3
Random Walks

Those cannot remember the past are condemned to repeat it.
—Santayanna

Random walks entered mathematics early on through the analysis of gambling and other games of chance. To cite a typical example, let X_0 denote the initial fortune of a certain gambler and let X_n stand for the amount won (if $X_n \geq 0$) or lost (if $X_n \leq 0$) the nth time that the gambler places a bet. In the simplest gambling situations, the X_n's are i.i.d., and the gambler's fortune at time n is described by the partial sum $S_n = \sum_{j=0}^{n} X_j$. The stochastic process $S = (S_n;\ n \geq 0)$ is called a one-dimensional random walk and lies at the heart of modern, as well as classical, probability theory. This chapter is a study of some properties of systems of such walks.

The main problem addressed here is, *under what conditions does the random walk return to* 0 *infinitely often?* To see how this may come up, suppose the gambler plays *ad infinitum* and has an unbounded credit line. We then wish to know under what conditions the gambler can break even, infinitely many times, as he or she plays on. In the language of the theory of Markov chains, we wish to know when the state 0 is *recurrent*.

The analogous problem for systems of random walks is more intricate and is the subject of much of this chapter: Suppose the X_j's are i.i.d. random vectors in d-space. Then, the d-dimensional random walk models the movement of a small particle in a homogeneous medium. Suppose we have N particles, each of which paints every point that it visits. If each individual particle uses a distinct color, under what conditions do the N random lines created by the N random particles cross paths infinitely many times? These are some of the main problems that are taken up in this chapter.

1 One-Parameter Random Walks

The stochastic process $S = (S_n;\ n \geq 1)$ is a **random walk** if it has stationary, independent increments. To put it another way, we consider independent, identically distributed random variables $X_1, X_2, \ldots$, all taking values in $\mathbb{R}^d$, and define the corresponding random walk $n \mapsto S_n$ as $S_n = \sum_{i=1}^n X_i$ ($n = 1, 2, \ldots$). Clearly, $X_1 = S_1$, and for all $n \geq 2$, $X_n = S_n - S_{n-1}$, when $n \geq 2$. Thus, we are justified in calling the X_i's the **increments** of S. This is a review section on one-parameter random walks; we develop the theory with an eye toward multiparameter extensions that will be developed in the remainder of this chapter.

1.1 Transition Operators

Suppose $S = (S_n;\ n \geq 1)$ is a d-dimensional random walk with increments $X = (X_n;\ n \geq 1)$. For all $n \geq 1$, define $\mathcal{F}_n$ to be the σ-field generated by $X_1, \ldots, X_n$. It is simple to see that $\mathcal{F}_n$ is precisely the σ-field generated by $S_1, \ldots, S_n$. In the notation of Chapter 1, we have shown that $\mathcal{F} = (\mathcal{F}_n;\ n \geq 1)$ is the history of the stochastic process S.

It is always the case that the study of the stochastic process S is equivalent to the analysis of probabilities of the form

$$\mathbb{P}(S_{n_1} \in E_1, S_{n_2} \in E_2, \ldots, S_{n_k} \in E_k),$$

where $k, n_1, \ldots, n_k \geq 1$ are integers and $E_1, \ldots, E_k$ are measurable subsets of $\mathbb{R}^d$. These probabilities are called the **finite-dimensional distributions** of S. It turns out that the finite-dimensional distributions of the random walk S are completely determined by the collection $\mathbb{P}(X_1 + x \in E)$, where $E \subset \mathbb{R}^d$ is measurable and $x \in \mathbb{R}^d$. A precise form of such a statement is called the Markov property; we shall come to this later. Bearing this discussion in mind, we define for all measurable functions $f : \mathbb{R}^d \to \mathbb{R}$, all $n \geq 1$, and $x \in \mathbb{R}^d$,

$$\mathcal{T}_n f(x) = \mathbb{E}\big[f(S_n + x)\big].$$

In particular, note that for all Borel sets $E \subset \mathbb{R}^d$, $\mathcal{T}_1 \mathbb{1}_E(x) = \mathbb{P}(X_1 + x \in E)$. Thus, once we know the operator $\mathcal{T}_n$, we know how to compute these probabilities. We begin our study of random walks by first analyzing these operators.

Note that $\mathcal{T}_n$ is a **bounded linear operator**: For all bounded measurable $f, g : \mathbb{R}^d \to \mathbb{R}$, $n \geq 1$, $x \in \mathbb{R}^d$ and all $\alpha, \beta \in \mathbb{R}$,

(i) $\sup_{x \in \mathbb{R}^d} |\mathcal{T}_n f(x)| \leq \sup_{x \in \mathbb{R}^d} |f(x)|$;

(ii) $\mathcal{T}_n(\alpha f + \beta g)(x) = \alpha \mathcal{T}_n f(x) + \beta \mathcal{T}_n g(x)$; and

(iii) $x \mapsto \mathcal{T}_n f(x)$ is measurable.

Next, we interpret $\mathcal{T}_n$ in terms of the conditional distributions of S.

Lemma 1.1.1 *For all $n, k \geq 1$ and all bounded measurable $f : \mathbb{R}^d \to \mathbb{R}$,*

$$\mathbb{E}\big[f(S_{k+n}) \,\big|\, \mathcal{F}_k\big] = \mathbb{E}\big[f(S_{k+n}) \,\big|\, S_k\big] = \mathcal{T}_n f(S_k), \qquad a.s.$$

In particular, for all $x \in \mathbb{R}^d$, $n, k \geq 1$, and all bounded measurable $f : \mathbb{R}^d \to \mathbb{R}$, $\mathcal{T}_{n+k} f(x) = \mathcal{T}_n(\mathcal{T}_k f)(x) = \mathcal{T}_k(\mathcal{T}_n f)(x)$.

In functional-analytic language, $(\mathcal{T}_n;\ n \geq 1)$ is a **semigroup** of operators. To see what the above lemma means, take $f = \mathbb{1}_E$ for some Borel set $E \subset \mathbb{R}^d$. The above says that if k denotes the current time,

1. given the present position S_k, any future position S_{k+n} is conditionally independent of the past positions $S_1, \ldots, S_{k-1}$; and
2. $\mathcal{T}_n \mathbb{1}_E(S_k)$ is the conditional probability of making a transition to E in n steps, given $\mathcal{F}_k$.

Motivated by this, we call $\mathcal{T}_n$ the ***n*-step transition operator of** S.

Proof of Lemma 1.1.1 Note that $S_{k+n} - S_k = \sum_{j=k+1}^{k+n} X_j$ is (a) independent of $\mathcal{F}_k$; and (b) has the same distribution as $S_n = \sum_{j=1}^{n} X_j$. Thus,

$$\begin{aligned}\mathbb{E}\big[f(S_{k+n}) \,\big|\, \mathcal{F}_k\big] &= \mathbb{E}\big[f(S_{k+n} - S_k + S_k) \,\big|\, \mathcal{F}_k\big] = \int f(x + S_k)\, \mathbb{P}(S_n \in dx) \\ &= \mathcal{T}_n f(S_k),\end{aligned}$$

almost surely. From this, we also can conclude the equality regarding the conditional expectation $\mathbb{E}[f(S_{k+n}) \,|\, S_k]$. Applying the preceding to $f(\bullet + x)$, we obtain $\mathbb{E}[f(x + S_{k+n}) \,|\, \mathcal{F}_k] = \mathcal{T}_n f(x + S_k)$, almost surely. Taking expectations, we deduce that $\mathcal{T}_{k+n} f(x) = \mathcal{T}_k(\mathcal{T}_n f)(x)$. The rest follows from reversing the roles of k and n. $\square$

Digression If we define $S_0 = 0$, then for any $x \in \mathbb{R}^d$, we can, and should, think of $x + S$ as our random walk started at x. In particular, S itself should be thought of as the random walk started at the origin. The above lemma suggests the following interpretation: Given the position of the process at time k, the future trajectories of our walk are those of a random walk started at S_k. The following is a more precise formulation of this and is a version of the so-called **Markov property** of S that was alluded to earlier.

Theorem 1.1.1 (The Markov Property) *Fix integers $k \geq 1$, $n \geq 2$ and bounded measurable functions $f_1, \ldots, f_n : \mathbb{R}^d \to \mathbb{R}$. Then, the following holds with probability one:*

$$\mathbb{E}\Big[\prod_{\ell=1}^{n} f_\ell(S_{k+\ell} - S_k) \,\Big|\, \mathcal{F}_k\Big] = \mathbb{E}\Big[\prod_{\ell=1}^{n} f_\ell(S_\ell)\Big].$$

In other words, for any $k \geq 1$, the process $n \mapsto S_{n+k} - S_k$ is (i) independent of $\mathcal{F}_k$; and (ii) has the same finite-dimensional distributions as the original process S.

Recalling that we think of $n \mapsto S_n + x$ as a random walk with increments $X_1, X_2, \ldots$ that starts at $x \in \mathbb{R}^d$, we readily obtain the following useful interpretation of the above.

Corollary 1.1.1 *Suppose $k \geq 1$ is a fixed integer. Then, conditionally on $\sigma(S_k)$, $(S_{k+n};\ k \geq 0)$ is a random walk whose increments have the same distribution as X_1. Moreover, the σ-field generated by $(S_{k+n};\ n \geq 0)$ is conditionally independent of $\mathcal{F}_k$, given $\sigma(S_k)$.*

See Section 3.6 of Chapter 1 for information on conditional independence.

***Exercise* 1.1.1** Carefully prove Corollary 1.1.1. □

Proof of Theorem 1.1.1 Since it depends only on $X_{k+1}, \ldots, X_{k+n}$, the random variable $\prod_{\ell=1}^n f_\ell(S_{k+\ell} - S_k)$ is independent of $(X_1, \ldots, X_k)$ and hence of $\mathcal{F}_k$. (Why?) As a result, with probability one,

$$\mathbb{E}\Big[\prod_{\ell=1}^{n} f_\ell(S_{k+\ell} - S_k) \,\Big|\, \mathcal{F}_k\Big] = \mathbb{E}\Big[\prod_{\ell=1}^{n} f_\ell(S_{k+\ell} - S_k)\Big].$$

On the other hand, the sequence $(X_{k+1}, \ldots, X_{k+n})$ has the same distribution as the sequence $(X_1, \ldots, X_n)$. After performing a little algebra, we can reinterpret this statement as follows: The distribution of the $\mathbb{R}^{nd}$-valued random vector $(S_{k+1}-S_k, \ldots, S_{k+n}-S_k)$ is the same as that of $(S_1, \ldots, S_n)$. In particular, we have $\mathbb{E}[\prod_{\ell=1}^n f_\ell(S_{k+\ell} - S_k)] = \mathbb{E}[\prod_{\ell=1}^n f_\ell(S_\ell)]$, which proves the result. □

It is clear that Corollary 1.1.1 extends the conditional independence assertion of Lemma 1.1.1. However, the latter lemma also contains information on the transition operators, to which we now return.

Corollary 1.1.2 *The transition operators, in fact $\mathcal{T}_1$, uniquely determine the finite-dimensional distributions and vice versa.*

Proof By the very definition of $\mathcal{T}_n$, if we know all finite-dimensional distributions, we can compute $\mathcal{T}_n f(x)$ for all measurable $f : \mathbb{R}^d \to \mathbb{R}_+$, all $n \geq 1$, and all $x \in \mathbb{R}^d$. The converse requires an honest proof. Consider the following proposition:

(Π_n) For all measurable $f_1, \ldots, f_n : \mathbb{R}^d \to \mathbb{R}_+$, $\mathbb{E}[\prod_{\ell=1}^n f_\ell(S_\ell)]$ can be computed from $\mathcal{T}_1$.

Our goal is to show that (Π_n) holds for all $n \geq 1$. We will prove this by using induction on n: Lemma 1.1.1 shows that (Π_1) is true. Thus, we suppose that $(\Pi_1), \ldots, (\Pi_{n-1})$ hold and venture to prove (Π_n). By Lemma 1.1.1, for all measurable $f_1, \ldots, f_n : \mathbb{R}^d \to \mathbb{R}_+$,

$$\mathbb{E}\Big[\prod_{\ell=1}^{n} f_\ell(S_\ell) \,\Big|\, \mathcal{F}_{n-1}\Big] = \prod_{\ell=1}^{n-1} f_\ell(S_\ell) \cdot \mathcal{T}_1 f(S_{n-1}) = \prod_{\ell=1}^{n-1} g_\ell(S_\ell),$$

where $g_i = f_i$ for all $1 \leq i \leq n-2$ and $g_{n-1}(x) = f_{n-1}(x) \cdot \mathcal{T}_1 f_{n-1}(x)$. Taking expectations, we see that $\mathbb{E}[\prod_{\ell=1}^{n} f_\ell(S_\ell)] = \mathbb{E}[\prod_{i=1}^{n-1} g_i(S_i)]$. By Π_{n-1}, this can be written entirely in terms of $\mathcal{T}_1$, thus proving (Π_n). $\square$

Exercise **1.1.2** Find an explicit recursive formula for $\mathbb{E}[\prod_{\ell=1}^{n} f_\ell(S_\ell)]$ in terms of $\mathcal{T}_1$. $\square$

1.2 The Strong Markov Property

Let $S = (S_k;\ k \geq 1)$ denote a d-dimensional random walk with history $\mathcal{F} = (\mathcal{F}_k;\ k \geq 1)$ and increment process $X = (X_k;\ k \geq 1)$. The **strong Markov property** of S states that for any finite stopping time T (with respect to the filtration $\mathcal{F}$), the stochastic process $(S_{k+T} - S_T;\ k \geq 1)$ is independent of $\mathcal{F}_T$ and has the same finite-dimensional distributions as the process S. Roughly speaking, this means that the process $(S_{k+T}; k \geq 1)$ is conditionally independent of $\mathcal{F}_T$ given S_T and is, in distribution, the random walk S started at S_T.

Theorem 1.2.1 (The Strong Markov Property) *Suppose T is a stopping time with respect to $\mathcal{F}$. Given integers $n, k \geq 1$ and bounded, measurable $f_1, \ldots, f_n : \mathbb{R}^d \to \mathbb{R}$,*

$$\mathbb{E}\Big[\prod_{\ell=1}^{n} f_\ell(S_{T+\ell} - S_T) \,\Big|\, \mathcal{F}_T\Big] \mathbb{1}_{(T<\infty)} = \mathbb{E}\Big[\prod_{\ell=1}^{n} f_\ell(S_\ell)\Big] \mathbb{1}_{(T<\infty)}, \qquad a.s.$$

Remarks (i) Given the transition operators, the above expression can be computed using Corollaries 1.1.1 and 1.1.2; see Exercise 1.1.2.

(ii) It is important to realize that the stopping time condition cannot be removed in general, as the following clearly shows.

Exercise **1.2.1** Consider the simple walk on $\mathbb{Z}^1$. Here, the increments $X_1, X_2, \ldots$ take the values ± 1 with probability $\frac{1}{2}$ each. Consider the $\mathbb{N}_0 \cup \{\infty\}$-valued random variable $L = \sup(k \geq 0 :\ S_k \leq -\frac{1}{2}k)$, where $\sup \varnothing = 0$. That is, L designates the last time that the random walk goes below the line $y = -\frac{1}{2}x$.

(i) Show that with probability one, $L < \infty$ and that L is *not* a stopping time with respect to the history of the process S.

(ii) Verify that $\mathcal{F}_L$ is a σ-field and that the process $j \mapsto S_{j+L} - S_L$ is independent of $\mathcal{F}_L$, where $\mathcal{F}_L = \big(A \in \vee_n \mathcal{F}_n : \ A \cap (L \le j) \in \mathcal{F}_j, \text{ for all } j \ge 0\big)$ is defined as if L were a stopping time.

(iii) Show that the stochastic process $j \mapsto S_{L+j} - S_L$ does *not* have the same finite-dimensional distributions as S.

This is a part of a deep result of Williams (1970, 1974).
(HINT: for part (i), you can use a limit theorem; for part (ii), condition on the value of L.) □

Proof of Theorem 1.2.1 For all $\ell \ge 1$, $S_{T+\ell} - S_T = \sum_{j=T+1}^{T+\ell} X_j$. Since for all $j \ge 1$, the event $(T = j)$ is $\mathcal{F}_T$-measurable,

$$\begin{aligned}\mathbb{E}\Big[\prod_{\ell=1}^{n} f_\ell(S_{T+\ell} - S_T)\,\Big|\,\mathcal{F}_T\Big]\mathbb{1}_{(T<\infty)} &= \sum_{j=1}^{\infty}\mathbb{E}\Big[\prod_{\ell=1}^{n} f_\ell(S_{T+\ell} - S_T)\,\Big|\,\mathcal{F}_T\Big]\mathbb{1}_{(T=j)}\\ &= \sum_{j=1}^{\infty}\mathbb{E}\Big[\prod_{\ell=1}^{n} f_\ell(S_{j+\ell} - S_j)\,\Big|\,\mathcal{F}_T\Big]\mathbb{1}_{(T=j)},\end{aligned}$$

almost surely. Regarding $j \ge 1$ as fixed, define $Y = \prod_{\ell=1}^{n} f_\ell(S_{j+\ell} - S_j)$ and for all $k \ge 1$, let $M_k = \mathbb{E}[Y \,|\, \mathcal{F}_k]$. By Theorem 1.1.1, Chapter 1, with probability one, $M_j \mathbb{1}_{(T=j)} = M_T \mathbb{1}_{(T=j)} = \mathbb{E}[Y|\mathcal{F}_T]\mathbb{1}_{(T=j)}$. Thus,

$$\mathbb{E}\Big[\prod_{\ell=1}^{n} f_\ell(S_{T+\ell} - S_T)\,\Big|\,\mathcal{F}_T\Big]\mathbb{1}_{(T<\infty)} = \sum_{j=1}^{\infty}\mathbb{E}\Big[\prod_{\ell=1}^{n} f_\ell(S_{j+\ell} - S_j)\,|\,\mathcal{F}_j\Big]\mathbb{1}_{(T=j)}.$$

By the stationarity and the independence of the increments of S, the above equals $\mathbb{E}[\prod_{\ell=1}^{n} f_\ell(S_\ell)]\mathbb{1}_{(T<\infty)}$, as desired. □

1.3 Recurrence

Suppose S is a d-dimensional random walk with increment process X and history $\mathcal{F}$. Throughout this section we assume that the X's are taking values in the d-dimensional integer lattice $\mathbb{Z}^d$.

A point $x \in \mathbb{Z}^d$ is said to be **recurrent** if $\mathbb{P}(S_k = x \text{ infinitely often}) > 0$. *When is a point $x \in \mathbb{Z}^d$ recurrent?* In this subsection we will resolve this when x is the origin of $\mathbb{Z}^d$. Since it is the starting position of the random walk, the origin is a very special point; see the Digression in Section 1.1. Recurrence properties of a general point $x \in \mathbb{Z}^d$ are discussed in Section 1.6 below.

Recalling that $\inf \varnothing = \infty$, let $\tau_1 = \inf(j \ge 1 : \ S_j = 0)$; that is, τ_1 is the first time the random walk visits 0. Iteratively define $\tau_{k+1} = \inf(j \ge 1 + \tau_k : \ S_j = 0)$, for $k \ge 1$. It is easy to see that $\tau_1, \tau_2, \ldots$ are stopping times. One should think of τ_1 $(\tau_2, \ldots)$ as the first (second, etc.) time the random walk visits the origin. Among other things, this sequence of visitation times has the following property.

Lemma 1.3.1 *Fix* $n, j \geq 1$. *On* $(\tau_n < \infty)$,

$$\mathbb{P}(\tau_{n+1} - \tau_n = j \,|\, \mathcal{F}_{\tau_n}) = \mathbb{P}(\tau_1 = j), \qquad a.s.$$

Suppose we knew that with probability one, $\tau_n < \infty$ for all $n \geq 1$. The above lemma asserts that in this case, $\tau_1, \tau_2 - \tau_1, \ldots$ is a sequence of independent, identically distributed random variables (why?). Since $\tau_n = \tau_1 + \sum_{j=2}^n (\tau_j - \tau_{j-1})$, $\tau = (\tau_n;\ n \geq 1)$ is then identified as a random walk with nonnegative increments.

Proof This is a consequence of the strong Markov property (see Theorem 1.2.1). In fact, since $S_{\tau_n} = 0$ on $\big(\tau_n < \infty\big)$,

$$\begin{aligned}
&\mathbb{P}\big(\tau_{n+1} - \tau_n = j \,\big|\, \mathcal{F}_{\tau_n}\big) \mathbb{1}_{(\tau_n < \infty)} \\
&= \mathbb{P}\big(S_{\tau_n + \ell} \neq 0 \text{ for all } 1 \leq \ell \leq j-1,\ S_{\tau_n + j} = 0 \,\big|\, \mathcal{F}_{\tau_n}\big) \mathbb{1}_{(\tau_n < \infty)} \\
&= \mathbb{P}\big(S_{\tau_n + \ell} - S_{\tau_n} \neq 0 \text{ for all } 1 \leq \ell \leq j-1,\ S_{\tau_n + j} - S_{\tau_n} = 0 \,\big|\, \mathcal{F}_{\tau_n}\big) \mathbb{1}_{(\tau_n < \infty)} \\
&= \mathbb{P}\big(S_\ell \neq 0 \text{ for all } 1 \leq \ell \leq j-1,\ S_j = 0\ \big) \mathbb{1}_{(\tau_n < \infty)} \\
&= \mathbb{P}\big(\tau_1 = j\big) \mathbb{1}_{(\tau_n < \infty)}.
\end{aligned}$$

The strong Markov property (Theorem 1.2.1) is used in the penultimate line. This proves the result. □

In particular, upon summing Lemma 1.3.1 over all integers $j \geq 1$, we arrive at the following: For all $n \geq 2$,

$$\begin{aligned}
\mathbb{P}(\tau_n < \infty) &= \mathbb{P}(\tau_n - \tau_{n-1} < \infty, \tau_{n-1} < \infty) \\
&= \mathbb{E}\Big[\mathbb{P}(\tau_n - \tau_{n-1} < \infty \,|\, \mathcal{F}_{\tau_{n-1}}) \mathbb{1}_{(\tau_{n-1} < \infty)}\Big] \\
&= \mathbb{P}(\tau_1 < \infty) \cdot \mathbb{P}(\tau_{n-1} < \infty).
\end{aligned}$$

By induction,

$$\mathbb{P}(\tau_n < \infty) = \big\{\mathbb{P}(\tau_1 < \infty)\big\}^n. \tag{1}$$

With the unambiguous understanding that $\infty \leq \infty$, we can deduce that the τ_n's are nondecreasing. Continuity properties of probability measures then imply that

$$\mathbb{P}(0 \text{ is recurrent}) = \lim_{n \to \infty} \mathbb{P}(\tau_n < \infty) = \lim_{n \to \infty} \big\{\mathbb{P}(\tau_1 < \infty)\big\}^n.$$

Taking equation (1) into account, we have proven the following:

Proposition 1.3.1 *The following are equivalent:*

(i) 0 *is recurrent;*

(ii) $\mathbb{P}(S_k = 0$ *infinitely often*$) = 1$*; and*

(iii) $\mathbb{P}(\tau_1 < \infty) = 1$.

Informally, we are stating that if starting from the origin we are sure of returning to the origin, then we will do so infinitely many times. This is an example of the strong Markov property at its finest.

1.4 Classification of Recurrence

A natural question is, *how do the finite-dimensional distributions of a $\mathbb{Z}^d$-valued random walk influence the recurrence of the point* 0*?* For all integers $n \geq 1$, define

$$R_n = 1 + \sum_{k=1}^{n} \mathbb{1}_{(S_k=0)}.$$

Recalling the Digression of Section 1.1, we think of S as starting from the origin, so that at time 0, S is at 0. Viewed as such, R_n denotes the total number of visits to the origin by time n. Note that $R_\infty = \lim_{n\to\infty} R_n$ is a random variable taking values in $\mathbb{N}\cup\{\infty\}$. Proposition 1.3.1 can be restated as follows: $\mathbb{P}(R_\infty = \infty) \in \{0, 1\}$. Moreover, this probability is 1 if and only if 0 is recurrent.

The key to our analysis of recurrence turns out to be $\mathbb{E}[R_\infty] = 1 + \sum_{k=1}^{\infty} \mathbb{P}(S_k = 0)$. In fact, we have the following result, due to G. Pólya, K. L. Chung, and W. H. J. Fuchs, which appeared in Chung and Fuchs (1951) in full generality; see (Pólya 1921; Chung and Ornstein 1962) for some related results. Supplementary Exercise 9 contains a complete statement of the above results: the so-called Chung–Fuchs theorem.

Theorem 1.4.1 (The Pólya Criterion) *The point* 0 *is recurrent if and only if* $\sum_{k=1}^{\infty} \mathbb{P}(S_k = 0) = \infty$.

Informally, S will hit 0 infinitely often if it is expected to do so. For our proof, we need the the following simple and powerful lemma, first found in Paley and Zygmund (1932).

Lemma 1.4.1 (Paley–Zygmund Lemma) *Suppose Z is an almost surely nonnegative random variable. Then for all $\varepsilon \in]0, 1[$,*

$$\mathbb{P}\big(Z \geq \varepsilon\mathbb{E}[Z]\big) \geq (1-\varepsilon)^2\, \frac{\{\mathbb{E}[Z]\}^2}{\mathbb{E}[Z^2]},$$

provided that all of the mentioned expectations exist.

***Exercise* 1.4.1** Prove the Paley–Zygmund lemma.
(HINT: Apply the Cauchy–Schwarz inequality to $\mathbb{E}[Z\mathbb{1}_{(Z\geq\varepsilon\mathbb{E}[Z])}]$.) □

***Exercise* 1.4.2** If Z is a nonnegative random variable that is also in $L^2(\mathbb{P})$, show that $\mathbb{P}(Z=0) \le \mathrm{Var}(Z)/\{\mathbb{E}[Z]\}^2$, where Var denotes the variance. □

***Exercise* 1.4.3** Suppose $\mathsf{E}_1, \mathsf{E}_2, \ldots$ are measurable events such that $\sum_j \mathbb{P}(\mathsf{E}_j) = +\infty$. Prove that whenever

$$\liminf_{n\to\infty} \frac{\sum_{j=1}^n \sum_{k=1}^n \mathbb{P}(\mathsf{E}_j \cap \mathsf{E}_k)}{\left\{\sum_{j=1}^n \mathbb{P}(\mathsf{E}_j)\right\}^2} < \infty,$$

then $\mathbb{P}(\mathsf{E}_n$ infinitely often$) > 0$. This is from (Chung and Erdős 1952; Kochen and Stone 1964).
(HINT: Consider the first two moments of $J_n = \sum_{j=1}^n \mathbb{1}_{\mathsf{E}_j}$.) □

Proof of Theorem 1.4.1 We have already made the observation that $R_\infty \ge 1$ and $\mathbb{E}[R_\infty - 1] = \sum_{k=1}^\infty \mathbb{P}(S_k = 0)$. (Since $R_\infty = \lim_n R_n$, a.s., this is a consequence of the monotone convergence theorem of measure theory.) Thus, $\sum_k \mathbb{P}(S_k = 0) < \infty$ if and only if $\mathbb{E}[R_\infty - 1] < \infty$. Consequently, $\sum_k \mathbb{P}(S_k < \infty) < \infty$ certainly implies that $R_\infty < \infty$, a.s.; that is to say that 0 is not recurrent. Next, we suppose that $\sum_k \mathbb{P}(S_k = 0) = \infty$. It is clear that $\mathbb{E}[R_n - 1] = \sum_{k=1}^n \mathbb{P}(S_k = 0)$ and that this sequence explodes as $n \to \infty$. We now estimate $\mathbb{E}[(R_n-1)^2]$, viz.,

$$\begin{aligned}\mathbb{E}\big[(R_n-1)^2\big] &= \mathbb{E}\Big[\sum_{k=1}^n \mathbb{1}_{(S_k=0)}\Big] + 2\mathbb{E}\Big[\sum\sum_{1\le k<\ell\le n} \mathbb{1}_{(S_k=0)}\mathbb{1}_{(S_\ell=0)}\Big] \\ &= \mathbb{E}[R_n - 1] + 2\sum\sum_{1\le k<\ell\le n} \mathbb{P}(S_k=0)\mathbb{P}(S_{\ell-k}=0),\end{aligned}$$

by the Markov property (Theorem 1.1.1). Relabeling the last summation and possibly adding more nonnegative terms, we arrive at the estimate

$$\mathbb{E}\big[(R_n-1)^2\big] \le \mathbb{E}[R_n-1] + 2\big(\mathbb{E}[R_n-1]\big)^2.$$

Since $R_n - 1 \in \mathbb{N}_0$, $(R_n - 1 > 0) = (R_n \ge 2)$. Applying Lemma 1.4.1 first, and then the above estimate, in this order, we arrive at the following:

$$\mathbb{P}(\tau_1 \le n) = \mathbb{P}(R_n \ge 2) \ge \frac{\big(\mathbb{E}[R_n-1]\big)^2}{\mathbb{E}[R_n-1] + 2\big(\mathbb{E}[R_n-1]\big)^2},$$

where $\tau_1 = \inf\big(j \ge 1 : \; S_j = 0\big)$. Since $\lim_n \mathbb{E}[R_n] = \infty$, this implies that $\mathbb{P}(\tau_1 < \infty) \ge \frac{1}{2}$. By Proposition 1.3.1, whenever $\mathbb{P}(\tau < \infty)$ is positive, it is, in fact, 1. This completes our proof. □

While it was meant to bring forth a powerful technique, our demonstration of Theorem 1.4.1 is not the fastest method for getting there, as we see next.

***Exercise* 1.4.4** Let N denote the total number of returns to zero. That is, $N = \sum_{k=0}^{\infty} \mathbb{1}_{(S_k=0)}$. Show that N is a geometric random variable with mean p^{-1}, where $p = \mathbb{P}(\exists k \geq 1 : \ S_k = 0)$. Use this to verify Pólya's criterion.
(Hint: Show that for all $k \geq 1$, $\mathbb{P}(N \geq k) = p\mathbb{P}(N \geq k-1)$.) $\square$

1.5 Transience

When a point $x \in \mathbb{Z}^d$ is not recurrent for the $\mathbb{Z}^d$-valued random walk S, we say that it is **transient**. It is easy to see that $0 \in \mathbb{Z}^d$ is transient if and only if

$$\lim_{n\to\infty} \mathbb{P}(S_k = 0 \text{ for some } k \geq n) = 0.$$

Thus, a natural measure for the strength of the transience of the origin is the rate at which $\mathbb{P}(S_k = 0 \ \text{ for some } \ k \geq n)$ goes to 0 as n goes to infinity. The following sheds much light on this rate.

Theorem 1.5.1 *If the origin is transient for the $\mathbb{Z}^d$-valued random walk S, the following holds for every integer $n \geq 1$:*

$$\frac{1}{2}\mathcal{T} \leq \mathbb{P}\big(S_k = 0 \textit{ for some } k \geq n\big) \leq 8\mathcal{T},$$

where

$$\mathcal{T} = \frac{\sum_{j=n}^{\infty} \mathbb{P}(S_j = 0)}{1 + \sum_{j=1}^{\infty} \mathbb{P}(S_j = 0)}.$$

This theorem makes the point that as $n \to \infty$, $\mathbb{P}(S_k = 0 \text{ for some } k \geq n)$ goes to zero like a constant multiple of $\sum_{j \geq n} \mathbb{P}(S_j = 0)$.

Remarks

1. This can be sharpened; see Supplementary Exercise 1.

2. Throughout this subsection we implicitly use the notation of Section 1.3 and Section 1.4.

3. It can be shown that $\mathbb{P}(\tau_1 = \infty) = \{1 + \sum_{k=1}^{\infty} \mathbb{P}(S_k = 0)\}^{-1}$; see Supplementary Exercise 1. This is the probability of never hitting 0.

Proof By transience and by Theorem 1.4.1, $\sum_{j=1}^{\infty} \mathbb{P}(S_j = 0) < \infty$. For all $n \geq 1$, let

$$Z = \sum_{j=n}^{\infty} \mathbb{1}_{(S_j=0)} = R_\infty - R_{n-1},$$

where $R_0 = 1$. Clearly, $\mathbb{E}[Z] = \sum_{j=n}^{\infty} \mathbb{P}(S_j = 0)$, which we know is finite. Recall our proof of Theorem 1.4.1; the method used there to estimate

$\mathbb{E}[(R_n - 1)^2]$ can be used here to show that

$$\mathbb{E}[Z^2] \leq 2 \sum_{\ell=n}^{\infty} \mathbb{P}(S_\ell = 0) \cdot \Big\{ 1 + \sum_{j=1}^{\infty} \mathbb{P}(S_j = 0) \Big\}. \tag{1}$$

Since $(Z > 0) \subseteq (S_k = 0 \text{ for some } k \geq n)$, we obtain the lower bound from the Paley–Zygmund lemma (Lemma 1.4.1).

For the upper bound on the probability, define

$$M_k = \mathbb{E}\Big[\sum_{j=n}^{\infty} \mathbb{1}_{(S_j=0)} \,\Big|\, \mathcal{F}_k \Big], \qquad k \geq n.$$

It is not hard to check that $M = (M_k;\ k \geq n)$ is a martingale. Moreover, for all $k \geq n$,

$$M_k \geq \mathbb{E}\big[\sum_{j=k}^{\infty} \mathbb{1}_{(S_j=0)} \,\big|\, \mathcal{F}_k \big] \cdot \mathbb{1}_{(S_k=0)} = \Big\{ 1 + \sum_{j=1}^{\infty} \mathbb{P}(S_{j+k} - S_k = 0 \,|\, \mathcal{F}_k) \Big\} \cdot \mathbb{1}_{(S_k=0)}.$$

We have used the monotone convergence theorem to write the conditional expectation and the sum of the conditional probabilities. By the Markov property (Corollary 1.1.1), $M_k \geq \{1 + \sum_{j=1}^{\infty} \mathbb{P}(S_j = 0)\} \cdot \mathbb{1}_{(S_k=0)}$, almost surely. Taking suprema over all $k \geq n$ and squaring, we obtain the following:

$$\mathbb{1}_{(S_k=0 \text{ for some } k \geq n)} \leq \Big\{ 1 + \sum_{j=1}^{\infty} \mathbb{P}(S_j = 0) \Big\}^{-2} \cdot \sup_{k \geq n} M_k^2. \tag{2}$$

By Doob's strong $(2,2)$ inequality (Theorem 1.4.1, Chapter 1),

$$\mathbb{E}\Big[\sup_{k \geq n} M_k^2 \Big] \leq 4 \sup_{k \geq n} \mathbb{E}[M_k^2].$$

Therefore, by taking expectations in equation (2), we obtain

$$\mathbb{P}(S_k = 0 \text{ for some } k \geq n) \leq 4 \Big\{ 1 + \sum_{k=1}^{\infty} \mathbb{P}(S_j = 0) \Big\}^{-2} \sup_{k \geq n} \mathbb{E}[M_k^2].$$

Jensen's inequality shows that for any $k \geq n$, $\mathbb{E}[M_k^2] \leq \mathbb{E}[Z^2]$. Consequently, equation (1) implies the result. $\square$

1.6 Recurrence of Possible Points

We now return to the question of when a general point $x \in \mathbb{Z}^d$ is recurrent. To illustrate the potential complications, consider the following simple example.

Example Suppose $d = 1$ and the $X_1, X_2, \ldots$ are independent, identically distributed random variables taking the values ± 2 with probability $\frac{1}{2}$ each. Let $S = (S_k;\ k \geq 1)$ denote the random walk whose increments are $X_1, X_2, \ldots$; i.e., $S_n = X_1 + \cdots + X_n$, for all $n \geq 1$. It should be absolutely clear that the point $x = 1$ is not recurrent. In fact, odd values can never be visited by S, and even values can. On the other hand, by the central limit theorem, $\limsup_n S_n = -\liminf_n S_n = +\infty$, almost surely. A little thought reveals that for any even number x, there are infinitely many n's such that $S_n = x$. $\square$

***Exercise* 1.6.1** Use the central limit theorem to show that, in the above example, $\limsup_n S_n = -\liminf_n S_n = +\infty$, a.s. $\square$

In the previous example we constructed a random walk for which all of the even numbers are recurrent, while the odd numbers can never be reached. This property turns out to be typical. To explore this phenomenon in greater depth, suppose S is a $\mathbb{Z}^d$-valued random walk. An $x \in \mathbb{Z}^d$ is **possible** if there exists an integer $k \geq 1$ such that $\mathbb{P}(S_k = x) > 0$. If x is not possible, it is deemed **impossible.** Clearly, impossible points are not, and can never be, visited. Therefore, any discussion of recurrence must be reduced to the possible points. What do the possible points of a random walk look like? Below is a prefatory result that will be elaborated upon in the next section.

Lemma 1.6.1 *The collection of all possible points of a $\mathbb{Z}^d$-valued random walk is an additive semigroup of $\mathbb{Z}^d$.*

Proof Suppose the random walk is denoted by S and $x_1, x_2 \in \mathbb{Z}^d$ are possible for S. By definition, there exist $k_1, k_2 \in \mathbb{Z}^d$ such that $p_i = \mathbb{P}(S_{k_i} = x_i) > 0$ for $i = 1, 2$. Since $\mathbb{P}(S_{k_1+k_2} - S_{k_1} = x_2) = \mathbb{P}(S_{k_2} = x_2) = p_2$, by the Markov property (Corollary 1.1.1),

$$\mathbb{P}(S_{k_1+k_2} = x_2 + x_1) \geq \mathbb{P}(S_{k_1} = x_1, S_{k_1+k_2} - S_{k_1} = x_2) = p_1 p_2 > 0.$$

This proves the lemma. $\square$

The following is a very important exercise.

***Exercise* 1.6.2** Let S denote a random walk on $\mathbb{Z}^d$ whose increment process is X. We say that S is **symmetric** if X_1 and $-X_1$ have the same distributions. Prove that whenever S is a symmetric random walk on $\mathbb{Z}^d$, the set of its possible values forms an additive subgroup of $\mathbb{Z}^d$. In particular, argue that the origin is always possible. $\square$

Lemma 1.6.2 *The collection of all recurrent points is an additive subgroup of $\mathbb{Z}^d$. In particular, if there are any recurrent points, 0 is one of them.*

Proof We will show that whenever x and y are recurrent, so is $x-y$. Let τ denote the first hitting time of y. That is, $\tau = \inf(k \geq 1 : \ S_k = y)$, where $\inf \varnothing = \infty$. Thanks to the recurrence of y, τ is finite and $S_\tau = y$, a.s.; cf. Proposition 1.3.1. Consequently, the strong Markov property (Theorem 1.2.1) implies the following (why?):

$$\begin{aligned}
&\mathbb{P}(S_k = x - y \text{ for infinitely many } k \geq 1) \\
&\qquad = \mathbb{P}(S_{k+\tau} - S_\tau = x - y \text{ for infinitely many } k \geq 1) \\
&\qquad = \mathbb{P}(S_{k+\tau} = x \text{ for infinitely many } k \geq 1) \\
&\qquad = \mathbb{P}(S_k = x \text{ for infinitely many } k \geq 1),
\end{aligned}$$

which is equal to one, thanks to the recurrence of x, together with Proposition 1.3.1. This completes our proof. $\square$

Theorem 1.6.1 *Suppose S is a $\mathbb{Z}^d$-valued random walk. If $x \in \mathbb{Z}^d$ is possible and $y \in \mathbb{Z}^d$ is recurrent, $x-y$ is recurrent. In particular, the following are equivalent:*

(i) 0 *is recurrent;*

(ii) all possible points x are recurrent with probability one.

Note that the condition (ii) subsumes the assumption that x is possible and that Theorem 1.6.1 extends Lemma 1.6.2.

Proof To begin, let us argue that the first assertion of the theorem implies the equivalence of (i) and (ii). Suppose (ii) holds, first. Then, for any possible point x, $0 = x - x$ is recurrent, by the first assertion of the theorem, thus proving (i). Conversely, if (i) holds, by the first assertion of the theorem and by Lemma 1.6.2, for any possible point x, $x = x - 0$ is recurrent. We have shown that $(i) \Leftrightarrow (ii)$ and are left to verify that for all possible points x and all recurrent points y, $x - y$ is recurrent. Holding such x and y fixed, define $\sigma_1 = \inf(k \geq 1 : \ S_k = y)$, $\sigma_2 = \inf(k \geq K_0 + \tau_1 : S_k = y)$, $\ldots$, where K_0 is a fixed constant that is to be chosen later on in this proof. (For now, you can think of $K_0 = 1$, in which case σ_j denotes the jth time the random walk hits y.) In general, for all $j \geq 1$, we define $\sigma_{j+1} = \inf(k \geq K_0 + \sigma_j : \ S_k = y)$, where $\inf \varnothing = \infty$, as usual. Since y is recurrent, $\sigma_j < \infty$ for all $j \geq 1$, with probability one. Now we define the events $\mathsf{E}_1, \mathsf{E}_2, \ldots$ as

$$\mathsf{E}_n = \big(S_k = x \text{ for some } \sigma_n < k < \sigma_{n+1}\big), \qquad n \geq 1.$$

As k varies between σ_n and σ_{n+1}, the process S_k makes a loop, starting from y and ending at y. This loop is called an **excursion** from y, and E_n denotes the event that in the nth excursion from y, the random walk hits

x at some point. Equivalently,

$$\mathsf{E}_n = \big(S_{k+\sigma_n} - S_{\sigma_n} = x - y \text{ for some } 1 \le k \le \sigma_{n+1} - \sigma_n\big).$$

You should check that as a consequence of the strong Markov property, $\mathsf{E}_1, \mathsf{E}_2, \ldots$ are independent events and all have the same probability $\mathbb{P}(\mathsf{E}_1)$; cf. Theorem 1.2.1. Now is the time to choose K_0. Since x is possible, by choosing K_0 large enough, we can ensure that $\mathbb{P}(\mathsf{E}_1) > 0$. (Why?) Thus, by the Borel–Cantelli lemma, $\mathbb{P}(\mathsf{E}_n \text{ infinitely often}) = 1$. In particular, x is recurrent and, thanks to Lemma 1.6.2, so is $x - y$, as desired. $\square$

1.7 Recurrence–Transience Dichotomy

Let S denote a $\mathbb{Z}^d$-valued random walk and P denote the collection of its possible values. According to Theorem 1.6.1, either all $x \in P$ are recurrent or they are all transient. This is the **recurrence–transience dichotomy**. The impossible values, of course, are never visited and have no effect on the structure of the random walk. On the other hand, at least in the presence of some recurrent values, all elements of P are recurrent and P is an additive group (Lemma 1.6.2 and Theorem 1.6.1).

Thus, when $P \neq \varnothing$, we can view S as a *Markov chain* on the group P. A little group theory will show that quite a bit more is true. Indeed, recall that $\mathbb{Z}^d$ is a free abelian group.[1] Since all subgroups of free abelian groups are free abelian,[2] Lemma 1.6.1 shows that P is itself a free abelian group. If $k \in \{1, \ldots, d\}$ denotes the rank of P, then P is *isomorphic* to $\mathbb{Z}^k$ (why an isomorphism and not just a homomorphism?). For us, this means that there exists a $k \times k$ invertible matrix $\mathbf{A}$ such that $\mathbf{A}P = \mathbb{Z}^k$. Since $S_n \in P$, a.s. for all $n \ge 1$, $\mathbf{A}S = \big(\mathbf{A}S_n;\ n \ge 1\big)$ is a random walk on $\mathbb{Z}^k$ and all points in $\mathbb{Z}^k$ are possible for this walk. Since $\mathbf{A}^{-1}$ exists, all statements about the P-valued Markov chain S translate to statements for the $\mathbb{Z}^k$-valued random walk $\mathbf{A}S$, and vice versa. Thus, it is no essential loss in generality to assume that S is itself a $\mathbb{Z}^d$-valued random walk for which all points in $\mathbb{Z}^d$ are possible.

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[1] Let $\mathfrak{G}$ be a class of groups. Consider some $G \in \mathfrak{G}$ whose generator is the set $g = \{x_i;\ i \in I\}$. Recall that G is **freely generated** by g (within the class $\mathfrak{G}$) if for any group $G' \in \mathfrak{G}$ that is generated by $\{y_i;\ i \in I\}$, the map $x_i \mapsto y_i$ extends to a homomorphism (i.e., operation-preserving) $G \to G'$. The cardinality of I is the **rank** of G, and G is **free** within $\mathfrak{G}$. A **free abelian group** is a group that is free within the class of all abelian groups. While general free groups do not have much rank structure in a "dimensional" sense, free abelian groups do.

[2] This is an immediate consequence of the free abelian group theorem: Each subgroup of a free *abelian* group is itself a free abelian group. (Why is it a consequence?) See Kargapolov and Merzljakov (1979, Theorem 7.1.4, Chapter 3),

Proposition 1.7.1 *Suppose S is a $\mathbb{Z}^d$-valued random walk and let φ denote the characteristic function of the increments $\varphi(\xi) = \mathbb{E}[e^{i\xi\cdot X_1}]$, $\xi \in \mathbb{R}^d$. Then,*

$$\sum_{k=1}^{\infty} \mathbb{P}(S_k = 0) = (2\pi)^{-d} \lim_{\lambda\uparrow 1} \int_{[-\pi,\pi]^d} \mathrm{Re}\left(\frac{\varphi(\xi)}{1-\lambda\varphi(\xi)}\right) d\xi.$$

Combining the above with Theorem 1.6.1, we conclude the following.

Corollary 1.7.1 *Suppose S is a $\mathbb{Z}^d$-valued random walk for which all points are possible. Let φ denote the characteristic function of the increments of S. Then, all $x \in \mathbb{Z}^d$ are transient, a.s., unless $\lim_{\lambda\uparrow 1} \int_{[-\pi,\pi]^d} \mathrm{Re}\{1-\lambda\varphi(\xi)\}^{-1}\, d\xi < \infty$, in which case all points are recurrent, a.s.*

Bearing in mind the discussion in the beginning of this subsection, what the above states is that for *any* random walk on $\mathbb{Z}^d$, either all possible points are recurrent, or all possible points are transient (why?). In the latter case, we say that the random walk is **recurrent** and in the former case, **transient**. It is important to point out that if S is a transient walk for which all points are possible, then with probability one, $\lim_{n\to\infty} |S_n| = \infty$. The converse also holds, as the following shows.

Exercise **1.7.1** S is transient if and only if $|S_n| \to \infty$, a.s. □

Proof of Proposition 1.7.1 By the inversion theorem of Fourier analysis on the torus (or by the inversion theorem for discrete random variables), for all $k \geq 1$, $\mathbb{P}(S_k = 0) = (2\pi)^{-d} \int_{[-\pi,\pi]^d} \{\varphi(\xi)\}^k\, d\xi$. Thus, for all $\lambda \in]0,1[$,

$$\sum_{k=1}^{\infty} \lambda^k \mathbb{P}(S_k = 0) = (2\pi)^{-d} \lambda \int_{[-\pi,\pi]^d} \mathrm{Re}\left(\frac{\varphi(\xi)}{1-\lambda\varphi(\xi)}\right) d\xi,$$

since the left-hand side is real-valued. (Check this calculation!) To finish, simply let $\lambda \uparrow 1$. □

In fact, the following (surprisingly) subtle fact holds:[3]

$$\lim_{\lambda\uparrow 1} \int_{[-\pi,\pi]^d} \mathrm{Re}\left(\frac{\varphi(\xi)}{1-\lambda\varphi(\xi)}\right) d\xi = \int_{[-\pi,\pi]^d} \mathrm{Re}\left(\frac{\varphi(\xi)}{1-\varphi(\xi)}\right) d\xi.$$

We will not have need for this.

[3] Cf. Ornstein (1969) and Stone (1969). For a more complete result, see Port and Stone (1971b, Theorem 16.2).

2 Intersection Probabilities

A collection of N (≥ 2) independent $\mathbb{Z}^d$-valued random walks $S^1, S^2, \ldots, S^N$ are said to **intersect** if there exists $t \in \mathbb{N}^N$ such that $S^1_{t^{(1)}} = \cdots = S^N_{t^{(N)}}$. If we think of S^i_k as the position of particle i at time k, then $S^1, \ldots, S^N$ intersect if and only if the particle trajectories cross at some point. It should be recognized that such intersections are different from the **collisions** of $S^1, \ldots, S^N$. The latter happens when there exists $k \in \mathbb{N}$ such that $S^1_k = S^2_k = \cdots = S^N_k$. In words, $S^1, \ldots, S^N$ intersect if the trajectories of $S^1, \ldots, S^N$ intersect, while they collide if the particles $S^1_k, \ldots, S^N_k$ collide at some time k.

In light of the development in Section 1, collision problems are simpler to analyze. For instance, two independent random walks S^1 and S^2 collide infinitely often if and only if 0 is recurrent for the random walk $k \mapsto S^1_k - S^2_k$. In this section we study the more intricate problem of intersections of independent random walks.

Define the multiparameter $\mathbb{Z}^{dN}$-valued process $S = \big(S_t;\ t \in \mathbb{N}^N\big)$ by

$$S_t = (S^1_{t^{(1)}}, \ldots, S^N_{t^{(N)}}), \qquad t \in \mathbb{N}^N.$$

This means that the first d coordinates of S_t match those of $S^1_{t^{(1)}}$, the second d coordinates of S_t are the coordinates of $S^2_{t^{(2)}}$, and so on. It is apparent that for any $m \geq 1$ (finite or infinite) the ranges of $S^1, \ldots, S^N$ intersect m times if and only if S hits the diagonal of $\mathbb{Z}^{Nd}$ m times. If we write any $x \in \mathbb{Z}^{Nd}$ as $x = (x^1, \ldots, x^N)$ with $x^i \in \mathbb{Z}^d$, then the diagonal of $\mathbb{Z}^{Nd}$ is the set $\operatorname{diag}(\mathbb{Z}^{Nd}) = \{x \in \mathbb{Z}^{Nd} :\ x^1 = \cdots = x^N\}$. In direct product notation, we can write $x \in \mathbb{Z}^{Nd}$ as $x = x^1 \otimes \cdots \otimes x^N$, where $x^i \in \mathbb{Z}^d$. (For example, $(1,2,3,4) = (1,2) \otimes (3,4) = 1 \otimes 2 \otimes 3 \otimes 4$.) Since $S_t = S^1_{t^{(1)}} \otimes \cdots \otimes S^N_{t^{(N)}}$, we sometimes write the stochastic process S as $S = S^1 \otimes \cdots \otimes S^N$ and refer to $S^1, \ldots, S^N$ as the **coordinate processes** of S. To write things more explicitly, consider $N = 2$. Then, $S = S^1 \otimes S^2$ is a two-parameter process defined by $S_{(i,j)} = (S^1_i, S^2_j)$, $i, j \geq 1$. This means that the first d coordinates of $S_{(i,j)}$ are the d coordinates of S^1_i, and the next d coordinates of $S_{(i,j)}$ are those of S^2_j.

Henceforth, we will assume that all points are possible for $S^1, \ldots, S^N$. See Section 1.7 for a discussion of this assumption and how it can be essentially made without loss of generality.

2.1 Intersections of Two Walks

Let S^1 and S^2 denote two independent random walks on $\mathbb{Z}^d$ and let $S = S^1 \otimes S^2$ denote the associated 2-parameter process. We are interested in knowing when S hits the diagonal of $\mathbb{Z}^{2d}$ finitely often. In other words, we ask, "*when is* $\sum_{j=n}^\infty \sum_{k=m}^\infty \mathbb{1}_{(S^1_j = S^2_k)}$ *finite for all choices of* $n, m \geq 1$*?*" At the time of writing this book, this question seems unanswerable for completely general walks S^1 and S^2. However, we will give a comprehensive answer when S^1 and S^2 are both symmetric, i.e., when S^1_j (respectively S^2_j) has the same distribution as $-S^1_j$ (respectively $-S^2_j$) for all $j \geq 1$; cf. Exercise 2.1.3 below for a further refinement.

According to the recurrence–transience dichotomy (Corollary 1.7.1 and its proceeding discussion), S^1 is either recurrent or transient, a.s. First, we address the easy case where S^1 (or equivalently, S^2) is recurrent.

Lemma 2.1.1 *If either of S^1 or S^2 is recurrent, then with probability one, there are infinitely many intersections.*

***Exercise* 2.1.1** Prove Lemma 2.1.1. □

According to Lemma 2.1.1, in our study of the intersections of S^1 and S^2 we can confine ourselves to the transient case.

Henceforth, S^1 and S^2 are symmetric walks, and $S^1_0 = S^2_0 = 0$.

Consider the function

$$G_\lambda(a,b) = \mathbb{E}\Big[\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \lambda^{j+k} \mathbb{1}_{(S^1_j + a = S^2_k + b)}\Big], \qquad \lambda \in]0,1[,\ a,b \in \mathbb{Z}^d. \tag{1}$$

Theorem 2.1.1 *Suppose S^1 and S^2 are symmetric, independent, transient random walks in $\mathbb{Z}^d$. Then, the following are equivalent:*

(i) $\lim_{\lambda\uparrow 1} G_\lambda(0,0) = +\infty$;

(ii) $\mathbb{P}(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \mathbb{1}_{(S^1_j = S^2_k)} < \infty) > 0$;

(iii) $\mathbb{P}(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \mathbb{1}_{(S^1_j = S^2_k)} < \infty) = 1$; *and*

(iv) $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \mathbb{P}(S^1_j = S^2_k) < \infty$.

The following technical lemma lies at the heart of Theorem 2.1.1 and seems to require symmetry.

Lemma 2.1.2 *Let φ^1 and φ^2 denote the characteristic functions of the increments of S^1 and S^2, respectively. Then, for all $\lambda \in]0,1[$,*

$$\sup_{a,b\in\mathbb{Z}^d} G_\lambda(a,b) = G_\lambda(0,0).$$

Proof By the inversion formula for characteristic functions,

$$\begin{aligned}
\mathbb{P}(S^1_j + a = S^2_k + b) &= (2\pi)^{-d}\int_{[-\pi,\pi]^d} e^{-i\xi\cdot(b-a)}\mathbb{E}[e^{i\xi\cdot S^1_j}]\,\mathbb{E}[e^{-i\xi\cdot S^2_k}]\,d\xi \\
&= (2\pi)^{-d}\int_{[-\pi,\pi]^d} e^{-i\xi\cdot(b-a)}\{\varphi^1(\xi)\}^j\{\varphi^2(-\xi)\}^k\,d\xi \\
&= (2\pi)^{-d}\int_{[-\pi,\pi]^d} e^{-i\xi\cdot(b-a)}\{\varphi^1(\xi)\}^j\{\varphi^2(\xi)\}^k\,d\xi.
\end{aligned}$$

In the last line, symmetry is used. Therefore,

$$G_\lambda(a,b) = (2\pi)^{-d} \int_{[-\pi,\pi]^d} e^{-i\xi\cdot(b-a)} \frac{1}{1-\lambda\varphi^1(\xi)} \, \frac{1}{1-\lambda\varphi^2(\xi)} \, d\xi. \tag{2}$$

On the other hand, $-1 \le \varphi^1(\xi), \varphi^2(\xi) \le 1$, which implies that $\{1 - \lambda\varphi^1(\xi)\}^{-1} \times \{1 - \lambda\varphi^2(\xi)\}^{-1}$ is nonnegative. Since $|e^{-i\xi\cdot(b-a)}| \le 1$, the lemma follows. $\square$

It may be helpful to note that the one-parameter version of this lemma *always* holds:

Exercise **2.1.2** Suppose S is a random walk on $\mathbb{Z}^d$ with $S_0 = 0$, and define for all $\lambda \in]0,1[$, $G_\lambda(a) = \mathbb{E}[\sum_{k=0}^\infty \lambda^k \mathbb{1}_{(S_k=a)}]$. Prove that even if S is not symmetric, $G_\lambda(a) \le G_\lambda(0)$ for all $a \in \mathbb{Z}^d$ and all $\lambda \in]0,1[$.
(HINT: Consider the first hitting time of a.) $\square$

Proof of Theorem 2.1.1 It is clear that $(iii) \Rightarrow (ii)$. Conversely, it is not hard to check that $(ii) \Rightarrow (iii)$, thanks to the Hewitt–Savage 0-1 law; cf. Exercise 1.7.5, Chapter 1. Since $(i) \Leftrightarrow (iv) \Rightarrow (iii)$ is clear, it remains to prove that if (iv) fails, then so will (iii).

Define for all $n \ge 1$,

$$J_\lambda = \sum_{j=0}^\infty \sum_{k=0}^\infty \lambda^{j+k} \mathbb{1}_{(S_j^1 = S_k^2)}.$$

Note that $\mathbb{E}[J_\lambda] = G_\lambda(0,0)$; cf. equation (2).

Since (iv) is assumed to fail, $\lim_{\lambda\uparrow 1} \mathbb{E}[J_\lambda] = +\infty$. Our strategy, then, is to show the existence of a nontrivial constant A_1 such that

$$\mathbb{E}[J_\lambda^2] \le A_1 \big(\mathbb{E}[J_\lambda]\big)^2, \qquad \lambda \in]0,1[. \tag{3}$$

Assuming this, we can finish our proof: Apply equation (3) and the Paley–Zygmund lemma (Lemma 1.4.1) to see that

$$\mathbb{P}\Big(\sup_{\lambda\in]0,1[} J_\lambda = +\infty\Big) \ge \lim_{\lambda\uparrow 1} \mathbb{P}\big(J_\lambda \ge \tfrac{1}{2}\mathbb{E}[J_\lambda]\big) \ge A_1^{-1},$$

which is positive. Thus, it remains to verify equation (3).

We can write $\mathbb{E}[J_\lambda^2] \le 2(T_1 + T_2)$, where

$$T_1 = \sum\sum_{i \le i'} \sum\sum_{j \le j'} \lambda^{i+i'+j+j'} \mathbb{P}(S_i^1 = S_j^2 \,,\, S_{i'}^1 = S_{j'}^2),$$
$$T_2 = \sum\sum_{i \le i'} \sum\sum_{j' \le j} \lambda^{i+i'+j+j'} \mathbb{P}(S_i^1 = S_j^2 \,,\, S_{i'}^1 = S_{j'}^2).$$

(Why?) Next, we write $T_1 = T_{11} + T_{12} + T_{13}$, where

$$T_{11} = \sum\sum_{i<i'} \sum\sum_{j<j'} \lambda^{i+i'+j+j'} \mathbb{P}(S_i^1 = S_j^2 \,,\, S_{i'}^1 = S_{j'}^2),$$

$$T_{12} = \sum_{i=0}^{\infty} \sum\sum_{j<j'} \lambda^{2i+j+j'} \mathbb{P}(S_i^1 = S_j^2 = S_{j'}^2),$$

$$T_{13} = \sum_{j=0}^{\infty} \sum\sum_{i<i'} \lambda^{i+i'+2j} \mathbb{P}(S_i^1 = S_j^2 = S_{i'}^1).$$

Similarly, we write $T_2 = T_{21} + T_{12} + T_{13}$, where

$$T_{21} = \sum\sum_{i<i'} \sum\sum_{j'<j} \lambda^{i+i'+j+j'} \mathbb{P}(S_i^1 = S_j^2 \,,\, S_{i'}^1 = S_{j'}^2).$$

We now estimate the T_{ij}'s in turn.

By the Markov property,

$$\begin{aligned} T_{11} &= \sum\sum_{i<i'} \sum\sum_{j<j'} \lambda^{i+i'+j+j'} \mathbb{P}(S_i^1 = S_j^2) \mathbb{P}(S_{i'-i}^1 = S_{j'-j}^2) \\ &= \sum\sum_{i<i'} \sum\sum_{j<j'} \lambda^{2i+2j+(i'-i)+(j'-j)} \mathbb{P}(S_i^1 = S_j^2) \mathbb{P}(S_{i'-i}^1 = S_{j'-j}^2) \\ &\le \big(\mathbb{E}[J_\lambda]\big)^2. \end{aligned}$$

On the other hand,

$$T_{12} \le \sum_{i=0}^{\infty} \sum\sum_{j<j'} \lambda^{i+j+j'} \mathbb{P}(S_i^1 = S_j^2) \mathbb{P}(S_{j'-j}^2 = 0) \le A_2 \mathbb{E}[J_\lambda],$$

where $A_2 = \sum_{i=0}^\infty \mathbb{P}(S_i^2 = 0)$. Of course, since S^2 is transient, $A_2 < +\infty$; cf. Theorem 1.4.1. In similar fashion we obtain $T_{13} \le A_3 \mathbb{E}[J_\lambda]$, where $A_3 = \sum_{i=0}^\infty \mathbb{P}(S_i^1 = 0)$ is finite as well. Since we have assumed (iv) of the theorem, our job is complete, once we show that there exists a nontrivial constant A_4 such that for all $n \ge 1$,

$$T_{21} \le A_4 \big(\mathbb{E}[J_\lambda]\big)^2. \tag{4}$$

Indeed, from this, equation (3), and hence the theorem, follows.

We observe that T_{21} equals

$$\begin{aligned} &\sum\sum_{i<i'} \sum\sum_{j'<j} \lambda^{i+i'+j+j'} \mathbb{P}\big(S_i^1 = S_{j'}^2 + [S_j^2 - S_{j'}^2] \,,\, S_i^1 + [S_{i'}^1 - S_i^1] = S_{j'}^2\big) \\ &= \sum\sum_{i<i'} \sum\sum_{j'<j} \lambda^{i+i'+j+j'} \mathbb{P}\big(S_i^1 = S_{j'}^2 + \bar{S}_{j-j'}^2 \,,\, S_i^1 + \bar{S}_{i'-i}^1 = S_{j'}^2\big), \end{aligned}$$

where $(\bar{S}_u^1, \bar{S}_v^2)$ is an independent copy of (S_u^1, S_v^2) for any two integers $u, v \geq 1$. This is a consequence of the Markov property; cf. Corollary 1.1.1. Consequently,

$$\begin{aligned} T_{21} &= \sum\sum_{i<i'}\sum\sum_{j'<j} \lambda^{i+i'+j+j'} \mathbb{P}\big(S_i^1 = S_{j'}^2 + \bar{S}_{j-j'}^2 \,,\, \bar{S}_{i'-i}^1 = \bar{S}_{j-j'}^2\big) \\ &\leq \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{u=0}^\infty \sum_{v=0}^\infty \lambda^{i+j+u+v} \mathbb{P}\big(S_i^1 = S_j^2 + \bar{S}_v^2 \,,\, \bar{S}_u^1 = -\bar{S}_v^2\big) \\ &= \sum_{u=0}^\infty \sum_{v=0}^\infty \lambda^{u+v} \mathbb{E}\big[G_\lambda(0, \bar{S}_v^2) \mathbb{1}_{(\bar{S}_u^1 = -\bar{S}_v^2)}\big], \end{aligned}$$

by independence and by equation (1). Thanks to Lemma 2.1.2,

$$\begin{aligned} T_{21} &\leq G_\lambda(0,0) \sum_{u=0}^\infty \sum_{v=0}^\infty \lambda^{u+v} \mathbb{P}(S_u^1 = -S_v^2) \\ &= \mathbb{E}[J_\lambda] \sum_{u=0}^\infty \sum_{v=0}^\infty \lambda^{u+v} \mathbb{P}(S_u^1 = -S_v^2) \\ &= \big(\mathbb{E}[J_\lambda]\big)^2. \end{aligned}$$

To follow up, the first line follows from the fact that $(\bar{S}_u^1, \bar{S}_v^2)$ has the same distribution as (S_u^1, S_v^2). The second line is from the definition of J_λ, and the third line follows from the symmetry hypothesis of the theorem. This verifies equation (4) and completes our task. □

Exercise **2.1.3** A characteristic function φ, on $\mathbb{R}^d$, is said to satisfy the **sector condition** if there exists a constant $A > 0$ such that

$$|\mathrm{Im}\varphi(\xi)| \leq A\big\{1 + |\mathrm{Re}\varphi(\xi)|\big\}, \qquad \xi \in \mathbb{R}^d.$$

Suppose S^1 and S^2 are independent random walks on $\mathbb{Z}^d$, whose increments have characteristic functions that satisfy the sector condition. Prove that Theorem 2.1.1 remains valid in this setting. □

Theorem 2.1.1 states that, under the given conditions, the trajectories[4] of S^1 and S^2 intersect infinitely many times if and only if $\sum_{j,k\geq 1} \mathbb{P}(S_j^1 = S_k^2) = \infty$. By a summability argument (see the described proof of Proposition 1.7.1), the latter can be written as follows.

Proposition 2.1.1 *We have*

$$\sum_{j,k=1}^\infty \mathbb{P}(S_j^1 = S_k^2) = (2\pi)^{-d} \lim_{\lambda \uparrow 1} \int_{[-\pi,\pi]^d} \frac{\varphi^1(\xi)}{1 - \lambda\varphi^1(\xi)} \frac{\varphi^2(\xi)}{1 - \lambda\varphi^2(\xi)}\, d\xi.$$

[4]Throughout this book, the *trajectories* of a stochastic process $(X_t;\ t \in T)$ are the realizations of the (random) function $t \mapsto X_t$, for any index set T.

***Exercise* 2.1.4** Verify Proposition 2.1.1. □

2.2 An Estimate for Two Walks

Let S^1 and S^2 be two independent $\mathbb{Z}^d$-valued random walks. According to Theorem 2.1.1, we can conclude that $\sum_{j,k\geq 1}\mathbb{P}(S_j^1 = S_k^2) < \infty$ is a necessary and sufficient condition for

$$\lim_{n,m\to\infty}\mathbb{P}(S_j^1 = S_k^2 \text{ for some } (j,k)\succcurlyeq(n,m)) = 0,$$

provided that the random walks are symmetric. We now explore the rate at which the above probability tends to 0, under the extra condition that there exists C_0 such that whenever $\mathbb{P}(S_i^1 = S_j^2) > 0$,

$$\mathbb{P}(S_i^1 = S_j^2 + a) \leq C_0\mathbb{P}(S_i^1 = S_j^2). \tag{1}$$

This is a unimodality-type condition and is verified, for instance, when S^1 and S^2 are so-called simple random walks; cf. Section 3.

Theorem 2.2.1 *Suppose S^1 and S^2 are two symmetric and independent $\mathbb{Z}^d$-valued random walks that satisfy condition (1). If $\sum_{j,k\geq 1}\mathbb{P}(S_j^1 = S_k^2) < \infty$, there exist nontrivial constants C_1 and C_2 such that for all $n,m\geq 1$,*

$$C_1\sum_{j=n}^{\infty}\sum_{k=m}^{\infty}\mathbb{P}(S_j^1 = S_k^2) \leq \mathbb{P}(S_j^1 = S_k^2 \text{ for some } (j,k)\succcurlyeq(n,m))$$
$$\leq C_2\sum_{j=n}^{\infty}\sum_{k=m}^{\infty}\mathbb{P}(S_j^1 = S_k^2).$$

Proof Define for all $n,m\geq 1$,

$$J_{n,m} = \sum_{j=n}^{\infty}\sum_{k=m}^{\infty}\mathbb{1}_{(S_j^1 = S_k^2)}.$$

Arguing as we did in Theorem 2.1.1, we can show that there exist nontrivial constants C_3 and C_4 such that for all $n,m\geq 1$, $\mathbb{E}[J_{n,m}^2] \leq C_3(\mathbb{E}[J_{n,m}])^2 + C_4\mathbb{E}[J_{n,m}]$; this uses (1), as well as symmetry. Since $\mathbb{E}[J_{n,m}]$ goes to zero as $n,m\to\infty$, we can deduce the existence of a finite constant C_1 such that

$$\mathbb{E}[J_{n,m}^2] \leq \frac{\mathbb{E}[J_{n,m}]}{C_1}. \tag{2}$$

The details are delegated to Supplementary Exercise 6. By the Paley–Zygmund lemma (Lemma 1.4.1),

$$\mathbb{P}(J_{n,m} > 0) \geq C_1\mathbb{E}[J_{n,m}],$$

which is the desired probability lower bound.

To demonstrate the corresponding upper bound, for all $n, m \geq 1$, let $\mathcal{F}_{n,m}$ define the σ-field generated by $((S_i^1, S_j^2);\ 1 \leq i \leq n, 1 \leq j \leq m)$. By Exercise 3.4.2 of Chapter 1, $\mathcal{F} = (\mathcal{F}_{n,m};\ n, m \geq 1)$ is a commuting filtration in the sense of Chapter 1. Fix $(n, m) \in \mathbb{N}^2$ and define

$$M_{p,q} = \mathbb{E}(J_{n,m} \mid \mathcal{F}_{p,q}), \qquad (p, q) \succcurlyeq (n, m).$$

By the Markov property (Corollary 1.1.1),

$$\begin{aligned}
M_{p,q} &\geq \sum_{i=p}^{\infty} \sum_{j=q}^{\infty} \mathbb{P}(S_i^1 = S_j^2 \mid \mathcal{F}_{p,q}) \mathbb{1}_{(S_p^1 = S_q^2)} \\
&= \Big\{ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mathbb{P}(S_{i+p}^1 - S_p^1 = S_{j+q}^2 - S_q^2 \mid \mathcal{F}_{p,q}) + 1 \Big\} \mathbb{1}_{(S_p^1 = S_q^2)} \\
&= \Big\{ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mathbb{P}(S_i^1 = S_j^2) + 1 \Big\} \mathbb{1}_{(S_p^1 = S_q^2)} \\
&= \sqrt{\frac{16}{C_2 C_1}}\ \mathbb{1}_{(S_p^1 = S_q^2)}.
\end{aligned}$$

(This defines C_2.) It is clear that $M = (M_t;\ t \in \mathbb{N}^2)$ is a two-parameter martingale with respect to the (commuting) filtration $\mathcal{F}$. Thus, by Cairoli's strong $(2,2)$ inequality (Theorem 2.3.1 and Corollary 3.5.1 of Chapter 1),

$$\begin{aligned}
\mathbb{P}(S_q^1 = S_q^2 \text{ for some } (p, q) \succcurlyeq (n, m)) &\leq \frac{C_2 C_1}{16}\ \mathbb{E}\Big[\sup_{(p,q) \succcurlyeq (n,m)} M_{p,q}^2 \Big] \\
&\leq C_2 C_1 \mathbb{E}[J_{n,m}^2].
\end{aligned}$$

The probability upper bound follows from this and equation (1). $\square$

2.3 Intersections of Several Walks

We are ready to consider the general problem of when and how often N independent random walks in $\mathbb{Z}^d$ intersect, when $N \geq 2$ is an arbitrary integer. This will be achieved by extending the two-parameter methods of Section 2.1 to N parameters.

Let $S^1, \ldots, S^N$ denote N independent $\mathbb{Z}^d$-valued random walks. The following can be proved in complete analogy to Lemma 2.1.1.

Lemma 2.3.1 *If any one of the coordinate processes is recurrent and if the trajectories of the remaining $N - 1$ coordinate processes intersect infinitely many times, then for all $t \in \mathbb{N}^N$, $\sum_{s \succcurlyeq t} \mathbb{1}_{(S^1_{s(1)} = \cdots = S^N_{s(N)})} = \infty$, almost surely.*

***Exercise* 2.3.1** Prove Lemma 2.3.1. □

In particular, we need to consider only the case where all of the coordinate processes are transient. By Theorem 1.4.1, this happens precisely when $\sum_{k=1}^{\infty} \mathbb{P}(S_k^i = 0) < \infty$, for all $i = 1, \ldots, N$, a condition that we will assume tacitly from now on.

Let $S_0^1 = \cdots = S_0^N = 0$ and define the N-variable version of equation (1) of Section 2.2 as

$$G_\lambda(a_1, \ldots, a_N) = \mathbb{E}\Big[\sum_{0 \le i_1, \ldots, i_N} \cdots \sum \lambda^{i_1 + \cdots + i_N} \mathbb{1}_{(S_{i_1}^1 + a_1 = S_{i_2}^2 + a_2 = \cdots = S_{i_N}^N + a_N)} \Big], \tag{1}$$

where $a_1, \ldots, a_N \in \mathbb{Z}^d$ and $\lambda \in]0,1[$. One can prove the following.

Proposition 2.3.1 *Suppose $S^1, \ldots, S^N$ are N symmetric and independent $\mathbb{Z}^d$-valued random walks whose increments have characteristic functions $\varphi^1, \ldots, \varphi^N$, respectively. Then, $G_\lambda(a_1, \ldots, a_N) \le G_\lambda(0, \ldots, 0)$ for all $a_1, \ldots, a_N \in \mathbb{Z}^d$. Moreover,*

$$G_\lambda(0, \ldots, 0) = (2\pi)^{-d(N-1)} \int_{[-\pi,\pi]^{d(N-1)}} F(\xi; \lambda)\, d\xi,$$

where for all $\xi \in [-\pi, \pi]^{d(N-1)}$ and all $\lambda \in]0,1[$,

$$F(\xi; \lambda) = \frac{1}{1 - \lambda \varphi^1\big(- \sum_{\ell=1}^{N-1} \xi^{(\ell)} \big)} \cdot \prod_{j=2}^{N-1} \frac{1}{1 - \lambda \varphi^j(\xi)}.$$

***Exercise* 2.3.2** Prove Proposition 2.3.1. □

Theorem 2.3.1 *Suppose $S^1, \ldots, S^N$ are symmetric, independent, $\mathbb{Z}^d$-valued, transient random walks. Then, the following are equivalent:*

(i) With positive probability, $\sum_{t \in \mathbb{N}^N} \mathbb{1}_{(S_{t(1)}^1 = \cdots = S_{t(N)}^N)} < \infty$;

(ii) With probability one, $\sum_{t \in \mathbb{N}^N} \mathbb{1}_{(S_{t(1)}^1 = \cdots = S_{t(N)}^N)} < \infty$; and

(iii) $\sum_{t \in \mathbb{N}^N} \mathbb{P}(S_{t(1)}^1 = \cdots = S_{t(N)}^N) < \infty$.

We provide only a sketch of the proof.

Sketch of Proof In light of the presented proof of Theorem 2.1.1, $(iii) \Rightarrow (ii) \Leftrightarrow (i)$ follows readily; it remains to show that if (iii) fails, then so does (ii).

For all $n \ge 1$, define

$$J_\lambda = \sum_{s \in \mathbb{N}_0^N} \lambda^{s^{(1)} + \cdots + s^{(N)}} \mathbb{1}_{(S_{s(1)}^1 = \cdots = S_{s(N)}^N)}, \qquad \lambda \in]0,1[.$$

Our goal is to show that if (iii) fails, $\sup_{\lambda\in\,]0,1[} J_\lambda = \infty$, with positive probability. It is this argument that we merely sketch. Since $\lim_{\lambda\uparrow 1} \mathbb{E}[J_\lambda] = +\infty$, it suffices to exhibit a finite C_1 such that $\mathbb{E}[J_\lambda^2] \le C_1(\mathbb{E}[J_\lambda])^2$ for all $\lambda \in\,]0,1[$. Once this is accomplished, the remainder of our argument follows our proof of Theorem 2.1.1 quite closely.

Clearly,

$$\mathbb{E}[J_\lambda^2] = \sum\sum_{i_1,i_1'\ge 0}\cdots\sum\sum_{i_N,i_N'\ge 0} \lambda^{\sum_{\ell=1}^N (i_\ell+i_\ell')}\mathbb{P}\big(S_{i_1}^1 = \cdots = S_{i_N}^N\,,\, S_{i_1'}^1 = \cdots = S_{i_N'}^N\big).$$

Let us consider the contribution to the above when $i_1 = i_1'$:

$$\begin{aligned}
&\sum_{i_1=0}^\infty \sum\sum_{i_2,i_2'\ge 0}\cdots\sum\sum_{i_N,i_N'\ge 0}\lambda^{2i_1+\sum_{\ell=2}^N(i_\ell+i_\ell')}\\
&\qquad\qquad \times\mathbb{P}\big(S_{i_1}^1 = S_{i_2}^2 = \cdots = S_{i_N}^N\,,\, S_{i_1}^1 = S_{i_2'}^2 = \cdots = S_{i_N'}^N\big)\\
&\qquad \le (N-1)!\sum_{i_1=0}^\infty\sum\sum_{0\le i_2\le i_2'}\cdots\sum\sum_{0\le i_N\le i_N'}\lambda^{\sum_{\ell=1}^N i_\ell + \sum_{\ell=2}^N (i_\ell'-i_\ell)}\\
&\qquad\qquad \times\mathbb{P}\big(S_{i_1}^1 = \cdots = S_{i_N}^N\big)\prod_{\ell=2}^N \mathbb{P}(S_{i_\ell'-i_\ell}^\ell = 0)\\
&\qquad \le C_2\mathbb{E}[J_\lambda],
\end{aligned}$$

for some finite constant C_2 that is independent of $\lambda\in[0,1]$. By symmetry,

$$\mathbb{E}[J_\lambda^2] \le C_3\mathbb{E}[J_\lambda] + \sum\sum_{\substack{0\le i_1,i_1'\\ i_1\ne i_1'}}\cdots\sum\sum_{\substack{0\le i_N,i_N'\\ i_N\ne i_N'}}\lambda^{\sum_{\ell=1}^N i_\ell+\sum_{\ell=2}^N(i_\ell'-i_\ell)}\mathcal{Q},$$

where $\mathcal{Q} = \mathbb{P}(S_{i_1}^1 = \cdots = S_{i_N}^N, S_{i_1'}^1 = \cdots = S_{i_N'}^N)$. A little thought shows that, over the range in question,

$$\mathcal{Q} = \mathbb{P}\begin{pmatrix} S_{i_1\wedge i_1'}^1 + \bar S_{i_1-(i_1\wedge i_1')}^1 = \cdots = S_{i_N\wedge i_N'}^N + \bar S_{i_N-(i_N\wedge i_N')}^N\\ \text{and}\\ S_{i_1\wedge i_1'}^1 + \bar S_{i_1'-(i_1'\wedge i_1)}^1 = \cdots = S_{i_N\wedge i_N'}^N + \bar S_{i_N'-(i_N'\wedge i_N)}^N\end{pmatrix},$$

where $(S_{u(1)}^1,\ldots,S_{u(N)}^N)$ and $(\bar S_{u(1)}^1,\ldots,\bar S_{u(N)}^N)$ are independent copies of one another for each $u\in\mathbb{N}^N$. Solving, we get

$$\mathcal{Q} \le \mathbb{P}\begin{pmatrix} S_{i_1\wedge i_1'}^1 + \bar S_{i_1-(i_1\wedge i_1')}^1 = \cdots = S_{i_N\wedge i_N'}^N + \bar S_{i_N-(i_N\wedge i_N')}^N\\ \text{and}\\ |\bar S_{|i_1'-i_1|}^1| = \cdots = |\bar S_{|i_N'-i_N|}^N|\end{pmatrix}.$$

The rest of the proof follows from changing variables ($j_\ell = |i_\ell' - i_\ell|$) and follows the $N=2$ argument very closely, except that we now use Proposition 2.3.1 in place of Lemma 2.1.2. $\square$

2.4 An Estimate for N Walks

We consider the problem of the previous subsection in the case where the number of intersections is finite, almost surely. Under the hypotheses of Theorem 2.3.1, this is to say that $\sum_{t\in\mathbb{N}^N}\mathbb{P}(S^1_{t^{(1)}} = \cdots = S^N_{t^{(N)}}) < \infty$. The question that we address now is, *how large is* $\mathbb{P}(S^1_{s^{(1)}} = \cdots = S^N_{s^{(N)}}$ *for some* $s \succcurlyeq t)$ *when* $t \in \mathbb{N}^N$ *is large, coordinatewise?* When $N = 2$, this was achieved in Theorem 2.2.1; the general case follows under the following unimodality analogue of equation (1) of Section 2.2: There exists a finite constant C_0 such that

$$\sup_{a_1,\ldots,a_N\in\mathbb{Z}^d}\mathbb{P}(S^1_{i_1}+a_1=\cdots=S^N_{i_N}+a_N)\le C_0\mathbb{P}(S^1_{i_1}=\cdots=S^N_{i_N}), \quad (1)$$

as long as the right-hand side is positive.

Theorem 2.4.1 *Suppose $S^1,\ldots,S^N$ are independent $\mathbb{Z}^d$-valued random walks and for all $t\in\mathbb{N}^N$, let $\psi(t)=\mathbb{P}(S^1_{t^{(1)}}=\cdots=S^N_{t^{(N)}})$, and assume that these walks satisfy condition (1) above. If $\sum_{t\in\mathbb{N}^N}\psi(t)<\infty$, there exist finite constants C_1 and C_2 such that for all $t\in\mathbb{N}^N$,*

$$C_1\sum_{s\succcurlyeq t}\psi(s)\le\mathbb{P}(S^1_{s^{(1)}}=\cdots=S^N_{s^{(N)}}\ \textit{for some}\ s\succcurlyeq t)\le C_2\sum_{s\succcurlyeq t}\psi(s).$$

One can prove this by finding a suitable N-parameter modification of the two-parameter argument used to prove Theorem 2.2.1.

***Exercise* 2.4.1** (Hard) Prove Theorem 2.4.1. □

3 The Simple Random Walk

Nearest-neighborhood random walks on $\mathbb{Z}^d$ are random walks that can move only to the nearest point in $\mathbb{Z}^d$. Indeed, let $(e_1,\ldots,e_d)$ denote the usual basis for $\mathbb{R}^d$. That is, for all $i,j\in\{1,\ldots,d\}$, $e^{(i)}_j$ equals 1 if $i=j$, and it equals 0 otherwise. Consider a $\mathbb{Z}^d$-valued random walk $S=(S_k;\ k\ge 1)$ with increments $X_1,X_2,\ldots$. We say that S is a **nearest-neighborhood random walk** if with probability one, $X_1\in\{\pm e_1,\ldots,\pm e_d\}$. Nearest-neighborhood random walks form some of the most common models for the motion of a randomly moving particle. An important member of this family of random walks is the simple random walk. A random walk S is said to be simple if it is truly unbiased in its motion. More precisely, S is a **simple random walk** if $\mathbb{P}(X_1=e_1)=\mathbb{P}(X_1=-e_1)=\cdots=\mathbb{P}(X_1=e_d)=\mathbb{P}(X_1=-e_d)=(2d)^{-1}$. In this section we put the general theory of Section 2 to test by way of explicit calculations.

Let us recall that for all $x\in\mathbb{R}^k$, $|x|=\max_{1\le\ell\le k}|x^{(\ell)}|$ and $\|x\|=\{\sum_{\ell=1}^k|x^{(\ell)}|^2\}^{\frac12}$ denote the ℓ^∞ and ℓ^2 norms of x, respectively.

3.1 Recurrence

We now wish to study the recurrence properties of a simple random walk S in $\mathbb{Z}^d$ with increments $X_1, X_2, \ldots$. The following elementary result is a first step in this direction.

Lemma 3.1.1 *All points are possible for a simple random walk.*

***Exercise* 3.1.1** Prove Lemma 3.1.1. $\square$

Thus, according to Corollary 1.7.1, S is either recurrent or transient. In order to decide which is the case, we first need a technical lemma.

Lemma 3.1.2 *The integral $\int_{[0,1]^d} \|\xi\|^{-\beta}\, d\xi$ is finite if and only if $\beta < d$.*

Proof Recall that $\|\xi\| = \{\sum_{j=1}^N (\xi^{(j)})^2\}^{\frac{1}{2}}$, while $|\xi| = \max_{1\le j\le N} |\xi^{(j)}|$. The traditional approach to this sort of problem is to estimate the integral in polar coordinates; we will do this in probabilistic language. First, note that $|\xi| \le \|\xi\| \le d^{\frac{1}{2}} |\xi|$. Therefore, $\int_{[0,1]^d} \|\xi\|^{-\beta}\, d\xi < \infty$ if and only if $\int_{[0,1]^d} |\xi|^{-\beta}\, d\xi < \infty$. Let U be a random variable that is uniformly picked on $[0,1]^d$. The problem is to decide when $\mathbb{E}\{|U|^{-\beta}\}$ is finite. On the other hand, a direct calculation shows that $\sum_{n\ge 1} \mathbb{P}(|U|^{-\beta} \ge n) = \sum_{n\ge 1} n^{-d/\beta}$, which is finite iff $d > \beta$. $\square$

Theorem 3.1.1 *Let S denote the simple random walk in $\mathbb{Z}^d$. Then S is recurrent if $d \le 2$; otherwise, S is transient.*

Proof Let φ denote the characteristic function of X_1. It is easy to check that

$$\varphi(\xi) = \frac{1}{d} \sum_{\ell=1}^{d} \cos(\xi^{(\ell)}), \qquad \xi \in \mathbb{R}^d. \tag{1}$$

Since $\varphi(\xi) \ge 0$ (and is, of course, real) for all $\xi \in [-1,1]^d$, we can apply the bounded and monotone convergence theorems to Proposition 1.7.1 to see that

$$\sum_{n=1}^{\infty} \mathbb{P}(S_n = 0) = (2\pi)^{-d} \int_{[-\pi,\pi]^d} \frac{\varphi(\xi)}{1-\varphi(\xi)}\, d\xi.$$

(Why?) Equivalently, we apply symmetry to deduce

$$1 + \sum_{n=1}^{\infty} \mathbb{P}(S_n = 0) = \pi^{-d} \int_{[0,\pi]^d} \Big(1 - \frac{1}{d} \sum_{\ell=1}^{d} \cos(\xi^{(\ell)})\Big)^{-1} d\xi.$$

By Theorem 1.4.1, it suffices to show that the above integral is finite if and only if $d \ge 3$. Owing to Taylor's theorem with remainder, for all y there exists a λ between 0 and y such that

$$\cos(y) = 1 - \frac{y^2}{2} + \frac{\lambda^4}{12}.$$

Hence, for all $y \in [0,1]$,

$$1 - \frac{y^2}{2} \le \cos(y) \le 1 - y^2\Big(\frac{1}{2} - \frac{1}{12}\Big) = 1 - \frac{5}{12}y^2. \tag{2}$$

This, in turn, implies the inequality

$$2d \int_{[0,1]^d} \|\xi\|^{-2}\, d\xi \le \int_{[0,1]^d} \Big(1 - \frac{1}{d}\sum_{\ell=1}^{d} \cos(\xi^{(\ell)})\Big)^{-1} d\xi \le \frac{12d}{5} \int_{[0,1]^d} \|\xi\|^{-2}\, d\xi.$$

Since d is an integer, by Lemma 3.1.2, $\int_{[0,1]^d}(1 - d^{-1}\sum_{\ell=1}^d \cos(\xi^{(\ell)}))^{-1}\, d\xi$ is finite if and only if $d \ge 3$. Our proof is concluded once we show that $\int_K (1 - d^{-1}\sum_{\ell=1}^d \cos(\xi^{(\ell)}))^{-1}\, d\xi < \infty$, where $K = [0,\pi]^d \setminus [0,1]^d$. To observe this, note that whenever $\xi \in K$, there is at least one $\ell \in \{1, \dots, N\}$ such that $\cos(\xi^{(\ell)}) \le \cos(1)$. For such ξ's, we can conclude that

$$1 - d^{-1}\sum_{\ell=1}^{d} \cos(\xi^{(\ell)}) \ge d^{-1}[1 - \cos(1)].$$

Since $\cos(1) < 1$,

$$\int_K \Big(1 - \frac{1}{d}\sum_{\ell=1}^{d} \cos\big(\xi^{(\ell)}\big)\Big)^{-1} d\xi \le \frac{d}{1-\cos(1)}\,\mathrm{Leb}(K). \tag{3}$$

Clearly, $\mathrm{Leb}(K) \le \pi^d < \infty$, which proves the result. □

Theorem 3.1.1 is deeply related to the following:

***Exercise* 3.1.2** (Hard) If S denotes the simple walk in $\mathbb{Z}^d$, then there exists a finite constant $C > 1$ such that for all $n \ge 1$,

$$C^{-1} n^{-\frac{d}{2}} \le \mathbb{P}(S_{2n} = 0) \le \sup_{a \in \mathbb{Z}^d} \mathbb{P}(S_{2n} = a) \le C n^{-\frac{d}{2}}.$$

(Hint: Use the inversion theorem for characteristic functions and write $\mathbb{P}(S_{2n} = 0)$ as $(2\pi)^{-\frac{d}{2}} \int_{[-\pi,\pi]^d} \mathbb{E}[e^{i\xi \cdot S_{2n}}]\, d\xi$. Use the fact that S has i.i.d. increments and expand this integral near $\xi = 0$. Alternatively, look at Durrett (1991) under "local central limit theorem.") □

***Exercise* 3.1.3** Use Exercise 3.1.2, together with Theorem 1.4.1, to construct an alternative proof of Theorem 3.1.1. □

3.2 Intersections of Two Simple Walks

Given two independent $\mathbb{Z}^d$-valued simple random walks, when do their trajectories intersect infinitely often? In other words, if the random walks are denoted by S^1 and S^2, when can we conclude that $\sum_{j,k \ge 1} \mathbb{1}_{(S^1_j = S^2_k)} = +\infty$?

Theorem 3.2.1 *Suppose S^1 and S^2 are independent simple random walks in $\mathbb{Z}^d$. With probability one, the trajectories of S^1 and S^2 intersect infinitely often if and only if $d \leq 4$.*

Proof When $d \leq 2$, S^1 and S^2 are recurrent; cf. Theorem 3.1.1. By Lemma 2.1.1, we can assume with no loss of generality that $d \geq 3$, i.e., that S^1 and S^2 are transient. Let φ denote the characteristic function of the increments of S^1 and/or S^2, since they have the same distribution. By Corollary 1.7.1, $\lim_{\lambda \uparrow 1} \int_{[-\pi,\pi]^d} \{1 - \lambda\varphi(\xi)\}^{-1} \, d\xi < \infty$. Since $\varphi(\xi) \geq 0$ for all $\xi \in [-1, 1]^d$, the bounded and monotone convergence theorems together show us that

$$\int_{[-\pi,\pi]^d} \frac{1}{1 - \varphi(\xi)} \, d\xi < \infty.$$

Once again applying the bounded and monotone convergence theorems, this time via Proposition 2.1.1, we obtain the following:

$$\begin{aligned}(2\pi)^d \sum_{j,k=1}^{\infty} \mathbb{P}(S_j^1 = S_k^2) &= \int_{[-\pi,\pi]^d} \frac{[\varphi(\xi)]^2}{\{1 - \varphi(\xi)\}^2} \, d\xi \\ &= \int_{[-\pi,\pi]^d} \{1 - \varphi(\xi)\}^{-2} \, d\xi \\ &\quad - \int_{[-\pi,\pi]^d} \{1 + \varphi(\xi)\}\{1 - \varphi(\xi)\}^{-1} \, d\xi.\end{aligned}$$

The second integral is finite. In fact, it is positive and bounded above by $2 \int_{[-\pi,\pi]^d} \{1 - \varphi(\xi)\}^{-1} \, d\xi < +\infty$. Thanks to symmetry and by Theorem 2.1.1, it suffices to show that $\int_{[0,\pi]^d} \{1 - \varphi(\xi)\}^{-2} \, d\xi < \infty$ if and only if $d \geq 5$.

Following the demonstration of Theorem 3.1.1, we split the integral in two parts: where $\xi \in [0, 1]^d$ and where $\xi \in K = [0, \pi]^d \setminus [0, 1]^d$. As in the derivation of equation (3) of Section 3.1, $\int_K \{1 - \varphi(\xi)\}^{-2} \, d\xi \leq d^2 \pi^d \{1 - \cos(1)\}^{-2}$, which is always finite. It remains to show that $\int_{[0,1]^d} \{1 - \varphi(\xi)\}^{-2} \, d\xi$ is finite if and only if $d \geq 5$.

Using equation (2) of Section 3.1,

$$(2d)^2 \int_{[0,1]^d} \|\xi\|^{-4} \, d\xi \leq \int_{[0,1]^d} \{1 - \varphi(\xi)\}^{-2} \, d\xi \leq \left(\frac{12d}{5}\right)^2 \int_{[0,1]^d} \|\xi\|^{-4} \, d\xi.$$

We obtain the result from Lemma 3.1.2. $\square$

The next question that we address is, *when do three or more independent simple random walks intersect infinitely many times?* When $d \geq 5$, the above theorem states that the answer is *never, a.s.* On the other hand, when $d \leq 2$, Theorem 3.1.1 implies that the random walks in question are recurrent; Lemmas 2.1.1 and 2.3.1 together show that any number of such random walks will intersect infinitely many times, a.s. Thus, the only dimensions of interest are $d = 3$ and $d = 4$. In the next two subsections we will study these in detail.

***Exercise* 3.2.1** Use Exercise 3.1.3 and Theorem 1.4.1 together to find an alternative proof of Theorem 3.2.1. □

***Exercise* 3.2.2** Show that if S^1 and S^2 denote two independent simple walks in $\mathbb{Z}^d$ where $d \geq 5$, there exists a finite constant $C > 1$ such that for all $n \geq 1$,

$$C^{-1} n^{-\frac{1}{2}(d-4)} \leq \mathbb{P}(S^1_i = S^2_j \text{ for some } i, j \geq n) \leq C n^{-\frac{1}{2}(d-4)}.$$

(HINT: Use Exercise 3.1.3 and Theorem 2.2.1.) □

3.3 Three Simple Walks

By Theorem 3.2.1 of the previous subsection, two independent $\mathbb{Z}^4$-valued simple random walks will intersect infinitely many times. We now address the problem for three such walks.

Theorem 3.3.1 *Suppose S^1, S^2, and S^3 are independent $\mathbb{Z}^d$-valued simple random walks. The trajectories of S^1, S^2, and S^3 will a.s. intersect infinitely often if and only if $d \leq 3$.*

Our proof relies on two technical lemmas regarding the function $E^d_\beta : \mathbb{R}^d \to \mathbb{R}_+$ that is defined as follows:

$$E^d_\beta(y) = \int\limits_{\substack{\xi \in \mathbb{R}^d: \\ \|\xi\| \leq 1}} \|y - \xi\|^{-\beta} \|\xi\|^{-\beta} \, d\xi, \qquad y \in \mathbb{R}^d. \tag{1}$$

Lemma 3.3.1 *Suppose $\beta < d < 2\beta$. Then, there are two finite and positive constants C_1 and C_2 that depend only on β and d such that for all $y \in \mathbb{R}^d$ with $\|y\| \leq 1$,*

$$C_1 \|y\|^{d-2\beta} \leq E^d_\beta(y) \leq C_2 \|y\|^{d-2\beta}.$$

Proof Fix some $y \in \mathbb{R}^d$ with $\|y\| \leq 1$. Evidently,

$$E^d_\beta(y) \geq \int\limits_{\|\xi\| \leq \|y\|} \|\xi - y\|^{-\beta} \cdot \|\xi\|^{-\beta} \, d\xi.$$

Over the region of integration, $\|\xi - y\| \leq \|\xi\| + \|y\| \leq 2\|y\|$. Hence,

$$E^d_\beta(y) \geq 2^{-\beta} \|y\|^{-\beta} \int\limits_{\|\xi\| \leq \|y\|} \|\xi\|^{-\beta} \, d\xi = 2^{d-\beta} \int\limits_{\|\zeta\| \leq 1} \|\zeta\|^{-\beta} d\zeta \cdot \|y\|^{d-2\beta},$$

which gives the desired lower bound with $C_1 = 2^{d-\beta} \int_{\|\zeta\| \le 1} \|\zeta\|^{-\beta} d\zeta$. (By Lemma 3.1.2, C_1 is finite and positive.) Next, we proceed with the upper bound. Write

$$E_\beta^d(y) = T_1 + T_2 + T_3,$$

where

$$\begin{aligned} T_1 &= \int_{\substack{\|\xi - y\| \le \frac{1}{2}\|y\| \\ \|\xi\| \le 1}} \|\xi - y\|^{-\beta} \cdot \|\xi\|^{-\beta} \, d\xi, \\ T_2 &= \int_{\substack{\|\xi - y\| > \frac{1}{2}\|y\| \\ \|\xi\| \le 2\|y\| \wedge 1}} \|\xi - y\|^{-\beta} \|\xi\|^{-\beta} \, d\xi, \\ T_3 &= \int_{\substack{\|\xi - y\| > \frac{1}{2}\|y\| \\ 2\|y\| \le \|\xi\| \le 1}} \|\xi - y\|^{-\beta} \|\xi\|^{-\beta} \, d\xi. \end{aligned}$$

We estimate the above in order. When $\|\xi - y\| \le \frac{1}{2}\|y\|$, by the triangle inequality, $\|\xi\| \ge \frac{1}{2}\|y\|$. Thus,

$$T_1 \le 2^\beta \|y\|^{-\beta} \int_{\|\zeta\| \le \frac{1}{2}\|y\|} \|\zeta\|^{-\beta} \, d\zeta = 2^{2\beta - d} \int_{\|\zeta\| \le 1} \|\zeta\|^{-\beta} \, d\zeta \ \|y\|^{d - 2\beta}.$$

By Supplementary Exercise 7,

$$T_1 \le \frac{2^{2\beta - d} d\omega_d}{d - \beta} \ \|y\|^{d - 2\beta}, \tag{2}$$

where ω_d denotes the d-dimensional Lebesgue measure of the ball $\{z \in \mathbb{R}^d : \|z\| \le 1\}$. Similarly,

$$T_2 \le 2^\beta \|y\|^{-\beta} \int_{\|\zeta\| \le 2\|y\|} \|\zeta\|^{-\beta} d\zeta = 2^d \int_{\|\zeta\| \le 1} \|\zeta\|^{-\beta} d\zeta \ \|y\|^{d - 2\beta}.$$

Another application of Supplementary Exercise 7 leads us to the bound

$$T_2 \le \frac{2^d d\omega_d}{d - \beta} \ \|y\|^{d - 2\beta}. \tag{3}$$

It remains to estimate T_3. First, we note that if $\|\xi - y\| \ge \frac{1}{2}\|y\|$ and $\|\xi\| \ge 2\|y\|$, then certainly $\|\xi - y\| \le \|\xi\| + \|y\| \le \frac{3}{2}\|\xi\|$. Thus,

$$\begin{aligned} T_3 &\le \left(\frac{3}{2}\right)^\beta \int_{\|\xi - y\| \ge \frac{1}{2}\|y\|} \|\xi - y\|^{-2\beta} \, d\xi \\ &\le \left(\frac{3}{2}\right)^\beta \int_{\|\zeta\| > \frac{1}{2}\|y\|} \|\zeta\|^{-2\beta} d\zeta \\ &\le 3^\beta 2^{-d} \|y\|^{d - 2\beta} \int_{\|\zeta\| > 1} \|\zeta\|^{-2\beta} \, d\zeta. \end{aligned}$$

To finish, we are left to show that $\int_{\|\zeta\|>1} \|\zeta\|^{-2\beta} d\zeta < \infty$. This is easy to do: Since $d < 2\beta$, by Supplementary Exercise 7,

$$\int_{\|\zeta\|>1} \|\zeta\|^{-2\beta}\, d\zeta = d\omega_d \int_1^{\infty} r^{d-1-2\beta}\, dr = \frac{d\omega_d}{2\beta - d}.$$

To summarize, we have shown that $T_3 \le 3^{\beta} 2^{-d} \omega_d d (2\beta - d)^{-1} \|y\|^{d-2\beta}$. Combining this with (3) and (4), we obtain $E^d_{\beta} \le C_2 \|y\|^{d-2\beta}$ with

$$C_2 = d\omega_d \Big\{ \frac{2^d + 2^{2\beta - d}}{d - \beta} + \frac{3^{\beta} 2^{-d}}{2\beta - d} \Big\}.$$

Since C_2 is clearly finite and positive, this concludes our proof. $\square$

Going over the above argument with some care, we can also decide what happens when $d = 2\beta$.

Lemma 3.3.2 *There exists a finite and positive constant C that depends only on d such that for all $y \in \mathbb{R}^d$ with $\|y\| \le 1$, $E^d_{\frac{d}{2}}(y) \le C \ln\big(4/\|y\|\big)$.*

Proof In the notation of our proof of Lemma 3.3.1, write $E^d_{d/2}(y) = T_1 + T_2 + T_3$. Since they still hold for $d = 2\beta$, equations (2) and (3) together show that $T_1 + T_2 \le C_1 \le C_1 \ln(4/\|y\|)$, with $C_1 = (2^{d+1} + 2)\omega_d$. Still proceeding with our proof of Lemma 3.3.1 and using $\beta = \frac{d}{2}$, we obtain,

$$\begin{aligned} T_3 &\le \Big(\frac{3}{2}\Big)^{\frac{d}{2}} \int_{2 \ge \|\xi\| \ge \frac{1}{2}\|y\|} \|\xi\|^{-d}\, d\xi \\ &= d\omega_d \Big(\frac{3}{2}\Big)^{\frac{d}{2}} \int_{\frac{1}{2}\|y\|}^{2} r^{-1}\, dr \\ &= d\omega_d \Big(\frac{3}{2}\Big)^{\frac{d}{2}} \ln\Big(\frac{4}{\|y\|}\Big). \end{aligned}$$

We have used Supplementary Exercise 7 once more and obtained the desired result with $C = C_1 + (\frac{3}{2})^{\frac{d}{2}} d\omega_d$. $\square$

***Exercise* 3.3.1** Prove that Lemma 3.3.2 is sharp, up to a constant. That is, prove that $\liminf_{\|y\| \to 0^+} \{\ln(1/\|y\|)\}^{-1} E^d_{\frac{\beta}{2}}(y) > 0$. $\square$

We are ready for the following.

Proof of Theorem 3.3.1 When $d \le 2$, the simple random walk is recurrent (Theorem 3.1.1). Thus, Lemmas 2.1.1 and 2.3.1 tell us that the trajectories of S^1, S^2, and S^3 intersect infinitely many times. (Why?) On the other hand, if $d \ge 5$, then by Theorem 3.2.1, the trajectories of S^1 and

S^2 intersect only finitely many times. In particular, so do the trajectories of S^1, S^2, and S^3. Thus, it remains to focus our attention on $d \in \{3,4\}$.

Let $\mathfrak{S} = (2\pi)^{2d} \sum_{i,j,k=1}^{\infty} \mathbb{P}\big(S_i^1 = S_j^2 = S_k^3\big)$. Thanks to Theorem 2.3.1, we need to show that $\mathfrak{S} < \infty$ when $d = 4$ while $\mathfrak{S} = \infty$ when $d = 3$. In order to do this, we begin with the identity

$$\mathfrak{S} = \lim_{\lambda \uparrow 1} \int_{[-\pi,\pi]^{2d}} \frac{\varphi(\xi_1+\xi_2)}{1-\lambda\varphi(\xi_1+\xi_2)} \frac{\varphi(\xi_1)}{1-\lambda\varphi(\xi_1)} \frac{\varphi(\xi_2)}{1-\lambda\varphi(\xi_2)} \, d\xi_1 \, d\xi_2.$$

(We have implicitly used the fact that φ is real-valued. Why?) While for every $\xi_1, \xi_2 \in [-1,1]^d$, $\varphi(\xi_1), \varphi(\xi_2) \geq 0$, it is not always true that $\varphi(\xi_1 + \xi_2) \geq 0$. To regain positivity, we split the above integral into two parts: Let I_1 denote the above integral taken over $[-\frac{1}{2}, \frac{1}{2}]^{2d}$, and I_2 the integral over $K = [-\pi,\pi]^{2d} \setminus [-\frac{1}{2}, \frac{1}{2}]^{2d}$. We estimate I_2 first. Since cosines are bounded above by 1,

$$|I_2| \leq \lim_{\lambda \uparrow 1} \int_K \frac{1}{1-\lambda\varphi(\xi_1+\xi_2)} \frac{1}{1-\lambda\varphi(\xi_1)} \frac{1}{1-\lambda\varphi(\xi_2)} \, d\xi_1 \, d\xi_2.$$

Note that whenever $\xi_1, \xi_2 \in K$, then for all $1 \leq \ell \leq d$,

(a) $\cos(\xi_1^{(\ell)} + \xi_2^{(\ell)}) \leq \cos(\frac{1}{2}) < 1$;

(b) $\cos(\xi_1^{(\ell)}) \leq \cos(\frac{1}{2}) < 1$; and

(c) $\cos(\xi_2^{(\ell)}) \leq \cos(\frac{1}{2}) < 1$.

Hence,

$$|I_2| \leq (2\pi)^{2d} \big\{1 - \cos(\tfrac{1}{2})\big\}^{-3} < \infty.$$

Thus, we need to show that $|I_1|$ is finite when $d = 4$ and is infinite when $d = 3$. This is where positivity comes into play: If $\xi_1, \xi_2 \in [-\frac{1}{2}, \frac{1}{2}]^{2d}$, then $\varphi(\xi_1), \varphi(\xi_2)$, and $\varphi(\xi_1 + \xi_2)$ are all nonnegative. By the monotone convergence theorem,

$$I_1 = \int_{[-\frac{1}{2},\frac{1}{2}]^{2d}} \frac{\varphi(\xi_1+\xi_2)}{1-\varphi(\xi_1+\xi_2)} \frac{\varphi(\xi_1)}{1-\varphi(\xi_1)} \frac{\varphi(\xi_2)}{1-\varphi(\xi_2)} \, d\xi_1 \, d\xi_2.$$

Moreover, if $\xi \in [-\frac{1}{2}, \frac{1}{2}]$, then $0 < \cos(\frac{1}{2}) \leq \cos(\xi) \leq 1$. We have arrived at the bound $\{\cos(\frac{1}{2})\}^3 I_1' \leq I_1 \leq I_1'$, where

$$I_1' = \int_{[-\frac{1}{2},\frac{1}{2}]^{2d}} \big(1-\varphi(\xi_1+\xi_2)\big)^{-1} \big(1-\varphi(\xi_1)\big)^{-1} \big(1-\varphi(\xi_2)\big)^{-2} \, d\xi_1 \, d\xi_2.$$

Since $I_1' \geq 0$, we want to show that I_1' is finite if $d = 4$ but is infinite if $d = 3$. By equations (1) and (2) of Section 3.1, $(2d)^3 I_1'' \leq I_1' \leq (\frac{12d}{5})^3 I_1''$, where

$$I_1'' = \int_{[-\frac{1}{2},\frac{1}{2}]^{2d}} \|\xi_1+\xi_2\|^{-2} \|\xi_1\|^{-2} \|\xi_2\|^{-2} \, d\xi_1 \, d\xi_2.$$

Our goal now is to show that I_1'' is finite if $d = 4$ and is infinite if $d = 3$. By Fubini's theorem and symmetry,

$$\begin{aligned} I_1'' &= \int_{[-\frac{1}{2},\frac{1}{2}]^{2d}} \|\xi_1 - \xi_2\|^{-2} \|\xi_1\|^{-2} \|\xi_2\|^{-2} \, d\xi_1 \, d\xi_2 \\ &\leq \int_{[-\frac{1}{2},\frac{1}{2}]^{2d}} E_2^d(\xi_1) \|\xi_1\|^{-2} \, d\xi_1. \end{aligned}$$

If $d = 4$, by Lemma 3.3.2 there exists a finite and positive constant C_1 such that

$$I_1'' \leq C_1 \int_{[-1,1]^4} \ln\Big(\frac{4}{\|\xi\|}\Big) \|\xi\|^{-2} \, d\xi.$$

Since $\ln(4/\|\xi\|) \leq 4/\|\xi\|$ for all $\xi \in \mathbb{R}^4$ with $\|\xi\| \leq 1$,

$$I_1'' \leq C_1 \int_{[-1,1]^d} \|\xi\|^{-3} \, d\xi,$$

which is finite, thanks to Lemma 3.1.2. If $d = 3$, by Lemma 3.3.2 there exists a finite positive constant C_2 such that

$$I_1'' \geq C_2 \int_{[-\frac{1}{2},\frac{1}{2}]^3} \|\xi\|^{-3} \, d\xi.$$

Since $d = 3$, Lemma 3.1.2 shows us that $I_1'' = \infty$. This concludes our proof. $\square$

3.4 Several Simple Walks

Throughout, let us fix an integer $N \geq 4$ and consider N independent simple walks, $S^1, \ldots, S^N$, all taking values in $\mathbb{Z}^d$. If $d \leq 2$, such random walks are recurrent (Theorem 3.1.1). By Lemma 2.1.1, when $d \leq 2$, the trajectories of $S^1, \ldots, S^N$ intersect infinitely often, a.s. Next, suppose $d \geq 4$. In this case, the trajectories of S^1, S^2, and S^3 intersect finitely often, a.s. (Theorem 3.3.1). Therefore, the same holds for $S^1, \ldots, S^N$. The only case that remains to be analyzed is $d = 3$.

Theorem 3.4.1 *The trajectories of four or more independent simple walks in $\mathbb{Z}^3$ will almost surely intersect at most finitely many times.*

Our proof is an imitation of those in the previous sections but requires one more technical lemma.

Lemma 3.4.1 *For all $y \in \mathbb{R}^3$ define*

$$F(y) = \int_{\substack{\xi \in \mathbb{R}^3: \\ \|\xi\| \leq 1}} \|\xi - y\|^{-1} \|\xi\|^{-2} \, d\xi.$$

Then, for all $y \in \mathbb{R}^3$ with $\|y\| \leq 1$, $F(y) \leq 20\pi \ln(4/\|y\|)$.

Proof We follow closely the arguments used in the given proofs of Lemmas 3.3.1 and 3.3.2. Write $F(y) = T_1 + T_2 + T_3$, where

$$T_1 = \int_{\substack{\|\xi - y\| \le \frac{1}{2}\|y\| \\ \|\xi\| \le 1}} \|\xi - y\|^{-1} \|\xi\|^{-2}\, d\xi,$$

$$T_2 = \int_{\substack{\|\xi - y\| > \frac{1}{2}\|y\| \\ \|\xi\| \le 2\|y\| \wedge 1}} \|\xi - y\|^{-1} \|\xi\|^{-2}\, d\xi,$$

$$T_3 = \int_{\substack{\|\xi - y\| > \frac{1}{2}\|y\| \\ 2\|y\| \le \|\xi\| \le 1}} \|\xi - y\|^{-1} \|\xi\|^{-2}\, d\xi.$$

We estimate each as in the demonstrations of Lemmas 3.3.1 and 3.3.2. To estimate T_1, use $\|\xi - y\| \ge \|y\|/2$ to obtain

$$T_1 \le 4\|y\|^{-2} \int_{\|\xi - y\| \le \frac{1}{2}\|y\|} \|\xi - y\|^{-1}\, d\xi\| = \int_{\|r\| \le 1} \|r\|^{-1} dr.$$

By Supplementary Exercise 7, $T_1 \le 2\pi \le 2\pi \ln(4/\|y\|)$. We have used the elementary fact that $\omega_3 = \frac{4\pi}{3}$. Likewise,

$$T_2 = 2\|y\|^{-1} \int_{\|\xi\| \le 2\|y\|} \|\xi\|^{-2}\, d\xi = 4 \int_{\|\xi\| \le 1} \|\xi\|^{-2}\, d\xi = 8\pi.$$

Since $8\pi \le 8\pi \ln(4/\|y\|)$, it remains to show that $T_3 \le 9\pi \ln(4/\|y\|)$. Use $\|\xi - y\| \le 3\|\xi\|$ to obtain

$$T_3 \le \frac{9}{4} \int_{1 \ge \|\xi - y\| \ge \frac{1}{2}\|y\|} \|\xi - y\|^{-3}\, d\xi \le \frac{9}{4} \int_{2 \ge \|\zeta\| \ge \frac{1}{2}\|y\|} \|\zeta\|^{-3} d\zeta.$$

By Exercise 3.4.1 below this equals $9\pi \ln(4/\|y\|)$, as desired. □

***Exercise* 3.4.1** For any $\varepsilon \in]0, 2[$, compute $\int_{2 \ge \|\zeta\| \ge \varepsilon} \|\zeta\|^{-3}\, d\zeta$. □

***Exercise* 3.4.2** Show that Lemma 3.4.1 is sharp, up to a constant. That is, $\liminf_{\|y\| \to 0^+} F(y)/\ln(1/\|y\|) > 0$. □

We are ready to prove the theorem.

Proof of Theorem 3.4.1 It suffices to consider only $N = 4$ and to show that

$$\sum_{i,j,k,\ell=0}^{\infty} \mathbb{P}(S_i^1 = S_j^2 = S_k^3 = S_\ell^4) < \infty,$$

where $S_0^1 = S_0^2 = S_0^3 = S_0^4 = 0$. However, symmetry and Proposition 2.3.1 together show that this is the same as showing that

$$\lim_{\lambda \uparrow 1} \int_{[-\pi,\pi]^9} \frac{\varphi(\xi_1 + \xi_2 + \xi_3)}{1 - \lambda\varphi(\xi_1 + \xi_2 + \xi_3)} \prod_{j=1}^{3} \frac{\varphi(\xi_j)}{1 - \lambda\varphi(\xi_j)}\, d\xi_1\, d\xi_2\, d\xi_3 < \infty.$$

We split the above integral into two parts. Let I_1 be the integral over $[-\frac{1}{3}, \frac{1}{3}]^9$ and I_2 the integral over $K = [-\pi, \pi]^9 \setminus [-\frac{1}{3}, \frac{1}{3}]^9$. The same argument used to prove Theorem 3.3.1 goes through unhindered to show that

$$|I_2| \le (2\pi)^9 \big[1 - \cos(\tfrac{1}{3})\big]^{-4} < \infty.$$

It suffices to show that I_1 is finite. When $\xi_i \in [-\frac{1}{3}, \frac{1}{3}]^3$ $(i = 1, 2, 3)$, $\varphi(\xi_i)$ is positive $(i = 1, 2, 3)$. Moreover, so is $\varphi(\xi_1 + \xi_2 + \xi_3)$. By the monotone convergence theorem,

$$\begin{aligned} I_1 &= \int_{[-\frac{1}{3},\frac{1}{3}]^9} \frac{\varphi(\xi_1+\xi_2+\xi_3)}{1-\varphi(\xi_1+\xi_2+\xi_3)} \prod_{j=1}^{3} \frac{\varphi(\xi_j)}{1-\varphi(\xi_j)}\, d\xi_1\, d\xi_2\, d\xi_3 \\ &\le \iiint\limits_{\|\xi_1\|,\|\xi_2\|,\|\xi_3\|\le 1} \big(1-\varphi(\xi_1+\xi_2+\xi_3)\big)^{-1} \prod_{\ell=1}^{3} \big(1-\varphi(\xi_\ell)\big)^{-1}\, d\xi_1\, d\xi_2\, d\xi_3. \end{aligned}$$

Employing equations (1) and (2) of Section 3.1, we deduce that $I_1 \le (\frac{36}{5})^4 J$, where

$$J = \iiint\limits_{\|\xi_1\|,\|\xi_2\|,\|\xi_3\|\le 1} \|\xi_1+\xi_2+\xi_3\|^{-2} \|\xi_1\|^{-2} \|\xi_2\|^{-2} \|\xi_3\|^{-2}\, d\xi_1\, d\xi_2\, d\xi_3.$$

We propose to show that $J < \infty$. Using symmetry and the definition of E_β^d (equation (1) of Section 3.3),

$$J \le \iint\limits_{\|\xi_1\|,\|\xi_2\|\le 1} E_2^3(\xi_1+\xi_2) \|\xi_1\|^{-2} \|\xi_2\|^{-2}\, d\xi_1\, d\xi_2.$$

Lemma 3.3.1 can be applied with $d = 3$ and $\beta = 2$ to show us the existence of a positive and finite constant C such that $J \le C \int_{\|\xi\|\le 1} F(\xi) \|\xi\|^{-2}\, d\xi$. By Supplementary Exercise 7, and by Lemma 3.4.1 above,

$$J \le 20\pi C \int_{\|\xi\|\le 1} \ln\Big(\frac{4}{\|\xi\|}\Big) \|\xi\|^{-2}\, d\xi,$$

which is finite, by Supplementary Exercise 7. □

4 Supplementary Exercises

1. Show that the inequalities of Theorem 1.5.1 can be sharpened to the following: $\mathbb{P}(S_k = 0 \text{ for some } k \ge n) = Q_n\{1 + Q_1\}^{-1}$, where $Q_n = \sum_{n=1}^{\infty} \mathbb{P}(S_j = 0)$.

2. Refine an aspect of Exercise 3.1.2 by showing that when S denotes the simple walk on $\mathbb{Z}^d$, $\lim_{n\to\infty}(2n)^{\frac{d}{2}}\mathbb{P}(S_{2n}=0)=(2\pi)^{-\frac{d}{2}}$.
This is a part of the local central limit theorem. You should compare this to the classical central limit theorem of A. de Moivre and P.-S. Laplace by looking at the density function of a mean-zero Gaussian random variable with the same variance as S_{2n}.

3. Let S denote a transient random walk on $\mathbb{Z}^d$ with $S_0=0$ and define T_x to be the first time S hits x. That is, $T_x=\inf(k\geq 0:\ S_k=x)$. In the notation of Section 1.4, show that $\mathbb{E}[R_{T_x}]=\sum_{k=0}^{\infty}\{\mathbb{P}(S_k=0)-\mathbb{P}(S_k=-x)\}$.
(HINT: By transience, $R_\infty<\infty$, a.s. Now we can write $R_\infty=\sum_{k=0}^{T_x-1}\mathbf{1}_{(S_k=0)}+\sum_{k=T_x}^{\infty}\mathbf{1}_{(S_k=0)}$ and use the strong Markov property.)

4. Show that for any random walk S on $\mathbb{Z}^d$ and for all integers $n,k\geq 1$, $\mathbb{E}[R_n^k]\leq k!\{\mathbb{E}[R_n]\}^k$. In particular, obtain the large deviation bound

$$\mathbb{P}\Big(\frac{R_n}{\mathbb{E}[R_n]}\geq\lambda\Big)\leq\frac{1}{1-\delta}\,e^{-\delta\lambda},\qquad \lambda>0,$$

where δ is an arbitrary number strictly between 0 and 1.

5. (*Mixing*) Much of the theory for independent random variables goes through with fewer hypotheses than independence. We explore one such possibility in this exercise.

A sequence of random variables $\xi_1,\xi_2,\dots$ is said to be **φ-mixing** if

$$\sup_{i\geq 1}\ \sup_{\substack{E\in\mathcal{F}_{[i+n,\infty[}\\ F\in\mathcal{F}_{[1,i]}}}\Big|\mathbb{P}(E\,|\,F)-\mathbb{P}(E)\Big|\leq\varphi(n),$$

where $\mathcal{F}_A$ is the σ-field generated by $\{\xi_i;\ i\in A\}$, and $\lim_{n\to\infty}\varphi(n)=0$. Note that if the ξ_i's are independent, then they are φ-mixing for any φ that vanishes at infinity.

(i) Prove that the tail σ-field $\mathcal{T}=\cap_n\mathcal{F}_{[n,\infty[}$ is trivial.

(ii) Show that whenever $\sum_n\varphi(n)<+\infty$,

$$\mathbb{P}(\xi_n=0\text{ infinitely often})=\begin{cases}0, & \text{if }\sum_{n=1}^{\infty}\mathbb{P}(\xi_n=0)<+\infty,\\ 1, & \text{if }\sum_{n=1}^{\infty}\mathbb{P}(\xi_n=0)=+\infty.\end{cases}$$

6. Verify equation (1) of Section 2.3.

7. Suppose U is chosen uniformly at random from $\mathbb{D}_m=\{\xi\in\mathbb{R}^m:\ \|\xi\|\leq 1\}$.

(i) Show that the density function of $\|U\|$ at $x\in[0,1]$ is mx^{m-1}.

(ii) Use the previous part to prove the following integration-by-parts formula: For all integrable functions $f:[0,1]\to[0,1]$,

$$\int_{\mathbb{D}_m}f(\|u\|)\,du=m\omega_m\cdot\int_0^1 s^{m-1}f(s)\,ds,$$

where ω_m denotes Lebesgue's (m-dimensional) measure of $\mathbb{D}_m$.

(iii) Show that

$$\omega_m = \begin{cases} \dfrac{m\pi^{\frac{m}{2}}}{(m/2)!} & \text{if } m \text{ is even,} \\ \dfrac{2^{\frac{1}{2}(m+1)}\pi^{\frac{1}{2}(m-1)}}{1 \cdot 3 \cdot 5 \cdots (m-2)}, & \text{if } m \text{ is odd and } m > 1. \end{cases}$$

8. (Hard) Let S denote the simple walk on $\mathbb{Z}^2$ and let $S_0 = 0$.

(i) When $d = 1$, use Supplementary Exercise 2 to deduce that with probability one,

$$\lim_{n\to\infty} \frac{1}{\ln n} \sum_{k=1}^{n} \frac{\mathbf{1}_{(S_k=0)}}{k^{\frac{1}{2}}} = \frac{1}{2\sqrt{2\pi}}.$$

(ii) Prove that when $d = 2$,

$$\lim_{n\to\infty} \frac{1}{\ln\ln n} \sum_{k=2}^{n} \frac{\mathbf{1}_{(S_k=0)}}{\ln k} = \frac{1}{4\pi}.$$

This is due to Erdős and Taylor (1960a, 1960b).
(Hint: For part (i), start by proving that the expected value of the limit theorem holds. Then, prove that the variance of the given sum is bounded by $C \ln n$, for some finite constant $C > 0$. Use the Borel–Cantelli lemma to obtain the a.s. convergence along the subsequence $n_k = \exp(k^2)$. To conclude part (i), estimate the sum for $n_k \le n \le n_{k+1}$ by the end values of n. Part (ii) is proved similarly, but the variance estimate is now given by a bound of $C \ln\ln n$, and the subsequence should be changed to $n_k = \exp(e^{k^2})$.)

9. (Hard) Suppose $X_1, X_2, \ldots$ denote i.i.d. random variables that take their values in $\mathbb{R}^d$ and define the corresponding random walk $S_n = \sum_{j=1}^n X_j$ $(n \ge 1)$. We say that 0 is recurrent if for all $\varepsilon > 0$, $\mathbb{P}(|S_n| < \varepsilon \text{ infinitely often}) > 0$.

(i) Verify that when $\mathbb{P}(X_1 \in \mathbb{Z}^d) = 1$, our two notions of recurrence are one and the same.

(ii) Show that 0 is recurrent if and only if for all $\varepsilon > 0$, $\mathbb{P}(|S_n| < \varepsilon \text{ infinitely often}) = 1$.

(iii) Define $S_0 = 0$ and prove that for all $n \ge 1$ and all $\varepsilon > 0$,

$$\sum_{j=0}^{n} \mathbb{P}(|S_j| \le 2\varepsilon) \le 16^d \sum_{j=0}^{n} \mathbb{P}(|S_j| \le \varepsilon).$$

(iv) Show that the following are all equivalent:

(a) 0 is recurrent;

(b) for some $\varepsilon > 0$, $\sum_{j=1}^{\infty} \mathbb{P}(|S_j| \le \varepsilon) = +\infty$;

(c) for all $\varepsilon > 0$, $\sum_{j=1}^{\infty} \mathbb{P}(|S_j| \le \varepsilon) = +\infty$.

(Hint: For part (iii), cover $[-2\varepsilon, 2\varepsilon]^d$ with 16^d cubes of side $\frac{1}{2}\varepsilon$ and apply the Markov property.)

10. Given a transient random walk S on $\mathbb{Z}^d$ with $S_0 = 0$, define for each $a \in \mathbb{Z}^d$, $u(a) = \mathbb{E}[\sum_{k=0}^{\infty} \mathbf{1}_{(S_k+a=0)}]$.

(i) Check that $u(0) = \mathbb{E}[R_\infty]$ and show that $u(a)$ is finite for all $a \in \mathbb{Z}^d$.

(ii) Show that $m \mapsto u(S_m)$ is a supermartingale.
(HINT: Apply Lemma 1.1.1 to $f(x) = \mathbf{1}_{\{0\}}(x)$.)

11. (Hard) Let S denote the simple walk on $\mathbb{Z}^d$.

(i) In the case $d \geq 3$, prove that there are finite positive constants $C_1 < C_2$ such that for all $n \geq 1$,

$$C_1 n^{-\frac{1}{2}(d-2)} \leq \mathbb{P}(S_i = 0 \text{ for some } i \geq n) \leq C_2 n^{-\frac{1}{2}(d-2)}.$$

(ii) Let $d = 2$ and suppose $c_1, c_2, \ldots$ is a nondecreasing sequence such that $\lim_{n\to\infty} c_n = +\infty$ and $\limsup_{n\to\infty} c_n/n < \infty$. Show that when $d = 2$, there exist finite positive constants $C_1 < C_2$ such that for all $n \geq 1$,

$$C_1 \frac{c_n}{n \ln c_n} \leq \mathbb{P}(S_i = 0 \text{ for some } n \leq i \leq n + c_n) \leq C_2 \frac{c_n}{n \ln c_n}.$$

(HINT: For the lower bound, consider the first two moments of $\sum_{j=n}^{n+c_n} \mathbf{1}_{(S_j=0)}$. For the upper bound, estimate the conditional expectation of $\sum_{j=n}^{n+2c_n} \mathbf{1}_{(S_j=0)}$, given $\mathcal{F}_m$, where m is between n and $n + c_n$.)

In different forms and to various extents, this can be found in Benjamini et al. (1995), Erdős and Taylor (1960a, 1960b), Lawler (1991), and Révész (1990).

12. Let $\tau_1, \tau_2, \ldots$ denote the first, second, $\ldots$ hitting times of 0 by a $\mathbb{Z}^d$-valued random walk S. The goal of this exercise is an exact computation of the distribution of τ_1.

(i) Show that for all $\lambda > 0$ and for all integers $n \geq 1$, $\mathbb{E}[e^{-\lambda\tau_n}] = (\mathbb{E}[e^{-\lambda\tau_1}])^n$.

(ii) Let $S_0 = \tau_0 = 0$ and for all $\lambda > 0$, define $V_\lambda = \sum_{k=0}^{\infty} e^{-\lambda k} \mathbf{1}_{(S_k=0)}$. Show that $V_\lambda = \sum_{n=0}^{\infty} e^{-\lambda\tau_n}$ and conclude the following identity for the Laplace transform of τ_1: $\mathbb{E}[e^{-\lambda\tau_1}] = 1 - \{\sum_{k=0}^{\infty} e^{-\lambda k} \mathbb{P}(S_k = 0)\}^{-1}$.

(iii) Show that when S is the simple walk on $\mathbb{Z}^d$,

$$\lim_{\lambda\to 0^+} \lambda^{\frac{1}{2}} \sum_{k=0}^{\infty} e^{-\lambda k} \mathbb{P}(S_k = 0) = \sqrt{2}$$

when $d = 1$, and when $d = 2$,

$$\lim_{\lambda\to 0^+} \frac{1}{\ln(\frac{1}{\lambda})} \sum_{k=0}^{\infty} e^{-\lambda k} \mathbb{P}(S_k = 0) = \frac{1}{2\pi}.$$

(HINT: Consider the distribution function $F(k) = \sum_{j\leq k} \mathbb{P}(S_j = 0)$. Apply the Tauberian theorem Theorem 2.1.1, Appendix B, together with Supplementary Exercise 2.)

Such results are a part of the folklore of random walks; for instance, read Chung and Hunt (1949) with care. In the above forms, they can be found in Khoshnevisan (1994), where you can also find further applications to measure the zero set of random walks.

13. (Continued from Supplementary Exercise 12)

(i) Let S denote the simple walk on $\mathbb{Z}^d$. In the notation of Supplementary Exercise 12, show that when $d = 1$, τ_n/n^2 converges in distribution to a nonnegative random variable τ_∞ whose Laplace transform is $\mathbb{E}[e^{-\zeta\tau_\infty}] = \exp(-\sqrt{\zeta})$.
(Hint: Use the convergence theorem for Laplace transforms (cf. Theorem 1.2.1, Appendix B). The random variable τ_∞ is the so-called **stable** random variable of index $\frac{1}{2}$ and will reappear later in Section 3.2, Chapter 10.)

(ii) Conclude that when $d = 1$, $R_n/\sqrt{n}$ converges in distribution to the absolute value of a standard Gaussian random variable.
(Hint: Since τ is the inverse function to R, roughly speaking, $\mathbb{P}(R_n \geq \lambda\sqrt{n}) = \mathbb{P}(\tau_{\lambda\sqrt{n}} \leq n)$. You need to make this work by a series of inequalities.)

14. (Hard) Suppose S^1 and S^2 are two independent simple walks on $\mathbb{Z}^4$. Consider a nondecreasing sequence $c_1, c_2, \ldots$ such that $\lim_{n\to\infty} c_n = +\infty$ and $\limsup_{n\to\infty} c_n/n < \infty$. Show the existence of two positive finite constants $C_1 < C_2$ such that for all $n \geq 1$,

$$C_1\left(\frac{c_n}{n}\right)^2 \cdot \frac{1}{\ln c_n} \leq \mathbb{P}(S_i^1 = S_j^2 \text{ for some } n \leq i, j \leq n + c_n) \leq C_2\left(\frac{c_n}{n}\right)^2 \cdot \frac{1}{\ln c_n}.$$

(You should first study Supplementary Exercise 11.)

5 Notes on Chapter 3

Section 1 The references (Ornstein 1969; Spitzer 1964; Révész 1990; Revuz 1984) are excellent resources for the fine and general structure of one-parameter random walks, Markov chains, and their connections to ergodic theory and potential theory.

The argument of Section 1.7 that reduces attention to the set of possible points is quite old, but often goes unmentioned when $d > 1$, perhaps to avoid discussions relating to free abelian groups.

Much of the material of this section, and, in fact, chapter, can be extended to random walks on locally compact abelian groups. A comprehensive account of the potential-theoretic aspects of this can be found in Port and Stone (1971a, 1971b).

The basic message of the investigations of recurrence for random walks is that a point is recurrent for the walk if and only if the walk is expected to hit that point infinitely often. The number of times the random walk hits a given point is the so-called local time at that point. There are limit theorems associated with such local times; they can be viewed as refinements of the notion of recurrence, among other things; see Bass and Khoshnevisan (1993b, 1993c, 1995), Borodin (1986, 1988), Csáki and Révész (1983), Csörgő and Révész (1984, 1985, 1986), Kesten and Spitzer (1979), Jacod (1998), Khoshnevisan (1992, 1993), Knight (1981), Perkins (1982), and Révész (1981).

Section 2 In the probability literature, the study of the intersections of random walk trajectories goes back at least to Dvoretzky and Erdős (1951), as well as Erdős and Taylor (1960b, 1960a), and Dvoretzky et al. (1950, 1954, 1958, 1957). Related results, together with references to the physics literature, can be found in (Madras and Slade 1993; Lawler 1991).

In this section we essentially showed that the intersections are recurrent if and only if the walks are expected to intersect infinitely many times, at least as long as all of the intervening walks are symmetric. At this time it is not known whether Theorem 2.3.1 holds without any symmetry, or sector-type, hypotheses.

Further analysis of the number of intersections of random walks leads to a so-called intersection local time that is the main subject of Le Gall et al. (1989), Le Gall and Rosen (1991), Lawler (1991), Rosen (1993), and Stoll (1987, 1989). Some very general results can be found in (Bass and Khoshnevisan 1992a; Dynkin 1988).

Many of the quantitative results of this section are new.

Section 3 The results of this section are all classical and can be found in the pre-60's references cited under Section 2 above. For further refinements, see Lawler (1991). Many of the presented proofs in this section are new. Further related works, but in a genuine multiparameter context, can be found in Etemadi (1977).

A variant of Exercise 3.2.1 can be found in Lawler (1991, Theorem 3.3.2).

Section 4 A variant of Supplementary Exercise 14 can be found in Lawler (1991, Theorem 3.3.2).

Supplementary Exercise 5 seems to be new. However, much is known about sums of mixing random variables. A good starting place for this is Billingsley (1995).

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