}_\sharp^*)_1$. To this end, choose a finite–dimensional subspace $\mathcal{G}$ of $\mathcal{R}_\sharp$ such that

$$\inf\{\|f_k - g\| : g \in \mathcal{G}\} < \varepsilon \quad (1 \le k \le N)$$

([3], page 308, Lemma (2.5)); for each k $(1 \le k \le N)$, then choose $g_k \in \mathcal{G}$ such that $\|f_k - g_k\| < \varepsilon$. The dual space $\mathcal{G}^*$ of $\mathcal{G}$ is a finite–dimensional Banach space with respect to the usual norm defined by

$$\|v\|' = \sup\{|v(g)| : g \in \mathcal{G},\ \|g\| \le 1\}.$$

Since $K \subset (\mathcal{R}_\sharp^*)_1$, and $(\mathcal{R}_\sharp^*)_1$ is a subset of the unit ball of $(\mathcal{G}^*)_1$, we can regard u and, for each $T \in \mathcal{R}_1$, the functional $\phi(T)$ as elements of $(\mathcal{G}^*)_1$. Now suppose that

$$\inf\left\{\|u - \phi(T)\|' : T \in \mathcal{R}_1\right\} \neq 0. \tag{5.1}$$

Note that K, being weak*–totally bounded, is located in $\mathcal{G}_1$. By Corollary (4.4) on page 341 of [3], there exists a linear functional ψ, with norm 1, on $\mathcal{G}^*$ such that

$$|\psi(u)| > \sup\{|\psi(v)| : v \in K\}.$$

Since $\mathcal{G}$ is finite–dimensional, ψ is weak*–uniformly continuous on $(\mathcal{G}^*)_1$; so, by Corollary (6.9) on page 357 of [3], there exists $g \in \mathcal{G}$ such that $\psi(v) = v(g)$ for all $v \in \mathcal{G}^*$. In particular,

$$\begin{aligned}|u(g)| &> \sup\{|v(g)| : v \in K\}\\ &= \sup\{|\phi(T)(g)| : T \in \mathcal{R}_1\}\\ &= \sup\{g(T) : T \in \mathcal{R}_1\}\\ &= \|g\|,\end{aligned}$$

which is absurd since $u \in (\mathcal{R}_\sharp^*)_1$. We conclude that (1) is false, and hence that

$$\|u - \phi(T_0)\|' < \frac{\varepsilon}{M+1}$$

for some $T_0 \in \mathcal{R}_1$, where

$$M = \max_{1 \le k \le N} \|g_k\| \, .$$

For each k $(1 \le k \le N)$ we now have

$$\begin{aligned} |(u - \phi(T_0))\,(f_k)| &\le |(u - \phi(T_0))\,(f_k - g_k)| + |(u - \phi(T_0))\,(g_k)| \\ &\le 2\,\|f_k - g_k\| + \|u - \phi(T_0)\|'\,\|g_k\| \\ &< 2\varepsilon + \frac{\varepsilon}{M+1} M \\ &< 3\varepsilon; \end{aligned}$$

in other words, $\phi(T_0) \in V$. Since $\phi(T_0) \in K$, this completes the proof that $\phi(\mathcal{R}_1)$ is dense in $(\mathcal{R}_\sharp^*)_1$.

Finally, the uniform continuity of the inverse mapping on K follows from the identity

$$|\langle Tx, y\rangle| = |\phi(T)\,(f_{x,y}))| \quad (x, y \in H;\ T \in \mathcal{R}_1)\, ,$$

with reference to the definitions of the weak*– and weak–operator topologies, and to Proposition 1.2.8 on page 19 of [12]. Q.E.D.

We proved in [7] (see also [8]) that, under the hypotheses of Theorem 1, the ultraweakly continuous linear functionals on $\mathcal{R}$ extend to ultraweakly continuous linear functionals on $\mathcal{B}(H)$ and are precisely those functionals f_A mapping T to Trace(TA), with A a trace–class operator on H. The norm of f_A on $\mathcal{R}$ is

$$\|f_A\|_{\mathcal{R}} = \sup\{|\mathrm{Trace}(TA)| : T \in \mathcal{R}_1\}\, ,$$

which in the case $\mathcal{R} = \mathcal{B}(H)$ equals the trace–class norm

$$\|A\|_2 = \mathrm{Trace}\,(A)$$

of A (see [4]).

Taken with Theorem 1, these observations lead to

Corollary 2 *Let $\mathcal{R}$ be a linear subset of $\mathcal{B}(H)$ such that $\mathcal{R}_1$ is totally bounded in the weak–operator topology τ_w. Let $\mathcal{T}(H)$ denote the set of trace–class operators on H, taken with the norm*

$$\|A\|_{\mathcal{R}} = \sup\{|\mathrm{Trace}(TA)| : T \in \mathcal{R}_1\},$$

Then

$$\Phi(T)(A) = \mathrm{Trace}(TA) \quad (T \in \mathcal{R},\ A \in \mathcal{T}(H))$$

defines a one–one linear mapping Φ of $\mathcal{R}$ into the dual space $\mathcal{T}(H)^*$ with the following properties.

(i) $\Phi(\mathcal{R}_1)$ *is dense in the unit ball $\mathcal{T}(H)_1^*$ of $\mathcal{T}(H)^*$.*

(ii) Φ *is uniformly continuous on $\mathcal{R}_1$.*

(iii) *the restriction of Φ^{-1} to $\Phi(\mathcal{R}_1)$ is uniformly continuous relative to the weak*–topology on $\Phi(\mathcal{R}_1)$ and the weak–operator topology on $\mathcal{R}_1$.*

Corollary 3 *Under the hypotheses of* Theorem 1 *and* Corollary 2, *the following conditions are equivalent.*

(i) $\mathcal{R}_1$ *is weak–operator complete.*

(ii) ϕ *maps $\mathcal{R}_1$ onto the unit ball of $\mathcal{R}_\sharp^*$.*

(iii) Φ *maps $\mathcal{R}_1$ onto the unit ball of $\mathcal{T}(H)^*$ relative to the norm $\|\cdot\|_{\mathcal{R}}$.*

PROOF. This is a special case of the following general lemma about metric spaces, whose straightforward proof we omit. Q.E.D.

Lemma 4 *Let X be a metric space, Y a complete metric space, and ϕ a one–one uniformly continuous mapping of X onto a dense subset of Y such that ϕ^{-1} is uniformly continuous on $\phi(X)$. Then X is complete if and only if $\phi(X) = Y$.*

Classically, any von Neumann algebra—that is, weak–operator closed *–subalgebra of $\mathcal{B}(H)$—can be identified, via the mapping ϕ, with the dual of its predual $\mathcal{R}_\sharp$ ([12], Vol. 2, page 482). If this were provable constructively, then we could use Theorem 1 to prove that $\mathcal{B}_1(H)$ is τ_w–complete, which, as mentioned above, we cannot do within constructive mathematics. Thus Theorem 1 appears to be the best general constructive result of its type.

http://www.springer.com/978-1-4020-0152-9

Reuniting the Antipodes - Constructive and Nonstandard Views of the Continuum

Symposium Proceedings, San Servolo, Venice, Italy, May 16–22, 1999

Schuster, P.; Berger, U.; Osswald, H. (Eds.)

2001, XIII, 329 p., Hardcover
ISBN: 978-1-4020-0152-9