Preface

Background

I was an eighteen-year-old freshman when I began studying analysis. I had arrived at Columbia University ready to major in physics or perhaps engineering. But my seduction into mathematics began immediately with Lipman Bers' calculus course, which stood supreme in a year of exciting classes. Then after the course was over, Professor Bers called me into his office and handed me a small blue book called *Principles of Mathematical Analysis* by W. Rudin. He told me that if I could read this book over the summer, understand most of it, and prove it by doing most of the problems, then I might have a career as a mathematician. So began twenty years of struggle to master the ideas in "Little Rudin."

I began because of a challenge to my ego but this shallow reason was quickly forgotten as I learned about the beauty and the power of analysis that summer. Anyone who recalls taking a "serious" mathematics course for the first time will empathize with my feelings about this new world into which I fell. In school, I restlessly wandered through complex analysis, analytic number theory, and partial differential equations, before eventually settling in numerical analysis. But underlying all of this indecision was an ever-present and ever-growing appreciation of analysis. An appreciation that still sustains my intellect even in the often cynical world of the modern academic professional.

But developing this appreciation did not come easy to me, and the presentation in this book is motivated by my struggles to understand the most basic concepts of analysis. To paraphrase J. von Neumann, it is not that we understand mathematics, rather mathematics just becomes familiar with practice. We often understand a difficult concept by considering special cases that make the concept concrete. In turn, our understanding of a concept is shaded by the special cases we consider. After learning about mathematics in specific contexts, it is easy to become convinced that this is the natural and best setting in which to teach these ideas.

I think this is especially true of analysis. I view analysis as the art and the science of estimation. That the practice of analysis is an art is understood by anyone who tries to explain an " epsilon-delta" proof of differentiation to a calculus student. At certain points, the natural response to "Why did you do that?" is "It's obvious, don't you see?" By the science of estimation, I refer to the need for the mathematical rigor that guarantees that any estimates obtained are meaningful and that plausible arguments are true.

Neither an art nor a science can be taught effectively in the abstract. Concepts and techniques that are perfectly well motivated in practical settings simply become a "bag of tricks" in the abstract. Moreover, technical difficulties often become overwhelming when there are no concrete examples to motivate the issues or provide a compelling reason to spend time on the complications. Too often, the mind lacks the firepower to leap past abstract technical mathematics to imagine how the underlying ideas might be used.

Consequently, I present the basic ideas of real analysis in the context of a fundamental problem of applied mathematics, which is approximating solutions of physical models. This approach is natural to me because of my research interests in numerical analysis and applied mathematics. I am a numerical analyst because my first reaction to being faced with a difficult analytic concept is to compute examples. I believe this "experimental" approach to understanding mathematics is natural for many people. So as much as practicable, I present analysis from a constructive point of view. Many major theorems are proved using constructive arguments that can be implemented on a computer and verified by computation. The theorems themselves are motivated in the context of solving models of physical situations that beg for computational solution. I believe that students who implement these proofs and solve the practical problems in this book will develop a "hands-on" understanding of analysis that will serve them well in the future.

Motivation

I have three overt reasons for writing this book, and one covert reason.

First whenever I teach numerical analysis, I am annoyed by the amount of time I spend on topics from basic calculus. From the point of view of scientists and engineers, modern calculus is very unsatisfactory. Students spend much of their time practicing skills that are rarely used and are never taught some fundamental ideas that come up repeatedly. One consequence is that students studying science and engineering spend a large portion of their time in upper-level mathematics courses on elementary topics at the expense of sophisticated material for which they truly need a mathematician's help.

Second, teaching a modern calculus course is a frustrating experience for many analysts. Calculus should be a course in real analysis because that is what it is. But the current trend in teaching calculus is to avoid anything to do with analysis and instead concentrate on solving practically unimportant "exact answer" problems. The conventional wisdom is that analysis is too hard (or put cynically, students are too dumb to learn real mathematics). But having met many bright students over the years, I have found this rationale increasingly questionable. Rather, this trend might originate in the observation that teaching rigorous mathematics to young students requires significant effort and ingenuity from the instructor.

Third, teaching introductory real analysis using a modern abstract approach, even from a beautiful book like Rudin's, is far from optimal. As noted, I have serious doubts as to the effectivness of an abstract approach to teaching analysis. Moreover, this approach has some serious consequences. First of all, it perpetuates the faulty notion that there is some difference between "pure" analysis and the "dirty" topics important to numerical analysis and applied mathematics. This seeds the prejudices of pure and applied mathematicians that are so unfortunate for mathematics. Moreover, it makes the typical introductory real analysis course unattractive to the brightest students in science and engineering, who could benefit from taking such a course.

This book attempts to place the basic ideas of real analysis and numerical analysis together in an applied setting that is both accessible and motivational to young students of all technical persuasions.

This goal reflects my covert reason for writing this book. Namely, this book is a personal statement about how I believe people learn mathematics and how mathematics should therefore be taught.

Usage

This book begins by considering the solution of algebraic models with numeric roots. The discussion leads naturally from the integers through rational numbers and induction to the construction of the real numbers. Interwoven is a thorough discussion of functions, and the high point of this part of the book is the theory of the fixed point iteration for solving nonlinear equations. The next part of the book is concerned with models that involve derivatives and whose solutions are functions. Modeling and the analysis of functions motivates the introduction of the derivative, while the solution of the simplest differential models motivates the introduction of the integral. We investigate the properties of these operations thoroughly; and then as a practical application, we derive and analyze the basic transcendental functions as solutions of some classic differential equations. This part concludes with a discussion of Newton's method for solving root problems. With the basic material about numbers and functions in hand, the book turns to more detailed analysis of functions, including investigations of continuity, sequences of functions, and approximation theory. The book concludes by discussing the solution of nonlinear differential equations by means of the essentially important Contraction Mapping Principle and Arzela's theorem about equicontinuous functions.

While these are classic topics, the material in this book is not arranged in the usual order found in most real analysis texts. There are two reasons. One of the few tenets of teaching I have managed to hold after twenty years is to introduce only one new concept at a time and only introduce a concept when it is needed. Consequently, material in this book is introduced in an order motivated by the practical problem of solving models rather than by the formal style of building the subject from the ground up. Three important examples are the introduction and use of Lipschitz continuity well before other notions of continuity, the introduction of differentiation via the linearization of a function, and the introduction of integration as an approximation method for solving a differential equation rather than as a way of computing the area under a curve. Each of these choices yields distinct pedagogical benefits in terms of motivating ideas and teaching students how to *do* analysis.

The order of the material in this book is also dictated by the goal of presenting constructive arguments. For example, assuming Lipschitz continuity makes it much easier to give constructive proofs for several fundamental results like the Mean Value Theorem. Hence, the most general notion of continuity and general versions of some fundamental results are not presented until the final third of the book, where the discussion becomes more abstract and sophisticated as well as less constructive.

This book is aimed at two kinds of courses. First, there is the honors calculus sequence typically taken by freshman planning on a technical major. These students often have advanced placement credit in calculus. Second, there is the introductory course in real analysis offered to mathematics majors that have completed calculus. This book has been used successfully for both kinds of courses at Georgia Tech and Colorado State University. Much of this material has also been successfully tested at Chalmers University of Technology in Sweden.

To use this book for such courses, it is necessary to be selective on the material covered. For a freshman honors calculus course, I lecture on material in Chapters 1–4, 5–7 (briefly), 8–15, and finally calculus material proper in Chapters 16–30 and 35. I conclude by covering selected material in Chapters 31 and 36–38. A calculus course that follows this syllabus certainly omits several topics covered in a standard course, like a detailed discussion of integration techniques and various standard "applications." I have not found that my students suffer from this. For an advanced calculus/introductory real analysis course, I lecture on material in Chapters 3, 4, 8–15, 16, 18–23, 25–27, 28 and 29 very briefly, 32–35. I then lecture on a selection of material from Chapters 36–41.

The material is supplemented by exercises that range from simple computations to estimates to computational projects. When I teach this material, I assign a mixture of course work, including in-class exams testing basic understanding, take-home problem sets covering the more difficult analytic problems, and "laboratory" projects performed using a computer and requiring a written report.

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