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SUPERVALUATIONAL FREE LOGIC AND THE LOGIC OF INFORMATION GROWTH

0. INTRODUCTION

The use of supervaluations in free logic was proposed by van Fraassen (see his (1966) and (1969)) as a useful device for saving the validity of all classical tautologies within free logic. Some years later, E. Bencivenga refined the idea and proposed an intuitively appealing semantics for supervaluational free logic (see, particularly, his works (1980) and (1986)). The main idea in this semantics is that free models (models in which some constants may lack any reference) may be variously completed by interpreting undefined constants in *all* possible ways. This leads us to consider *completions* of models, that is, totally defined models which complete free models in all possible ways. But this can be seen as having a partial information and increasing it in several manners. This is the main idea underlying this paper. We shall propose translations of supervaluational free logic to other logics which can be used to formalize information growth: modal logic and Data Logic.

It has been noticed (see, for instance, van Benthem (1986)) that partial information and its growth can be modelled within modal logic (particularly within extensions of S4 modal systems). The classical translation of intuitionistic propositional logic to modal logic is a good example, but not the only one: see, for instance, van Benthem (1986), and Barba (1989a). After the considerations given above, it seems natural to try a similar translation of supervaluational free logic to modal logic. Some results in this direction were obtained in Barba (1989), but the class of modal models used there was a very unusual one (from a modal point of view), as very restrictive conditions on models (K-models) were imposed. Our first main result in this paper will be a reduction of supervaluational free logic to a very natural quantified extension of the modal system S4.1.

Our second main goal will be concerned with Data Semantics. Data Semantics was proposed in Veltman (1981) as an attempt to capture the logic of possible growth of information. Since supervaluational semantics for free logic can be seen as a special case of information growth, it seems quite natural to reduce supervaluational free logic to Data Logic, and such

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a reduction will be carried out in this paper. However, Data Logic, as presented in Veltman (1981) and van Benthem (1986), is just a propositional logic, so we shall need to extend Data Semantics to a quantified language in order to achieve the proposed reduction.

But first of all let us recall the main ideas of Bencivenga's semantics.

1. SUPERVALUATIONAL SEMANTICS FOR FREE LOGIC

Let *SFL* be a first order language with individual constant symbols and identity, no free variables occurring in well formed formulas (wffs, for brevity) and an existence monadic predicate symbol $E!$.

A free structure is a pair $F = \langle E, f\rangle$ where E is a set of individuals and f is a function partially defined over constants and totally defined over predicates in the usual way. We shall follow Smullyan's (1968) strategy and make wide use of U-formulas all along this work. Given a domain U, a U-formula (for any given language) is a formula in which some constants have been replaced by elements of the domain. Both constants and members of U are called parameters in U. Then, when we have an interpretation function h (partially) defined for constants we can extend this function to a (partial) function over the set of parameters in a very natural way: if $r \in U$, then $h(r) = r$. We shall assume this kind of extension for interpretation functions in all semantic systems in this work. Now, with the ideas above applied to the domain E and the interpretation function f, our structure F yields a partial valuation V_1 for E-formulas in the following way:

(i) If $f(s_i)$ is defined for $1 \leq i \leq n$, then $V_1(Ps_1 \ldots s_n) = 1$ if $\langle f(s_i)\rangle \in f(P)$, and $V_1(Ps_1 \ldots s_n) = 0$ otherwise. (Here $\langle f(s_i)\rangle$ abbreviates the n-tuple $\langle f(s_1) \ldots f(s_n)\rangle$. We shall use this kind of abbreviation very often in the sequel).

(ii) If $f(s_i)$ is defined for $i = 1, 2$ then $V_1(s_1 = s_2) = 1$ if $f(s_1) = f(s_2)$, and $V_1(s_1 = s_2) = 0$ otherwise. If exactly one among $f(s_1)$ and $f(s_2)$ is defined, $V_1(s_1 = s_2) = 0$.

(iii) $V_1(E!s) = 1$ if $f(s)$ is defined, and otherwise $V_1(E!s) = 0$.

(iv) $V_1(\neg A) = 1$ if $V_1(A) = 0$, and $V_1(\neg A) = 0$ if $V_1(A) = 1$.

(v) If $V_1(A)$ and $V_1(B)$ are both defined, $V_1(A \wedge B) = 1$ if $V_1(A) = V_1(B) = 1$, and $V_1(A \wedge B) = 0$ otherwise.

(vi) $V_1(\forall xA) = 1$ if for every $r \in E$, $V_1(A[r/x]) = 1$, and $V_1(\forall xA) = 0$ if for some $r \in E$, $V_1(A[r/x]) = 0$.

(vii) $V_1(A)$ is undefined whenever none of the clauses above is satisfied.

It should be noticed that Bencivenga's clause (vi) differs from the one proposed here, as he adopts a substitutional interpretation of quantifiers, while we prefer a more standard one, as in Barba (1987) and (1989). This is in fact the only substantial difference between Bencivenga's semantics and the semantics presented here, all other variations being purely notational.

Now, given the free model F, we can consider completions $F' = \langle E', f' \rangle$ of F, which are classical models satisfying the conditions $E \subseteq E'$, $f(P) \subseteq f'(P)$ for every predicate P, and $f'(c) = f(c)$ whenever $f(c)$ is defined, for every constant c. Thus, in a sense, each completion of F is a different way in which the information contained in F could be increased. F' yields a classical valuation V_2 in the standard way, but this valuation may change the truth-values assigned by V_1 to some formulas, so we must define a valuation V_3 which combines V_1 and V_2 and respects the information supplied by V_1. This can be done as follows:

(i) Let A be any atomic E-formula. Then, if $V_1(A)$ is defined $V_3(A) = V_1(A)$, and otherwise $V_3(A) = V_2(A)$.
(ii) $V_3(\neg A) = 1$ if $V_3(A) = 0$, and $V_3(\neg A) = 0$ if $V_3(A) = 1$.
(iii) $V_3(A \wedge B) = 1$ if $V_3(A) = V_3(B) = 1$, and $V_3(A \wedge B) = 0$ otherwise.
(iv) $V_3(\forall xA) = 1$ if for every $r \in E$, $V_3(A[r/x]) = 1$, and $V_3(\forall xA) = 0$ if for some $r \in E$, $V_3(A[r/x]) = 0$.

Finally, the supervaluation SV determined by F is a partial valuation such that

$SV(A) = 1$ iff for every completion F' of F, $V_3(A) = 1$,
$SV(A) = 0$ iff for every completion F' of F, $V_3(A) = 0$.

Thus we can say that supervaluational truth in a free model is truth under all possible completions of the model, provided that the information supplied by the free model itself is not ignored (this is Bencivenga's principle of prevalence of reality). However, it can be noticed that the information contained in valuations V_2 would be more close to that contained in V_1 if we adopt a more restrictive definition of completions of a free model F. Suppose we impose on completions F' of F the following two conditions:

C1 For each constant c, if $f(c)$ is not defined, then $f(c) \in E' - E$.
C2 $f'(P) \cap E^n = f(P)$, for every n-ary predicate P.

Falsity of formulas $Ps_1 \ldots s_n$ and falsity of identities $s = t$ would then be preserved from valuations V_1 to valuations V_2. Let us explain what this means: we know that $V_1(Ps_1...s_n) = 1$ implies $V_2(Ps_1...s_n) = 1$, because $f(P) \subseteq f'(P)$. But $V_1(Ps_1...s_n) = 0$ does not, in general, imply $V_2(Ps_1...s_n) = 0$, because the case in which $f(s_i)$ is defined for $1 \leq i \leq n$ and $\langle f(s_i)\rangle \notin f(P)$ but $\langle f(s_i)\rangle \in f'(P)$ is not excluded. However, if C2 holds this can be no longer the case. If we now turn our attention to identities $s = t$ we note that V_1 and V_2 agree whenever both $f(s)$ and $f(t)$ are defined, but they may differ when exactly one of them is defined, because then $V_1(s = t) = 0$, but f' may be such that $f'(s) = f'(t)$, and thus $V_2(s = t) = 1$. This cannot happen if C1 holds, because then the undefined constant must take (in the completion) a value outside E, which is different from the value of the previously defined constant (which, of course, belongs to E). Completions satisfying C1 and C2 have the technical advantage that for any A of the form $Ps_1...s_n$ or $s = t$, if $V_1(A)$ is defined then $V_1(A) = V_2(A)$, which implies that in order to calculate $V_3(A)$ we only need to take into account the completion F' and its valuation V_2. Thus we can establish:

PROPOSITION 1: Let F be any free structure, F' a completion of F satisfying C1 and C2 and A any E-formula of the form $Ps_1...s_n$ or $s = t$. Then $V_3(A) = V_2(A)$, independently of whether all parameters s, t, $s_1...s_n$ are defined in F or not. Or, in other words, $V_3(Ps_1...s_n) = 1$ iff $\langle f'(s_i)\rangle \in f'(P)$, and $V_3(s = t) = 1$ iff $f'(s) = f'(t)$. ■

There may be philosophical reasons for not imposing any restrictions on completions (and this seems to be Bencivenga's opinion). But, fortunately, philosophical convictions and technical advantages do not conflict (at least in this case), because the supervaluation determined by completions satisfying C1 and C2 would be exactly the same as the one determined by all completions, as the following lemma proves:

LEMMA 1: Let F be a free model and F' any completion of F. Then there is a completion F^* of F satisfying C1 and C2 such that $V_3(A) = V_3^*(A)$ for every E-formula A.

PROOF: Let $Undef = \{c\colon f(c)$ is undefined$\}$, and define an equivalence relation $\approx$ on $Undef$ by the clause $c \approx d$ iff $f'(c) = f'(d)$, for $c, d \in Undef$. For each equivalence class $[c]$ let r_c be a different object not in E. Thus, for every $c, d \in Undef$, $r_c = r_d$ iff $c \approx d$. Define $F^* = \langle E^*, f^*\rangle$ as follows:

$E^* = E \cup \{r_c: c \in Undef\}$, and
$f^*(P) = f(P) \cup \{\langle t_i \rangle: t_j = r_c$, for some j, $1 \leq j \leq n$, and $c \in Undef$, and $\langle t'_i \rangle \in f'(P)\}$,

where $t'_j = f(c)$ if $t_j = r_c$, and otherwise $t'_j = t_j$. Recall that $\langle t_i \rangle$ is $\langle t_1 \ldots t_n \rangle$. The lemma can now be proved by induction. It should be kept in mind that we are dealing with E-formulas, not with E'- or E^*-formulas.

– Suppose A is $Ps_1 \ldots s_n$. If $f(s_i)$ is defined for $1 \leq i \leq n$, $V_3(A) = V_3{}^*(A) = V_1(A)$. Iff $f(s_j)$ is undefined for some j, such an s_j must be a constant $c \in Undef$, and then $V_3(A) = V_2(A) = 1$ iff $\langle f'(s_i) \rangle \in f'(P)$. Let $t_i = f^*(s_i)$ if $s_i \in Undef$, and otherwise let $t_i = f'(s_i) \in E$ (for, in this case, s_i is either a constant defined in F or a member of E). Let $t_i = f^*(s_i)$. Then, clearly, $\langle t_i \rangle$ and $\langle t'_i \rangle$ are as in definition of $f^*(P)$, and thus $V_3(A) = V_2(A) = 1$ iff $\langle t'_i \rangle = \langle f'(s_i) \rangle \in f'(P)$, iff $\langle t_i \rangle = \langle f^*(s_i) \rangle \in f^*(P)$, iff $V_3{}^*(A) = V_2{}^*(A) = 1$.

– Suppose A is $s_1 = s_2$. $V_3(A)$ $[V_3{}^*(A)]$ depends on F' $[F^*]$ only if neither $f(s_1)$ nor $f(s_2)$ are defined. But then both s_1 and s_2 are constants in $Undef$, and $V_3(A) = 1$ iff $f'(s_1) = f'(s_2)$, iff $s_1 \approx s_2$ iff $f^*(s_1) = f^*(s_2)$ iff $V_3{}^*(A) = 1$.

– The cases in which A is $E!s$ and the induction step are obvious and omitted. ■

According to this result, in so far as we are interested only in valuations V_3, we can assume that all completions satisfy conditions C1 and C2 above, and we shall make use of this assumption in the sequel in order to simplify proofs.

2. Reduction of *SFL* to QS4.1

First, we need a standard modal quantified language *MQL* and S4.1-Kripke models for it. Our QS4.1-models will be structures $M = \langle W, R, U^+, U, h \rangle$, where W is a non-empty set and $\langle W, R \rangle$ is a partial order such that for every $v \in W$ there is some w which is a dead end accessible from v, that is, vRw and wRw' implies $w = w'$. (We are using the expression 'dead end' in a sense slightly different from the usual one in modal logic. In our sense dead ends are reflexive points). U^+ is a set and U is a function assigning a subset $U(w)$ of U^+ to each possible world $w \in W$ and subject to a nested-domains condition: if vRw then $U(v) \subseteq U(w)$. Finally, h is an interpretation function such that $h(c) \in U^+$ (notice that we assume rigid terms, defined everywhere) for every constant c and $h(P, w) \in U(w)^n$, for every n-ary predicate P. Valuation functions (for U^+-formulas) are determined in the usual form, with the following clauses for atomic formulas and for the quantifier $\forall$:

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