CHAPTER 2

VERISIMILITUDE

In Chapter 1, I introduced Popper's comparative content definition, which is based on truth-value and logical strength, and showed how it excludes the comparison of two different false theories. After the publication of this peculiarity in 1974,[1] Miller and Kuipers searched independently for another way to formalize Popper's intuitions about verisimilitude. Their endeavours resulted in *comparative* content definitions (Kuipers calls his version the *naïve* comparative definition). Miller formulates a distance function which has as its codomain the original Boolean algebra instead of the real numbers. Both definitions are almost identical to the consequence definition also hinted at in the first chapter (subsection 1.4.1). Here we examine Miller's and Kuipers's content proposals.

In the present and next chapter, I shall present the content and likeness orderings of the elements of a finite Lindenbaum algebra. In Chapter 1, we saw that the definitions vary considerably regarding their applications and interpretations, and that each has its own level of abstraction. This complicates the comparison. Without exception, however, the definitions can all be used to order propositions, and therefore, the Lindenbaum algebra provides good grounds for a thorough basic comparison. There are also other reasons for choosing a Boolean algebra as the basic application for the comparison. The Boolean algebra relates to the algebras of sets such as the *Brouwerian* algebra and the *Stone space*, which form more sophisticated interpretations. Another important extension concerns modal algebras. We shall encounter all of these in due course. As it will turn out, already with respect to Boolean Algebras, the various definitions will produce different orderings.

In Section 2.1, I introduce the notion of a Lindenbaum algebra which is central to my comparison. Miller's (1978) definition, and Kuipers's comparative content definition are introduced in the second and third section. Their proposals are very similar, and both turn out to be weaker versions of the consequence definition. In Section 2.4, we discuss various ways to formulate the symmetric difference measure. The characteristics of Miller's and Kuipers's proposals are the subject of Section 2.5; and finally, in Section 2.6, I reformulate Kuipers's content definition in modal terms, and end with a modal version of the consequence definition that fully restores the falsity clause.

2.1. THE LINDENBAUM ALGEBRA

Let us consider a language $\mathscr{L}$ which has as a vocabulary, voc($\mathscr{L}$), a (possibly infinite) set of atomic propositions. As explained in Chapter 1, a Lindenbaum (-Tarski) algebra $\mathscr{B}$ of $\mathscr{L}$, is a Boolean algebra, the elements of which are equivalence classes of $\mathscr{L}$ sentences. For all practical purposes, we let one element of such an equivalence class represent the entire class, such that the elements of the algebra are single sentences. In this Boolean algebra, the 0 and 1 represent the contradiction and the tautology, respectively. Furthermore, the usual + and . represent the disjunction and the conjunction of $\mathscr{L}$; and the $\leq$ sign refers to derivability in $\mathscr{L}$. An element t is an *atom* in the Lindenbaum algebra $\mathscr{B}$ iff $a = 0$ obtains for all a such that $a < t$. A theory is complete if it renders all sentences of the language under consideration false or true. Thus, t is an atom in the algebra if and only if t is complete. Often, a Boolean algebra is isomorphic to an algebra of sets. Then, the $\leq$ sign represents the set inclusion, and the + and the . in $\mathscr{B}$ (the disjunction and conjunction of $\mathscr{L}$) represent the union and intersection of the set algebra, respectively.

EXAMPLE: As an example we sketch the Lindenbaum algebra, $\mathscr{B}$ of $\mathscr{L}[p, q]$. This $\mathscr{L}$ is a propositional language with descriptive vocabulary $\{p, q\}$. Recall that the elements of $\mathscr{B}$ are sets of equivalent sentences. In our representation, we let single sentences represent those equivalence classes; and the lower of two connected sentences implies the upper one. For instance $p \wedge q$ implies $\neg p \vee q$.

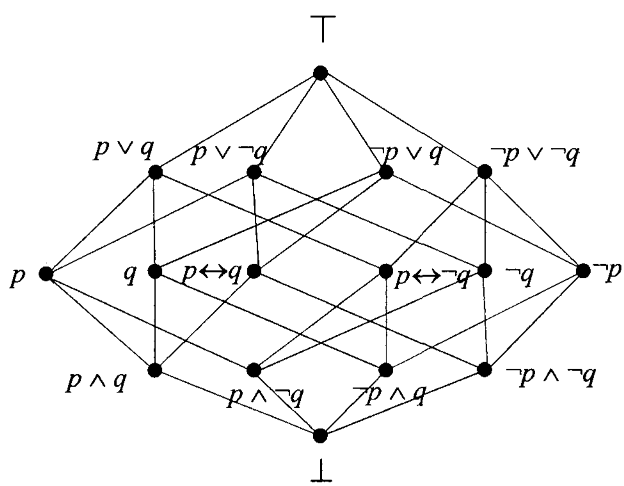

Fig. 1. Lindenbaum algebra $\mathscr{B}$ of $\mathscr{L}[p,q]$.

All the elements of the second layer are atoms of the algebra. They are the constituents of $\mathscr{L}[p,q]$. Of course, if voc($\mathscr{L}$) is finite, every theory of $\mathscr{L}$ is axiomatizable; however, this is different if voc($\mathscr{L}$) is infinite. Intuitively, this means that the atomic propositions p_i which are true according to the theory, can form such a haphazard set that it is impossible to enumerate all of them by any recursive method. With respect to a propositional language with an infinite vocabulary $\mathscr{L}[p_1, p_2, \ldots]$, the elements of the Lindenbaum algebra only represent the axiomatizable theories. The infinite Lindenbaum algebra does not contain atoms since for all elements a in the algebra, there is an element b with $0 < b < a$. Thus, if the truth of $\mathscr{L}[p_1, p_2, \ldots]$ is complete, then it is not an atom of the algebra, although it might be axiomatizable. *End Example*

2.2. MILLER'S CONTENT DEFINITION

Miller's verisimilitude definition is to be found in Miller (1978). Its main message is that on a Boolean (set) algebra, the symmetric difference is the only *normal, downward strictly monotone autometric* (I shall explain these notions below), and therefore it is the most appropriate distance measure on the algebra. It establishes a verisimilitude ordering as the distance between theories is inversely proportional to their similarity.

Miller's (1978) publication consists of three parts. The first one concerns the symmetric difference measure on a *Lindenbaum* algebra (the first and second section of Miller (1978)), and the second, an autometric on *Brouwerian* algebras, which represent richer languages than the propositional ones (his third section). Unfortunately, in contrast to Lindenbaum algebras, there is no autometric operation on a Brouwerian algebra that is normal and downward strictly monotone. Then, in the third part (the fourth section), Miller proposes a second strategy to extend his Lindenbaum approach. This model theoretical approach uses the autometric on the Boolean set algebra of the *Stone space* of the $\mathscr{L}$-models modulo elementary equivalence. Miller shows that "this representation effectively allows us to forget about Brouwerian algebras altogether."[2] The important results of his Brouwerian strategy, however, remain valid. Although I sketch the three parts of the paper, I shall take the model theoretic version of Miller's proposal as the outcome of his considerations (the impatient reader may skip the sequel and pick up the thread at definition 2.1. (page 42).[3]

2.2.1. The Symmetric Difference on the Lindenbaum Algebra

Miller's point of departure is to take literally the idea that theories "can be close to or distant from one another."[4] This distance function, which applies to theories, has to fulfill the customary axioms of a metric. Such a distance function $d(B,A)$ for theories must satisfy the customary conditions (subsection 1.5.2):

$$d(A, B) = 0 \text{ if and only if } A = B$$
$$d(A, B) = d(B, A)$$
$$d(A, C) \leq d(A, B) + d(B, C) \text{ (the triangle inequality)}$$

Obviously, just as in the Euclidian space, there are lots of distance functions fulfilling these axioms. How can the number of distance functions be reduced? This can be achieved, by choosing the original Boolean algebra as the codomain for the function instead of the real numbers. Miller uses Ellis's (1951) claim that the *symmetric difference* of a and b, defined by

$$\Delta(a, b) := (a.-b) + (b.-a)$$

fulfils the customary axioms of a distance function. Note that in a Boolean set algebra $\Delta(a, b)$ designates $(a-b) \cup (b-a)$ and in a Lindenbaum algebra, it means $a \leftrightarrow \neg b$. According to Ellis, a Boolean algebra $\mathcal{B}$ is *autometrized* if the codomain of its metric is $\mathcal{B}$ itself. The symmetric difference fulfils this demand. Thus, Miller calls the distance function $*$ on a Boolean algebra $\mathcal{B}$ with codomain $\mathcal{B}$ an *autometric*.

The symmetric difference is not the only possible autometric. Miller introduces two conditions that together are necessary and sufficient for an autometric to be equal to the symmetric difference. Following Ellis, he calls an autometric *normal* if there is an element e such that for all $a \in \mathcal{B}$: $a * e = a$. In a Lindenbaum algebra e is the contradiction. Next, he calls an autometric operation *downward strictly monotone* iff

$$c < b < a \Rightarrow b*a < c*a$$

Miller proves subsequently that if $*$ is a normal, downward strictly monotone autometric on the Boolean algebra $\mathcal{B}$, it is the symmetric difference. If being normal and downward strict monotonicity are necessary conditions for a verisimilitude definition, then Miller has found objective reasons to sort out the symmetric difference as the favourite metric on a Boolean algebra. Let τ be the complete truth of a propositional language; in other words, τ is an *atom* in the Lindenbaum algebra $\mathcal{B}$ ($\mathcal{B}$ is finite). The theory ψ is more verisimilar than φ if its distance to the truth is smaller than the distance between φ and the truth τ. In terms of the symmetric difference on $\mathcal{B}$ this definition reads:

(1) $\psi \,\Delta\, \tau \leq \varphi \,\Delta\, \tau$ (that is $\psi \leftrightarrow \neg\tau \vDash \varphi \leftrightarrow \neg\tau$, or $\varphi \leftrightarrow \tau \vDash \psi \leftrightarrow \tau$; see Observation 2.2, p. 52)

If we assume, following Miller, the completeness of the truth, this definition has three consequences of which the first two are also consequences of Popper's definition:[5]

(2) "the truth content of a false theory is closer to the truth than is the false theory itself," since the truth content of a false theory φ is equal to $\varphi \vee \tau$, and $\varphi \leftrightarrow \tau \vDash (\varphi \vee \tau) \leftrightarrow \tau$

(3) "the stronger of two comparable true theories is closer to the truth," since if $\tau \vDash \psi$ and $\tau \vDash \varphi$ (†), then $\psi \leftrightarrow \neg\tau$ equals $\psi \wedge \neg\tau$. Therefore, definition (2.1) reduces to $\psi \wedge \neg\tau \vDash \varphi$, which with (†), equals $\psi \vDash \varphi$

(4) "the stronger of two comparable false theories is closer to the truth", since if $\psi \vDash \neg\tau$, then $\psi \leftrightarrow \neg\tau \equiv \psi \vee \tau$. Thus definition (2.1) reduces to $\psi \vDash \varphi \vee \tau$; but as $\psi \vDash \tau$ it follows that $\psi \vDash \varphi$ (*the child's-play objection*).

It is the last conclusion that Miller finds "altogether less amusing". After all, it means that for false theories, and all actual theories are strictly speaking false, the verisimilitude relation between theories is identical to the relation of logical deduction.

In the next two subsections, we shall encounter Miller's endeavour to extend his autometric proposal for propositions, $\varphi \leftrightarrow \tau \vDash \psi \leftrightarrow \tau$, to the case in which the truth in not finitely axiomatizable. First, he seeks an adequate autometric on the Brouwerian algebra in which every element is a consequence class. After the failure of this attempt, he successfully considers the autometric on Stone space of the models of $\mathscr{L}$. First, I introduce Miller's first attempt; next I shall present the autometric on the Stone space.

2.2.2. No Symmetric Difference Measure on Brouwerian Algebra's

In the second part (section three) of Miller's paper, the answer to the question how theories should be ordered when couched in richer than finite propositional languages is formulated. Since, then, the set of all true propositions need not be finitely axiomatizable, Miller uses Tarski's *calculus of deductive systems*. A deductive system A is a set of sentences closed under logical deduction, $A = \mathrm{Cn}(A)$. In Tarski's calculus, the set inclusion partially orders the deductive systems, which form a lattice. Consequently, the greatest lower bound of A and B is $A \cap B$. The least upper bound is, contrary to one's first intuitions, not equal to $A \cup B$ since $\mathrm{Cn}(A) \cup \mathrm{Cn}(B) \neq \mathrm{Cn}(A \cup B)$. For instance, if $A := \mathrm{Cn}(\varphi \rightarrow \psi)$ and $B := \mathrm{Cn}(\varphi)$, then $\psi \notin \mathrm{Cn}(A) \cup \mathrm{Cn}(B)$ but $\psi \in \mathrm{Cn}(A \cup B)$. The least upper bound is defined by $\mathrm{Cn}\{a \wedge b \mid a \in A, b \in B\}$. The top of the algebra is equal to Sent($\mathscr{L}$), the set of all $\mathscr{L}$-sentences, and the bottom is the set of tautologies. Miller chooses to use the dual of Tarski's calculus to simplify the comparison with the Lindenbaum algebra top

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