4
Realizing and Omitting Types

4.1 Types

Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure and $A \subseteq M$. Let $\mathcal{L}_A$ be the language obtained by adding to $\mathcal{L}$ constant symbols for each $a \in A$. We can naturally view $\mathcal{M}$ as an $\mathcal{L}_A$-structure by interpreting the new symbols in the obvious way. Let $\mathrm{Th}_A(\mathcal{M})$ be the set of all $\mathcal{L}_A$-sentences true in $\mathcal{M}$. Note that $\mathrm{Th}_A(\mathcal{M}) \subseteq \mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$.

Definition 4.1.1 Let p be the set of $\mathcal{L}_A$-formulas in free variables $v_1, \ldots, v_n$. We call p an *n-type* if $p \cup \mathrm{Th}_A(\mathcal{M})$ is satisfiable. We say that p is a *complete n-type* if $\phi \in p$ or $\neg\phi \in p$ for all $\mathcal{L}_A$-formulas ϕ with free variables from $v_1, \ldots, v_n$. We let $S_n^{\mathcal{M}}(A)$ be the set of all complete n-types.

We sometimes refer to incomplete types as *partial types*. Also, we often write $p(v_1, \ldots, v_n)$ to stress that p is an n-type.

By the Compactness Theorem, we could replace “satisfiable” by “finitely satisfiable” in Definition 4.1.1.

Consider the example $\mathcal{M} = (\mathbb{Q}, <)$ where A is the set of natural numbers. Let $p(v)$ be the set of formulas $\{v > 1, v > 2, v > 3, \ldots\}$. If Δ is a finite subset of $p(v) \cup \mathrm{Th}_A(\mathcal{M})$, then we see that Δ is satisfiable by interpreting v as a sufficiently large element of $\mathbb{Q}$. By the Compactness Theorem, $p(v) \cup \mathrm{Th}_A(\mathcal{M})$ is satisfiable and $p(v)$ is a 1-type.

For the same structure, let $q(v) = \{\phi(v) \in \mathcal{L}_A : \mathcal{M} \models \phi(\frac{1}{2})\}$. For example the formula $v < 3$ is in $q(v)$, whereas $v > 2$ is not. For any $\mathcal{L}_A$-formula $\psi(v)$, either $\mathcal{M} \models \psi(\frac{1}{2})$ or $\mathcal{M} \models \neg\psi(\frac{1}{2})$. Thus, $q(v)$ is a complete 1-type.

The latter example can be generalized to produce complete types in arbitrary structures. If $\mathcal{M}$ is any $\mathcal{L}$-structure, $A \subset M$, and $\overline{a} = (a_1, \ldots, a_n) \in M^n$, let $\mathrm{tp}^{\mathcal{M}}(\overline{a}/A) = \{\phi(v_1, \ldots, v_n) \in \mathcal{L}_A : \mathcal{M} \models \phi(a_1, \ldots, a_n)\}$. Then, $\mathrm{tp}^{\mathcal{M}}(\overline{a}/A)$ is a complete n-type. We write $\mathrm{tp}^{\mathcal{M}}(\overline{a})$ for $\mathrm{tp}^{\mathcal{M}}(\overline{a}/\emptyset)$.

Definition 4.1.2 If p is an n-type over A, we say that $\overline{a} \in M^n$ *realizes* p if $\mathcal{M} \models \phi(\overline{a})$ for all $\phi \in p$. If p is not realized in $\mathcal{M}$ we say that $\mathcal{M}$ *omits* p.

In the examples given above, $p(v)$ is not realized in $\mathcal{M} = (\mathbb{Q}, <)$, whereas clearly $1/2$ realizes $q(v)$. In fact, there are many realizations of $q(v)$ in $\mathcal{M}$. Suppose that r is any rational number with $0 < r < 1$. We can construct an automorphism σ of $\mathcal{M}$ that fixes every natural number but $\sigma(1/2) = r$. Because σ fixes all elements of A, σ is also an $\mathcal{L}_A$-automorphism. By Theorem 1.1.10,

$$\mathcal{M} \models \phi(1/2) \Leftrightarrow \mathcal{M} \models \phi(r).$$

Thus, r also realizes $q(v)$.

In fact, the elements of $\mathbb{Q}$ that realize $q(v)$ are exactly the rational numbers s such that $0 < s < 1$. If $s \leq 0$, then the formula $0 < v$ is in $q(v)$ but $\mathcal{M} \models \neg(0 < s)$. Thus, s does not realize $q(v)$. Similarly, no $s \geq 1$ realizes $q(v)$.

The Compactness Theorem tells us that every type can be realized in an elementary extension.

Proposition 4.1.3 *Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $A \subseteq M$, and p an n-type over A. There is $\mathcal{N}$ an elementary extension of $\mathcal{M}$ such that p is realized in $\mathcal{N}$.*

Proof Let $\Gamma = p \cup \mathrm{Diag_{el}}(\mathcal{M})$. We claim that Γ is satisfiable.

Suppose that Δ is a finite subset of Γ. Without loss of generality, Δ is the single formula

$$\phi(v_1, \ldots, v_n, a_1, \ldots, a_m) \wedge \psi(a_1, \ldots, a_m, b_1, \ldots, b_l),$$

where $a_1, \ldots, a_m \in A$, $b_1, \ldots, b_l \in M \setminus A$, $\phi(\overline{v}, \overline{a}) \in p$, and $\mathcal{M} \models \psi(\overline{a}, \overline{b})$. Let $\mathcal{N}_0$ be a model of the satisfiable set of sentences $p \cup \mathrm{Th}_A(\mathcal{M})$. Because $\exists \overline{w}\ \psi(\overline{a}, \overline{w}) \in \mathrm{Th}_A(\mathcal{M})$,

$$\mathcal{N}_0 \models \phi(\overline{v}, \overline{a}) \wedge \exists \overline{w}\ \psi(\overline{a}, \overline{w}).$$

By interpreting $b_1, \ldots, b_l$ as witnesses to $\exists \overline{w}\ \psi(a_1, \ldots, a_m, \overline{w})$, we make $\mathcal{N}_0 \models \Delta$. Thus, Δ is satisfiable.

By the Compactness Theorem, Γ is satisfiable. Let $\mathcal{N} \models \Gamma$. Because $\mathcal{N} \models \mathrm{Diag_{el}}(\mathcal{M})$, the map that sends $m \in M$ to the interpretation of the constant symbol m in $\mathcal{N}$ is an elementary embedding. Let $c_i \in N$ be the interpretations of v_i. Then, $(c_1, \ldots, c_n)$ is a realization of p.

It is worth noting that if $\mathcal{N}$ is an elementary extension of $\mathcal{M}$, then $\mathrm{Th}_A(\mathcal{M}) = \mathrm{Th}_A(\mathcal{N})$. Thus $S_n^{\mathcal{M}}(A) = S_n^{\mathcal{N}}(A)$. This observation and Proposition 4.1.3 yield a characterization of complete types.

Corollary 4.1.4 *$p \in S_n^{\mathcal{M}}(A)$ if and only if there is an elementary extension $\mathcal{N}$ of $\mathcal{M}$ and $\overline{a} \in N^n$ such that $p = \mathrm{tp}^{\mathcal{N}}(\overline{a}/A)$.*

Proof If $\overline{a} \in N^n$, then $\mathrm{tp}^{\mathcal{N}}(\overline{a}/A) \in S_n^{\mathcal{N}}(A) = S_n^{\mathcal{M}}(A)$. On the other hand if $p \in S_n^{\mathcal{M}}(A)$, then, by Proposition 4.1.3, there is an elementary extension $\mathcal{N}$ of $\mathcal{M}$ and $\overline{a} \in \mathcal{M}$ realizing p. Because p is complete, if $\phi(\overline{v}) \in \mathcal{L}_A$, then exactly one of $\phi(\overline{v})$ and $\neg\phi(\overline{v})$ is in p. Thus, $\phi(\overline{v}) \in \mathrm{tp}^{\mathcal{N}}(\overline{a}/A)$ if and only if $\phi(\overline{v}) \in p$ and $p = \mathrm{tp}^{\mathcal{N}}(\overline{a}/A)$.

Complete types tell us what possible first-order properties elements can have in an elementary extension. What does it mean if two elements of a structure $\mathcal{M}$ realize the same complete type over A? Let us return to the example where $\mathcal{M} = (\mathbb{Q}, <)$ and A is the natural numbers. We showed that $a, b \in \mathbb{Q}$ realize the same complete 1-type over A if and only if there is an automorphism σ of $\mathcal{M}$ fixing A such that $\alpha(a) = b$. Although this is not true in general (see, for example, Exercises 4.5.1 and 4.5.9), it is if we allow passage to an elementary extension.

Proposition 4.1.5 *Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure and $A \subseteq M$. Let $\overline{a}, \overline{b} \in M^n$ such that $\mathrm{tp}^{\mathcal{M}}(\overline{a}/A) = \mathrm{tp}^{\mathcal{M}}(\overline{b}/A)$. Then, there is $\mathcal{N}$ an elementary extension of $\mathcal{M}$ and σ an automorphism of $\mathcal{N}$ fixing all elements of A such that $\sigma(\overline{a}) = \overline{b}$.*

If $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures and $B \subseteq M$, we say that $f : B \to N$ is a *partial elementary map* if and only if

$$\mathcal{M} \models \phi(\overline{b}) \Leftrightarrow \mathcal{N} \models \phi(f(\overline{b}))$$

for all $\mathcal{L}$-formulas ϕ and all finite sequences $\overline{b}$ from B. We will prove Proposition 4.1.5 by carefully iterating the following lemma and its corollary.

Lemma 4.1.6 *Let $\mathcal{M}, \mathcal{N}, B$ be as above and let $f : B \to N$ be partial elementary. If $b \in M$, there is an elementary extension $\mathcal{N}_1$ of $\mathcal{N}$ and $g : B \cup \{b\} \to \mathcal{N}_1$ a partial elementary map extending f.*

Proof Let $\Gamma = \{\phi(v, f(a_1), \ldots, f(a_n)) : \mathcal{M} \models \phi(b, a_1, \ldots, a_n), a_1, \ldots, a_n \in B\} \cup \mathrm{Diag}_{\mathrm{el}}(\mathcal{N})$.

Suppose that we find a structure $\mathcal{N}_1$ and an element $c \in N_1$ satisfying all of the formulas in Γ. Because $\mathcal{N}_1 \models \mathrm{Diag}_{\mathrm{el}}(\mathcal{N})$, $\mathcal{N}_1$ is an elementary extension of $\mathcal{N}$. It is also easy to see that we can extend f to a partial elementary map by $b \mapsto c$.

Thus, it suffices to show that Γ is satisfiable. By the Compactness Theorem it suffices to show that every finite subset of Γ is satisfiable in $\mathcal{N}$. Taking conjunctions, it is enough to show that if $\mathcal{M} \models \phi(b, a_1, \ldots, a_n)$, then $\mathcal{N} \models \exists v\ \phi(v, f(a_1), \ldots, f(a_n))$. But this is clear because $\mathcal{M} \models \exists v\ \phi(v, a_1, \ldots, a_n)$ and f is partial elementary.

Corollary 4.1.7 *If $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures, $B \subseteq M$ and $f : B \to N$ is a partial elementary map, then there is $\mathcal{N}'$ an elementary extension of $\mathcal{N}$ and $g : \mathcal{M} \to \mathcal{N}'$ an elementary embedding.*

Proof Let $\kappa = |M|$, and let $\{a_\alpha : \alpha < \kappa\}$ be an enumeration of M. Let $\mathcal{N}_0 = \mathcal{N}$, $B_0 = B$, and $g_0 = f$. Let $B_\alpha = B \cup \{a_\beta : \beta < \alpha\}$. We inductively build an elementary chain $(N_\alpha : \alpha < \kappa)$ and $g_\alpha : B_\alpha \to N_\alpha$ partial elementary such that $g_\beta \subseteq g_\alpha$ for $\beta < \alpha$.

If $\alpha = \beta+1$, and $g_\beta : B_\beta \to N_\beta$ is partial elementary, then, by Proposition 4.1.3, we can find $N_\beta \prec N_\alpha$ and $g_\alpha : B_\alpha \to N_\alpha$ extending g_β.

If α is a limit ordinal, let $N_\alpha = \bigcup_{\beta<\alpha} N_\beta$ and $g_\alpha = \bigcup_{\beta<\alpha} g_\beta$. By Lemma 2.3.11, $\mathcal{N}_\alpha$ is an elementary extension of N_β for $\beta < \alpha$ and f_α is a partial elementary map.

Let $\mathcal{N}' = \bigcup_{\alpha<\kappa} \mathcal{N}_\alpha$ and $g = \bigcup_{\alpha<\kappa} g_\alpha$. Again by Lemma 2.3.11, $\mathcal{N} \prec \mathcal{N}'$ and g is partial elementary. But $\mathrm{dom}(g) = M$, so g is an elementary embedding of $\mathcal{M}$ into $\mathcal{N}'$.

Proof of 4.1.5 Let $f : A \cup \{a\} \to A \cup \{b\}$ such that $f|A$ is the identity and $f(a) = b$. Because $\mathrm{tp}^{\mathcal{M}}(a/A) = \mathrm{tp}^{\mathcal{M}}(b/A)$, f is a partial elementary map. By Corollary 4.1.7 there is $\mathcal{N}_0$ an elementary extension of $\mathcal{M}$ and $f_0 : \mathcal{M} \to \mathcal{N}_0$ an elementary embedding extending f. We will build a sequence of elementary extensions

$$\mathcal{M} = \mathcal{M}_0 \prec \mathcal{N}_0 \prec \mathcal{M}_1 \prec \mathcal{N}_1 \prec \mathcal{M}_2 \prec \mathcal{N}_2 \ldots$$

and elementary embeddings $f_i : \mathcal{M}_i \to \mathcal{N}_i$ such that $f_0 \subseteq f_1 \subseteq f_2 \ldots$ and N_i is contained in the image of f_{i+1}. Having done this, let

$$\mathcal{N} = \bigcup_{i<\omega} \mathcal{N}_i = \bigcup_{i<\omega} \mathcal{M}_i$$

and $\sigma = \bigcup f_i$. By Lemma 2.3.11, $\mathcal{N}$ is an elementary extension of $\mathcal{M}$ and $\sigma : \mathcal{N} \to \mathcal{N}$ is an elementary map such that $\sigma|A$ is the identity and $\sigma(a) = b$. By construction σ is surjective. Thus, σ is the desired automorphism.

We now describe the construction. Given $f_i : \mathcal{M}_i \to \mathcal{N}_i$, we can view f_i^{-1} as a partial elementary map from the image of f_i into $\mathcal{M}_i \prec \mathcal{N}_i$. By Corollary 4.1.7, we can find $\mathcal{M}_{i+1}$ an elementary extension of $\mathcal{N}_i$ and extend f_i^{-1} to an elementary embedding $g_i : \mathcal{N}_i \to \mathcal{M}_{i+1}$. We can view g_i^{-1} as a partial elementary map from the image of g into $\mathcal{N}_i \prec \mathcal{M}_{i+1}$. Again by Corollary 4.1.7, we can find $\mathcal{N}_{i+1}$ an elementary extension of $\mathcal{M}_{i+1}$ and an elementary embedding $f_{i+1} : \mathcal{M}_{i+1} \to \mathcal{N}_{i+1}$ extending g_i^{-1}. Because $f_{i+1} \supseteq g_i^{-1}$ and $g_i \supseteq f_i^{-1}$, $f_{i+1} \supseteq f_i$. Because N_i is the domain of g_i, N_i is in the range of f_{i+1}.

Stone Spaces

There is a natural topology on the space of complete n-types $S_n^{\mathcal{M}}(A)$. For ϕ an $\mathcal{L}_A$-formula with free variables from $v_1, \ldots, v_n$, let

$$[\phi] = \{p \in S^{\mathcal{M}}(A) : \phi \in p\}.$$

If p is a complete type and $\phi \vee \psi \in p$, then $\phi \in p$ or $\psi \in p$. Thus $[\phi \vee \psi] = [\phi] \cup [\psi]$. Similarly, $[\phi \wedge \psi] = [\phi \cap \psi]$.

The *Stone topology* on $S_n^{\mathcal{M}}(A)$ is the topology generated by taking the sets $[\phi]$ as basic open sets. For complete types p, exactly one of ϕ and $\neg\phi$ is in p. Thus, $[\phi] = S_n^{\mathcal{M}}(A) \setminus [\neg\phi]$ is also closed. We refer to sets that are both closed and open as *clopen*.

The topology of the type spaces will eventually play an important role. The next lemmas summarize some of the basic topological properties.

Lemma 4.1.8 *i) $S_n^{\mathcal{M}}(A)$ is compact.*

ii) $S_n^{\mathcal{M}}(A)$ is totally disconnected, that is if $p, q \in S_n^{\mathcal{M}}(A)$ and $p \neq q$, then there is a clopen set X such that $p \in X$ and $q \notin X$.

Proof

i) It suffices to show that every cover of $S_n^{\mathcal{M}}(A)$ by basic open sets has a finite subcover. Suppose not. Let $C = \{[\phi_i(\overline{v})] : i \in I\}$ be a cover of $S_n^{\mathcal{M}}(A)$ by basic open sets with no finite subcover. Let

$$\Gamma = \{\neg\phi_i(\overline{v}) : i \in I\}.$$

We claim that $\Gamma \cup \mathrm{Th}_A(\mathcal{M})$ is satisfiable. If I_0 is a finite subset of I, then because there is no finite subcover of C, there is a type p such that

$$p \notin \bigcup_{i \in I_0} [\phi_i].$$

Let $\mathcal{N}$ be an elementary extension of $\mathcal{M}$ containing a realization $\overline{a}$ of p. Then

$$\mathcal{N} \models \mathrm{Th}_A(\mathcal{M}) \cup \bigwedge_{i \in I_0} \neg\phi_i(\overline{a}).$$

We have shown that Γ is finitely satisfiable and hence, by the Compactness Theorem, satisfiable.

Let $\mathcal{N}$ be an elementary extension of $\mathcal{M}$, and let $\overline{a} \in \mathcal{N}$ realize Γ. Then

$$\mathrm{tp}^{\mathcal{N}}(\overline{a}/A) \in S_n^{\mathcal{M}}(A) \setminus \bigcup_{i \in I} [\phi_i(\overline{v})],$$

a contradiction.

ii) If $p \neq q$, there is a formula ϕ such that $\phi \in p$ and $\neg\phi \in q$. Thus, $[\phi]$ is a basic clopen set separating p and q.

Natural operations on types often give rise to continuous operations on the type space.

Lemma 4.1.9 *i) If $A \subseteq B \subset M$ and $p \in S_n^{\mathcal{M}}(B)$, let $p|A$ be the set of $\mathcal{L}_A$-formulas in p. Then, $p|A \in S_n^{\mathcal{M}}(A)$ and $p \mapsto p|A$ is a continuous map from $S_n^{\mathcal{M}}(B)$ onto $S_n^{\mathcal{M}}(A)$.*

ii) If $f : \mathcal{M} \to \mathcal{N}$ is an elementary embedding and $p \in S_n^{\mathcal{M}}(A)$, let

$$f(p) = \{\phi(\overline{v}, f(\overline{a})) : \phi(\overline{v}, \overline{a}) \in p\}.$$

Then, $f(p) \in S_n^{\mathcal{N}}(f(A))$ and $p \mapsto f(p)$ is continuous.

iii) If $f : A \to \mathcal{N}$ is partial elementary, then $S_n^{\mathcal{M}}(A)$ is homeomorphic to $S_n^{\mathcal{N}}(f(A))$.

Proof

i) Because $p|A \cup \mathrm{Th}_A(\mathcal{M}) \subseteq p \cup \mathrm{Th}_B(\mathcal{M})$, $p|A \cup \mathrm{Th}_A(\mathcal{M})$ is satisfiable. Because $p|A$ is the set of all $\mathcal{L}_A$-formulas in p, $p|A$ is complete. If ϕ is an $\mathcal{L}_A$-formula, then

$$\{p \in S_n^{\mathcal{M}}(B) : \phi \in p\} = [\phi].$$

Thus, ϕ is continuous.

If $q \in S_n^{\mathcal{M}}(A)$, there is an elementary extension $\mathcal{N}$ of $\mathcal{M}$ and $\overline{a} \in N$ realizing q. Then, $p = \mathrm{tp}^{\mathcal{N}}(\overline{a}/B) \in S_n^{\mathcal{M}}(B)$ and $p|A = q$. Thus, the restriction map is surjective.

ii) Suppose that Δ is a finite subset of $f(p)$. Say

$$\Delta = \{\phi_1(\overline{v}, f(\overline{a})), \ldots, \phi_m(\overline{v}, f(\overline{a}))\}$$

where $\phi_1(\overline{v}, \overline{a}), \ldots, \phi_m(\overline{v}, \overline{a}) \in p$. Because $p \cup \mathrm{Th}_A(\mathcal{M})$ is consistent,

$$\mathcal{M} \models \exists \overline{v} \bigwedge_{i=1}^{m} \phi_i(\overline{v}, \overline{a}).$$

Because f is elementary,

$$\mathcal{N} \models \exists \overline{v} \bigwedge_{i=1}^{m} \phi_i(\overline{v}, f(\overline{a}))$$

and $f(p) \cup \mathrm{Th}_{f(A)}(\mathcal{N})$ is consistent. It is easy to see that $f(p)$ is complete.

Because

$$\{p \in S_n^{\mathcal{M}}(A) : \phi(\overline{v}, f(\overline{a})) \in f(p)\} = [\phi(\overline{v}, \overline{a})],$$

$p \mapsto f(p)$ is continuous.

iii) Exercise 4.5.12.

Definition 4.1.10 We say that $p \in S_n^{\mathcal{M}}(A)$ is *isolated* if $\{p\}$ is an open subset of $S_n^{\mathcal{M}}(A)$.

Isolated points will play an important role in Section 4.2.

Proposition 4.1.11 *Let $p \in S_n^{\mathcal{M}}(A)$. The following are equivalent.*

i) p is isolated.

ii) $\{p\} = [\phi(\overline{v})]$ for some $\mathcal{L}_A$-formula $\phi(\overline{v})$. We say that $\phi(\overline{v})$ isolates p.

iii) There is an $\mathcal{L}_A$-formula $\phi(\overline{v}) \in p$ such that for all $\mathcal{L}_A$-formulas $\psi(\overline{v})$, $\psi(\overline{v}) \in p$ if and only if

$$\mathrm{Th}_A(\mathcal{M}) \models \phi(\overline{v}) \rightarrow \psi(\overline{v}).$$

Proof

i) $\Rightarrow$ ii) If X is open, then

$$X = \bigcup_{i \in I} [\phi_i]$$

for some collection of formulas $(\phi_i : i \in I)$. If $\{p\}$ is open, then $\{p\} = [\phi]$ for some formula ϕ.

ii) $\Rightarrow$ iii) Suppose that $\{p\} = [\phi(\overline{v})]$. Suppose that $\psi(\overline{v}) \in p$. We claim that $\mathrm{Th}_A(\mathcal{M}) \models \phi(\overline{v}) \rightarrow \psi(\overline{v})$. If not, then there is an elementary extension $\mathcal{N}$ of $\mathcal{M}$ and $\overline{a} \in N$ such that $\mathcal{N} \models \phi(\overline{a}) \wedge \neg\psi(\overline{a})$. Let $q = \mathrm{tp}^{\mathcal{N}}(\overline{a}/A) \in S_n^{\mathcal{M}}(A)$. Because $\phi(\overline{v}) \in q$, $q = p$. But $\neg\psi(\overline{v}) \in q$, a contradiction.

If, on the other hand, $\psi(\overline{v}) \notin p$, then $\neg\psi(\overline{v}) \in p$ and, by the argument above, $\mathrm{Th}_A(\mathcal{M}) \models \phi(\overline{v}) \rightarrow \neg\psi(\overline{v})$. Because $\mathrm{Th}_A(\mathcal{M}) \cup \{\phi(\overline{v})\}$ is satisfiable, $\mathrm{Th}_A(\mathcal{M}) \not\models \phi(\overline{v}) \rightarrow \psi(\overline{v})$.

iii) $\Rightarrow$ i) We claim that $[\phi(\overline{v})] = \{p\}$. Clearly, $p \in [\phi(\overline{v})]$. Suppose that $q \in [\phi(\overline{v})]$ and $\psi(\overline{v})$ is an $\mathcal{L}_A$-formula. If $\psi(\overline{v}) \in p$, then $\mathrm{Th}_A(\mathcal{M}) \models \phi(\overline{v}) \rightarrow \psi(\overline{v})$ and $\psi(\overline{v}) \in q$. On the other hand, if $\psi(\overline{v}) \notin p$, then $\neg\psi(\overline{v}) \in p$ and, by the argument above, $\psi(\overline{v}) \notin q$. Thus $p = q$.

Examples

We conclude this section by giving concrete descriptions of $S_n^{\mathcal{M}}(A)$ for several important examples.

Example 4.1.12 *Dense Linear Orders*

Let $\mathcal{L} = \{<\}$. Let $\mathcal{M} = (M, <)$ be a dense linear order without endpoints and let $A \subseteq M$. Let $p \in S_1^{\mathcal{M}}(A)$. If $a \in A$, then, because p is a complete type, exactly one of the formulas $v = a$, $v < a$, or $v > a$ is in p.

<u>case 1</u>: p is realized in A.

In other words, the formula $v = a \in p$ for some $a \in A$. In this case, $p = \{\psi(v) : \mathcal{M} \models \psi(a)\}$ and p is isolated by the formula $v = a$.

<u>case 2</u>: Otherwise.

Let $L_p = \{a \in A : a < v \in p\}$ and $U_p = \{a \in A : v < a \in p\}$. If $a < v, v < b \in p$, then, because $p \cup \mathrm{Th}_A(\mathcal{M})$ is satisfiable, $a < b$. Thus,

$a < b$ for $a \in L_p$ and $b \in U_p$ and L_p and U_p determine a cut in the ordering $(A, <)$.

Also note that if A is the disjoint union of L and U where $a < b$ for $a \in L$ and $b \in U$, then $\mathrm{Th}_A(\mathcal{M}) \cup \{a < v : a \in L\}$ and $\{v < b : b \in U\}$ is satisfiable. Thus, there is a type p with $L_p = L$ and $U_p = U$.

We claim that the cut completely determines p; that is,

$$\{p\} = \bigcap_{a \in L_p} [a < v] \cap \bigcap_{a \in U_p} [v < b].$$

Suppose that $q \neq p$, $L_p = L_q$ and $U_p = U_q$. Because the only atomic formulas are $u = v$ and $u < v$, p and q determine the same cut in A, and they contain the same atomic formulas. Because quantifier-free formulas are Boolean combinations of atomic formulas, p and q contain the same quantifier-free formulas. Because every formula is equivalent to a quantifier-free formula, $p = q$.

Using the identification between types and cuts, we can give a complete description of all types in $S_1^{\mathbb{Q}}(\mathbb{Q})$.

For $a \in \mathbb{Q}$, let p_a be the unique type containing $v = a$.

Let $p_{+\infty}$ be the unique type p with $L_p = \mathbb{Q}$ and $U_p = \emptyset$, and let $p_{-\infty}$ be the unique type p with $L_p = \emptyset$ and $U_p = \mathbb{Q}$. For $r \in \mathbb{R} \setminus \mathbb{Q}$, let p_r be the unique type p with $L_p = \{a \in \mathbb{Q} : a < r\}$ and $U_p = \{b \in \mathbb{Q} : r < b\}$. Finally, for $c \in \mathbb{Q}$, let p_{c^+} be the unique type p with $L_p = \{a \in \mathbb{Q} : a \leq c\}$ and $U_p = \{b \in \mathbb{Q} : c < b\}$, and let p_{c^-} be the unique type p with $L_p = \{a \in \mathbb{Q} : a < c\}$ and $U_p = \{b \in \mathbb{Q} : c \leq b\}$. These are all possible types. Note in particular that $|S_1^{\mathbb{Q}}(\mathbb{Q})| = 2^{\aleph_0}$.

We return to the general case where $\mathcal{M} \models \mathrm{DLO}$ and $A \subseteq M$ is nonempty. Aside from the types realized by elements of A, what types in $S_1^{\mathcal{M}}(A)$ are isolated? Suppose that L_p has a largest element a and U_p has a smallest element b. Then $p \in [a < v < b]$. Moreover, $\mathrm{Th}_A(\mathcal{M}) \models a < v < b \rightarrow c < v < d$ for all $c \in L_p$ and $d \in U_p$. Thus, $a < v < b$ isolates p. Similarly, if $U_p = \emptyset$ and L_p has a greatest element a, then $a < v$ isolates p, and if U_p has a smallest element b and $L_p = \emptyset$, then $v < b$ isolates p.

We claim that these are the only possibilities. For example, suppose that $U_p \neq \emptyset$ and has no least element. Suppose that $\phi(v)$ isolates p. Because U_p and L_p determine p,

$$\mathrm{Th}_A(\mathcal{M}) \cup \{a < v : a \in L_p\} \cup \{v < b : v \in U_p\} \models \phi(v).$$

Thus, we can find $a \in L_p \cup \{-\infty\}$ and $b \in U_p$ such that

$$\mathrm{Th}_A(\mathcal{M}) \models \{a < v < b\} \rightarrow \phi(v).$$

There is $c \in U_p$ such that $c < b$. Because $a < c < b$, $\mathcal{M} \models \phi(c)$. But then the type containing $v = c$ is in $[\phi(v)]$ contradicting the fact that $[\phi(v)]$ isolates p. Other cases are similar. We summarize as follows.

Proposition 4.1.13 *Let $\mathcal{M} \models$DLO and let $A \subseteq M$ be nonempty. Types in $S_1^{\mathcal{M}}(A)$ not realized by elements of A correspond to cuts in the ordering of A. A nonrealized type p is nonisolated if either $U_p \neq \emptyset$ has no least element or $L_p \neq \emptyset$ has no greatest element.*

Example 4.1.14 *Algebraically Closed Fields*

Let $K \models$ ACF, and let $A \subseteq K$. We first argue that, without loss of generality, we may assume that A is a field. Let k be the subfield of K generated by A. If $p \in S_n^K(k)$, then $p|A \in S_n^K(A)$. We claim that the restriction map is a bijection. By Lemma 4.1.9, we know that it is surjective, so we need only show that it is one-to-one. Suppose that $q \in S_n^K(A)$. For $b_1, \ldots, b_l \in k$, there are $a_1, \ldots, a_m \in A$ such that for each i there is $q_i(\overline{X}) \in \mathbb{Z}(X_1, \ldots, X_m)$ such that $b_i = q_i(\overline{a})$. Thus, for any $f(X_1, \ldots, X_l) \in \mathbb{Z}[X_1, \ldots, X_l, \overline{Y}]$ there is $g \in \mathbb{Z}[X_1, \ldots, X_m, \overline{Y}]$ such that $f(b_1, \ldots, b_l, \overline{y}) = 0$ if and only if $g(a_1, \ldots, a_m, \overline{y}) = 0$ for any $\overline{y}$. Thus, by quantifier elimination, for any formula $\phi(\overline{v}, \overline{b})$ with $\overline{b} \in k$, there is a formula $\psi(\overline{v}, \overline{a})$ with $\overline{a} \in A$ such that

$$K \models \phi(\overline{v}, \overline{b}) \leftrightarrow \psi(\overline{v}, \overline{a}).$$

Thus, if $p, q \in S_l^K(k)$ and $p \neq q$, then $p|A \neq q|A$.

Let k be a subfield of K. We will show that n-types over k are determined by prime ideals in $k[X_1, \ldots, X_n]$. For $p \in S_n^K(k)$, let

$$I_p = \{f(\overline{X}) \in k[X_1, \ldots, X_n] : f(\overline{v}) = 0 \in p\}.$$

If $f, g \in I_p$, then $f + g \in I_p$, and if $f \in I_p$ and $g \in k[\overline{X}]$, then $fg \in I_p$. Thus, I_p is an ideal. If $f, g \in k[\overline{X}]$, then

$$K \models \forall \overline{v}\ f(\overline{v})g(\overline{v}) = 0 \rightarrow (f(\overline{v}) = 0 \vee g(\overline{v}) = 0).$$

Thus, if $fg \in I_p$, then either $f \in I_p$ or $g \in I_p$. Hence, I_p is a prime ideal.

On the other hand, suppose that $P \subset k[\overline{X}]$ is a prime ideal. There is a prime ideal $Q \subset K[\overline{X}]$ such that $Q \cap k[\overline{X}] = P$.[1] Let F be the algebraic closure of the fraction field of $K[\overline{X}]/Q$. By model-completeness, F is an elementary extension of K. Let $x_i = X_i/Q$ for $i = 1, \ldots, n$. For $f \in K[\overline{X}]$, $f(\overline{x}) = 0$ if and only if $f \in Q$. Thus, if $p = \mathrm{tp}^F(\overline{x}/k)$, then $I_p = P$. Thus, $p \mapsto I_p$ is a surjective map from $S_n^K(k)$ onto the prime ideals of $k[\overline{X}]$. We claim that $p \mapsto I_p$ is one-to-one. Suppose that $p, q \in S_n^K(k)$ and $p \neq q$.

[1] This follows, for example, from [68] 7.5, because $K[\overline{X}]$ is a faithfully flat $k[\overline{X}]$-algebra, but we sketch a more elementary proof. If $K[\overline{X}]P$ is the $K[\overline{X}]$ ideal generated by P, we first claim that $K[\overline{X}]P \cap k[\overline{X}] = P$. Let B be a basis for K as a k-vector space with $1 \in B$. B is also a basis for $K[\overline{X}]$ as a free $k[\overline{X}]$-module. If $f \in K[\overline{X}]$, then $f = \sum_{b \in B} f_b b$, where each $f_b \in k[\overline{X}]$ and all but finitely many $f_b = 0$. If $f \in K[\overline{X}]P$, then each $f_b \in P$. If $f \in K[\overline{X}]P \cap k[\overline{X}]$, then $f = f_1 \in P$. Let S be the multiplicatively closed set $k[\overline{X}] \setminus P$. Let $Q \subset K[\overline{X}]$ be maximal among the ideals containing P and avoiding S. Then, Q is a prime ideal and $Q \cap k[\overline{X}] = P$.

There is a formula $\phi \in p$ such that $\neg\phi \in q$. By quantifier elimination, we may assume that ϕ is

$$\bigvee_{i=1}^{m} \left[\bigwedge_{j=1}^{k} f_{i,j}(\overline{v}) = 0 \wedge \bigwedge_{l=1}^{s} g_{i,l}(\overline{v}) \neq 0 \right],$$

where $f_{i,j}, g_{i,l} \in k[\overline{X}]$. If $I_p = I_q$, then

$$f_{i,j}(\overline{v}) = 0 \in p \Leftrightarrow f_{i,j}(\overline{v}) = 0 \in q$$

and

$$g_{i,l}(\overline{v}) = 0 \in p \Leftrightarrow g_{i,l}(\overline{v}) = 0 \in q.$$

Thus, $\phi \in p$ if and only if $\phi \in q$.

Definition 4.1.15 For A a ring, the *Zariski spectrum* of A is the set of all prime ideals of A. We denote the Zariski Spectrum by $\mathrm{Spec}(A)$ and topologize $\mathrm{Spec}(A)$ by taking basic closed sets $\{P \in \mathrm{Spec}(A) : a_1, \ldots, a_m \in P\}$ for $a_1, \ldots, a_m \in A$. This is called the *Zariski topology* on $\mathrm{Spec}(A)$.

Proposition 4.1.16 *The map $p \mapsto I_p$ is a continuous bijection from $S_n^K(k)$ to $\mathrm{Spec}(k[X_1, \ldots, X_n])$.*

Proof We have shown that the map is one-to-one so we need only show that it is continuous. Suppose that $f_1, \ldots, f_m \in k[X_1, \ldots, X_n]$. Then, the inverse image of $\{P \in \mathrm{Spec}(k[\overline{X}]) : f_1, \ldots, f_m \in P\}$ is $\{p \in S_n^K(k) : f_1(\overline{v}) = 0 \wedge \ldots \wedge f_m(\overline{v}) = 0 \in p\}$, a clopen set. Thus, $p \mapsto I_p$ is continuous.

Although $p \mapsto I_p$ is continuous, it is not a homeomorphism. In particular, for $f \in k[\overline{X}] \setminus k$, $\{p \in S_n^K(k) : f(\overline{v}) = 0\}$ is clopen in $S_n^K(k)$, whereas the image in $\mathrm{Spec}(A)$ is closed but not open. Although the Stone topology is finer than the Zariski topology, we can use it when studying the Zariski topology.

Corollary 4.1.17 *The Zariski topology on $\mathrm{Spec}(k[\overline{X}])$ is compact.*

Proof This is clear because $S_n^K(k)$ is compact and $p \mapsto I_p$ is continuous.

Proposition 4.1.16 also allows us to count types.

Corollary 4.1.18 *Suppose that $K \models \mathrm{ACF}$ and k is a subfield of K. Then $|S_n^K(k)| = |k| + \aleph_0$.*

Proof By Hilbert's Basis Theorem, all ideals in $k[\overline{X}]$ are finitely generated. Thus, there are only $|k| + \aleph_0$ prime ideals.

4.2 Omitting Types and Prime Models

The Compactness Theorem allows us to build models realizing types. It is often also useful to build models that omit certain types. Let $\mathcal{L}$ be a language and T an $\mathcal{L}$-theory. For p an n-type consistent with T, we would like to know whether there is $\mathcal{M} \models T$ omitting p. It is not hard to give a necessary topological condition.

For T an $\mathcal{L}$-theory, we let $S_n(T)$ be the set of all complete n-types p such that $p \cup T$ is satisfiable. If T is complete and $\mathcal{M} \models T$, then $S_n(T) = S_n^{\mathcal{M}}(\emptyset)$. In particular, $S_n(T)$ is a totally disconnected compact topological space with basic open sets

$$[\phi] = \{p : \phi \in p\}.$$

For p a complete type, p is isolated in $S_n(T)$ if and only if $\{p\} = [\phi]$ for some ϕ. We can extend this notion to possibly incomplete types.

Definition 4.2.1 Let $\phi(v_1, \ldots, v_n)$ be an $\mathcal{L}$-formula such that $T \cup \{\phi(\overline{v})\}$ is satisfiable, and let p be an n-type. We say that ϕ *isolates* p if

$$T \models \forall \overline{v}(\phi(\overline{v}) \rightarrow \psi(\overline{v}))$$

for all $\psi \in p$.

Note that if p is a complete type and $\phi(\overline{v})$ isolates p, then

$$T \models \phi(\overline{v}) \rightarrow \psi(\overline{v}) \quad \Leftrightarrow \quad \psi(\overline{v}) \in p$$

for all formulas $\psi(\overline{v})$. In particular, for all formulas $\psi(\overline{v})$ exactly one of $T + \phi(\overline{v}) \wedge \psi(\overline{v})$ and $T + \phi(\overline{v}) \wedge \neg\psi(\overline{v})$ is satisfiable.

We can only omit an isolated type if we do not witness the isolating formula.

Proposition 4.2.2 *If $\phi(\overline{v})$ isolates p, then p is realized in any model of $T \cup \{\exists \overline{v}\ \phi(\overline{v})\}$. In particular, if T is complete, then every isolated type is realized.*

Proof If $\mathcal{M} \models T$ and $\mathcal{M} \models \phi(\overline{a})$, then $\overline{a}$ realizes p. If T is complete and $T \cup \{\phi(\overline{v})\}$ is satisfiable, then $T \models \exists \overline{v}\ \phi(\overline{v})$.

For countable languages, this is also a sufficient condition.

Theorem 4.2.3 (Omitting Types Theorem) *Let $\mathcal{L}$ be a countable language, T an $\mathcal{L}$-theory, and p a (possibly incomplete) nonisolated n-type over $\emptyset$. Then, there is a countable $\mathcal{M} \models T$ omitting p.*

Proof We will prove this by a modification of the Henkin construction used to prove the Compactness Theorem. Let $C = \{c_0, c_1, \ldots\}$ be countably many new constant symbols, and let $\mathcal{L}^* = \mathcal{L} \cup C$. As in the proof of the Compactness Theorem, we will build $T^* \supseteq T$, a complete $\mathcal{L}^*$-theory with the witness property, and build $\mathcal{M} \models T^*$ as in Lemma 2.1.7. We will arrange the construction such that, for all $d_1, \ldots, d_n \in C$, there is a formula $\phi(\overline{v}) \in p$ such that $T^* \models \neg\phi(d_1, \ldots, d_n)$. This will ensure that $d_1^{\mathcal{M}}, \ldots, d_n^{\mathcal{M}}$ does not realize p. Because every element of M is the interpretation of a constant symbol in C, $\mathcal{M}$ omits p.

We will construct a sequence $\theta_0, \theta_1, \theta_2, \ldots$ of $\mathcal{L}^*$-sentences such that

$$\models \theta_t \rightarrow \theta_s$$

for $t > s$ and $T^* = T \cup \{\theta_i : i = 0, 1, \ldots\}$ is a satisfiable extension of T.

Let $\phi_0, \phi_1, \phi_2, \ldots$ list all $\mathcal{L}^*$-sentences. To ensure that T^* is complete, we will either have

$$\models \theta_{3i+1} \rightarrow \phi_i$$

or

$$\models \theta_{3i+1} \rightarrow \neg\phi_i.$$

If ϕ_i is $\exists v\ \psi(v)$ and $\models \theta_{3i+1} \rightarrow \phi_i$, then

$$\models \theta_{3i+2} \rightarrow \psi(c)$$

for some $c \in C$. This will ensure that T^* has the witness property. Let $\overline{d}_0, \overline{d}_1, \ldots$ list all n-tuples from C. We will choose θ_{3i+3} to ensure that $\overline{d}_i^{\mathcal{M}}$ does not realize p in the canonical model of T^*.

stage 0: Let θ_0 be $\forall x\ x = x$.

Suppose that we have constructed θ_s such that $T \cup \theta_s$ is satisfiable. There are three cases to consider.

stage $s + 1 = 3i + 1$: (completeness) If $T \cup \{\theta_s, \phi_i\}$ is satisfiable then θ_{s+1} is $\theta_s \wedge \phi_i$; otherwise, θ_{s+1} is $\theta_s \wedge \neg\phi_i$. In either case $T \cup \theta_{s+1}$ is satisfiable.

stage $s + 1 = 3i + 2$: (witness property) Suppose that ϕ_i is $\exists v\ \psi(v)$ for some formula ψ and $T \models \theta_s \rightarrow \phi_i$. In this case, we want to find a witness for ψ. Let $c \in C$ be a constant that does not occur in $T \cup \{\theta_s\}$. Because only finitely many constants from C have been used so far, we can always find such a c. Let $\theta_{s+1} = \theta_s \wedge \psi(c)$. If $\mathcal{N} \models T \cup \{\theta_s\}$, then there is $a \in N$ such that $\mathcal{N} \models \psi(a)$. By letting $c^{\mathcal{N}} = a$, we have $\mathcal{N} \models \theta_{s+1}$. Thus, in this case $T \cup \{\theta_{s+1}\}$ is satisfiable.

If ϕ_i is not of the correct form or $T \not\models \theta_s \rightarrow \phi_i$, then let θ_{s+1} be θ_s.

stage $s + 1 = 3i + 3$: (omitting p) Let $\overline{d}_i = (e_1, \ldots, e_n)$. Let $\psi(v_1, \ldots, v_n)$ be the $\mathcal{L}$-formula obtained from θ_s by replacing each occurrence of e_i by v_i and then replacing every other constant symbol $c \in C \setminus \{e_0, \ldots, e_n\}$ occurring in θ_s by the variable v_c and putting a $\exists v_c$ quantifier in front. In particular, we get rid of all of the constants in θ_s from C either by replacing them by variables or by quantifying over them. For example, if θ_s is

$$\forall x \exists y\ cx + e_1 e_2 = y^2 + de_2,$$

where c, d, e_1, e_2 are distinct constants in C, then $\psi(v_1, v_2)$ would be the formula

$$\exists v_c \exists v_d \forall x \exists y \; v_c x + v_1 v_2 = y^2 + v_d v_2.$$

Because p is nonisolated, there is a formula $\phi(\overline{v}) \in p$ such that

$$T \not\models \forall \overline{v} \; (\psi(\overline{v}) \to \phi(\overline{v})). \qquad (*)$$

Let θ_{s+1} be $\theta_s \wedge \neg\phi(\overline{d}_i)$. We must argue that $T \cup \theta_{s+1}$ is satisfiable. By $(*)$ there is $\mathcal{N} \models T$ with $\overline{a} \in N$ such that

$$\mathcal{N} \models \psi(\overline{a}) \wedge \neg\phi(\overline{a}).$$

We can make $\mathcal{N}$ into a model of θ_{s+1} by interpreting the constants $c \in C \setminus \{e_1, \ldots, e_n\}$ as the witnesses to v_c and e_i as a_i.

This completes the construction. Let $T^* = T \cup \{\theta_0, \theta_1, \ldots\}$. Because $T \cup \{\theta_s\}$ is satisfiable for each s, T^* is satisfiable. If ϕ is any $\mathcal{L}$-sentence, then $\phi = \phi_i$ for some i, and at stage $3i+1$ we ensure that $T^* \models \phi$ or $T^* \models \neg\phi$. Thus, T^* is complete.

If $\psi(v)$ is an $\mathcal{L}$-formula and $T^* \models \exists v \; \psi(v)$, then there is an i such that ϕ_i is $\exists v \; \psi(v)$ and at stage $3i+2$ we ensure that $T^* \models \psi(c)$ for some $c \in C$. Thus, T^* has the witness property.

If $\mathcal{M}$ is the canonical model of T^* constructed as in Lemma 2.1.7, we claim that $\mathcal{M}$ omits p. Suppose that $\overline{a} \in M^n$. Because every element of M is the interpretation of a constant symbol, there is $\overline{d}_i$ such that $\overline{d}_i^{\mathcal{M}} = \overline{a}$. At stage $3i+3$, we ensure that $\mathcal{M} \models \neg\phi_i(\overline{d})$ for some $\phi_i \in p$. Thus $\overline{a}$ does not realize p.

The proof of the Omitting Types Theorem can be generalized to omit countably many types at once.

Theorem 4.2.4 *Let $\mathcal{L}$ be a countable language, and let T be an $\mathcal{L}$-theory. Let X be a countable collection of nonisolated types over $\emptyset$. There is a countable $\mathcal{M} \models T$ that omits all of the types $p \in X$.*

Proof (Sketch) Let $p_0, p_1, \ldots$ list X. Let C be as in the proof of Theorem 4.2.3, and let $\overline{d}_0, \overline{d}_1 \ldots$ list all finite sequences from C. Fix $\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, a bijection.

We do a Henkin-style argument as in the proof of Theorem 4.2.3. If $s = 0$, $3i+1$, or $3i+2$, we proceed exactly as above. If $i = \pi(m, n)$, then at stage $s = 3i+3$ we proceed as above to ensure that $\overline{d}_m$ does not realize p_n.

If $\mathcal{M}$ is the canonical model, we eventually ensure that no finite sequence from M realizes any of the types p_i.

The assumption of countability of $\mathcal{L}$ is necessary in the Omitting Types Theorem. Suppose that $\mathcal{L}$ is the language with two disjoint sets of constant symbols C and D, where C is uncountable and $|D| = \aleph_0$. Let T be the theory $\{a \neq b : a, b \in C, a \neq b\}$ and p be the type $\{v \neq d : d \in D\}$. Because every model of T is uncountable, there is always an element that is not the interpretation of a constant in D. Thus, every model of T realizes p. On the other hand, if $\phi(v)$ is any $\mathcal{L}$-formula, then, because only countably many constants from D occur in $T \cup \{\phi(v)\}$, there is $d \in D$ such that $T \cup \{\phi(d)\}$ is satisfiable. Thus, p is nonisolated.

The necessity of X being countable in Theorem 4.2.4 is more problematic. For example, if $\aleph_0 < \lambda < 2^{\aleph_0}$, we could ask whether for a countable T we can omit a family of λ nonisolated types. This turns out to depend on set theoretic assumptions (see Exercise 4.5.14).

We give one concrete application of the Omitting Types Theorem. Let $\mathcal{L} = \{+, \cdot, <, 0, 1\}$, and let PA be the axioms for Peano arithmetic. Suppose that $\mathcal{M}, \mathcal{N} \models \text{PA}$. We say that $\mathcal{N}$ is an *end extension* of $\mathcal{M}$ if $N \supset M$ and $a < b$ for all $a \in M$ and $b \in N \setminus M$.

Theorem 4.2.5 *If $\mathcal{M}$ is a countable model of PA, then there is $\mathcal{M} \prec \mathcal{N}$ such that $\mathcal{N}$ is a proper end extension of $\mathcal{M}$.*

Proof Consider the language $\mathcal{L}^*$ where we have constant symbols for all elements of M and a new constant symbol c. Let $T = \text{Diag}_{\text{el}}(\mathcal{M}) \cup \{c > m : m \in M\}$, and for $a \in M \setminus \mathbb{N}$ let p_a be the type $\{v < a, v \neq m : m \in M\}$. Any $\mathcal{N} \models T$ is a proper elementary extension of $\mathcal{M}$. If $\mathcal{N}$ omits each p_a, then $\mathcal{N}$ is an end extension of $\mathcal{M}$. By Theorem 4.2.4, it suffices to show that each p_a is nonisolated.

Suppose that $\phi(v)$ is an $\mathcal{L}^*$ formula isolating p_a. Let $\phi(v) = \theta(v, c)$, where θ is an $\mathcal{L}_M$-formula. Then

$$T \cup \theta(v, c) \models v < a.$$

Because $T \cup \{\theta(v, c)\}$ is satisfiable,

$$\mathcal{M} \models \forall x \exists y > x \exists v < a\ \theta(v, y).$$

The Pigeonhole Principle is provable in Peano arithmetic. Thus

$$\mathcal{M} \models [\forall x \exists y > x \exists v < a\ \theta(v, y)] \rightarrow \exists v < a \forall x \exists y > x\ \theta(v, y). \qquad (**)$$

Thus, there is $m < a$ such that

$$\mathcal{M} \models \forall x \exists y > x\ \theta(m, y).$$

We claim that $T \cup \{\theta(m, c)\}$ is satisfiable. If not, there is $n \in M$ such that

$$\text{Diag}_{\text{el}}(\mathcal{M}) + c > n \models \neg\theta(m, c)$$

contradicting $(**)$. Thus, $\phi(v)$ does not isolate p_a, a contradiction.

Prime and Atomic Models

We use the Omitting Types Theorem to study small models of a complete theory. For the remainder of this section, we will assume that $\mathcal{L}$ is a countable language and T is a complete $\mathcal{L}$-theory with infinite models.

Definition 4.2.6 We say that $\mathcal{M} \models T$ is a *prime model* of T if whenever $\mathcal{N} \models T$ there is an elementary embedding of $\mathcal{M}$ into $\mathcal{N}$.

For example, let $T = \mathrm{ACF}_0$. If $K \models \mathrm{ACF}_0$, and F is the algebraic closure of $\mathbb{Q}$, then there is an embedding of F into K. Because ACF_0 is model complete this embedding is elementary. Thus, F is a prime model of ACF_0. Similarly, RCF has a prime model, the real closure of $\mathbb{Q}$.

For a third example, consider $\mathcal{L} = \{+, \cdot, <, 0, 1\}$ and let T be $\mathrm{Th}(\mathbb{N})$, true arithmetic. If $\mathcal{M} \models T$, then we can view $\mathbb{N}$ as an initial segment of $\mathcal{M}$. We claim that this embedding is elementary. We use the Tarski–Vaught test (Proposition 2.3.5). Let $\phi(v, w_1, \ldots, w_m)$ be an $\mathcal{L}$-formula and let $n_1, \ldots, n_m \in \mathbb{N}$ such that $\mathcal{M} \models \exists v\ \phi(v, \overline{n})$. Let ψ be the $\mathcal{L}$-sentence

$$\exists v\ \phi(v, \underbrace{1 + \ldots + 1}_{n_1-\text{times}}, \ldots, \underbrace{1 + \ldots + 1}_{n_m-\text{times}}).$$

Then, $\mathcal{M} \models \psi$ and $\mathbb{N} \models \psi$ because $\mathcal{M} \equiv \mathbb{N}$. But then, for some $s \in \mathbb{N}$,

$$\mathbb{N} \models \phi(s, \underbrace{1 + \ldots + 1}_{n_1-\text{times}}, \ldots, \underbrace{1 + \ldots + 1}_{n_m-\text{times}})$$

and

$$\mathbb{N} \models \phi(\underbrace{1 + \ldots + 1}_{\text{s}-\text{times}}, \underbrace{1 + \ldots + 1}_{n_1-\text{times}}, \ldots, \underbrace{1 + \ldots + 1}_{n_m-\text{times}}).$$

Because the latter statement is an $\mathcal{L}$-sentence,

$$\mathcal{M} \models \phi(\underbrace{1 + \ldots + 1}_{\text{s}-\text{times}}, \underbrace{1 + \ldots + 1}_{n_1-\text{times}}, \ldots, \underbrace{1 + \ldots + 1}_{n_m-\text{times}})$$

and $\mathcal{M} \models \phi(s, n_1, \ldots, n_m)$. By the Tarski–Vaught test, $\mathbb{N} \prec \mathcal{M}$. Thus, $\mathbb{N}$ is a prime model of T.

Suppose $\mathcal{M}$ is a prime model of T. Suppose that $j : \mathcal{M} \to \mathcal{N}$ is an elementary embedding. If $\overline{a} \in M^n$ realizes $p \in S_n(T)$, then so does $j(\overline{a})$. If $p \in S_n(T)$ is nonisolated, there is $\mathcal{N}$ such that $\mathcal{N}$ omits p. If $\mathcal{M}$ realizes p, then we can not elementarily embed $\mathcal{M}$ into $\mathcal{N}$; thus, $\mathcal{M}$ must also omit p. In particular, if $\overline{a} \in M^n$, then $\mathrm{tp}^{\mathcal{M}}(\overline{a})$ must be isolated. This leads us to the following definition.

Definition 4.2.7 We say that $\mathcal{M} \models T$ is *atomic* if $\mathrm{tp}^{\mathcal{M}}(\overline{a})$ is isolated for all $\overline{a} \in M^n$.

We have just argued that prime models are atomic. For countable models, the converse is also true.

Theorem 4.2.8 *Let $\mathcal{L}$ be a countable language and let T be a complete $\mathcal{L}$-theory with infinite models. Then, $\mathcal{M} \models T$ is prime if and only if it is countable and atomic.*

Proof

($\Rightarrow$) We have argued that prime models are atomic. Because $\mathcal{L}$ is countable, T has a countable model. Thus, the prime model must be countable.

($\Leftarrow$) Let $\mathcal{M}$ be countable and atomic. Let $\mathcal{N} \models T$. We must construct an elementary embedding of $\mathcal{M}$ into $\mathcal{N}$. Let $m_0, m_1, \ldots, m_n, \ldots$ be an enumeration of M. For each i, let $\theta_i(v_0, \ldots, v_i)$ isolate the type of $(m_0, \ldots, m_i)$. We will build $f_0 \subseteq f_1 \subseteq \ldots$ a sequence of partial elementary maps from $\mathcal{M}$ into $\mathcal{N}$ where the domain of f_i is $\{m_0, \ldots, m_{i-1}\}$. Then, $f = \bigcup_{i=0}^{\infty} f_i$ is an elementary embedding of $\mathcal{M}$ into $\mathcal{N}$.

Let $f_0 = \emptyset$. Because $\mathcal{M} \equiv \mathcal{N}$, f_0 is partial elementary.

Given f_s, let $n_i = f(m_i)$ for $i < s$. Because $\theta_s(m_0, \ldots, m_s)$ and f_s is partial elementary,

$$\mathcal{N} \models \exists v\ \theta_s(n_0, \ldots, n_{s-1}, v).$$

Let $n_s \in N$ such that $\mathcal{N} \models \theta_s(n_0, \ldots, n_s)$. Because θ_s isolates $\mathrm{tp}^{\mathcal{M}}(m_0, \ldots, m_s)$,

$$\mathrm{tp}^{\mathcal{M}}(m_0, \ldots, m_s) = \mathrm{tp}^{\mathcal{N}}(n_0, \ldots, n_s).$$

Thus, $f_{s+1} = f_s \cup \{(m_s, n_s)\}$ is a partial elementary map.

Theorem 4.2.8 will lead to a criterion for the existence of prime models. We need one preparatory lemma.

Lemma 4.2.9 *Suppose that $(\overline{a}, \overline{b}) \in M^{m+n}$ realizes an isolated type in $S_{m+n}(T)$. Then $\overline{a}$ realizes an isolated type in $S_m(T)$. Indeed if $A \subseteq M$ and $(\overline{a}, \overline{b}) \in M^{m+n}$ realizes an isolated type in $S^{\mathcal{M}}_{m+n}(A)$, then $\mathrm{tp}^{\mathcal{M}}(\overline{a}/A)$ is isolated.*

Proof Let $\phi(\overline{v}, \overline{w})$ isolate $\mathrm{tp}^{\mathcal{M}}(\overline{a}, \overline{b}/A)$. We claim that $\exists w\ \phi(\overline{v}, \overline{w})$ isolates $\mathrm{tp}^{\mathcal{M}}(\overline{a}/A)$. Let $\psi(\overline{v})$ be any $\mathcal{L}_A$-formula such that $\mathcal{M} \models \psi(\overline{a})$. We must show that

$$\mathrm{Th}_A(\mathcal{M}) \models \exists \overline{w}\ (\phi(\overline{v}, \overline{w}) \rightarrow \psi(\overline{v})).$$

Suppose not. Then, there is $\overline{c} \in M^m$ such that

$$\mathcal{M} \models \exists \overline{w}\ (\phi(\overline{c}, \overline{w}) \wedge \neg\psi(\overline{c})).$$

Let $\overline{d} \in M^n$ such that $\mathcal{M} \models \phi(\overline{c}, \overline{d}) \wedge \neg\psi(\overline{c})$. Because $\phi(\overline{v}, \overline{w})$ isolates $\mathrm{tp}^{\mathcal{M}}(\overline{a}, \overline{b}/A)$,

$$\mathrm{Th}_A(\mathcal{M}) \models \phi(\overline{v}, \overline{w}) \rightarrow \psi(\overline{v}).$$

This is a contradiction because

$$\psi(\overline{v}) \in \mathrm{tp}^{\mathcal{M}}(\overline{a}/A) \subset \mathrm{tp}^{\mathcal{M}}(\overline{a}, \overline{b}/A).$$

An extension of this lemma is proved in Exercise 4.5.11.

Theorem 4.2.10 *Let $\mathcal{L}$ be a countable language and let T be a complete $\mathcal{L}$-theory with infinite models. Then, the following are equivalent:*
i) T has a prime model;
ii) T has an atomic model $\mathcal{M}$;
iii) the isolated types in $S_n(T)$ are dense for all n.

Proof We have already shown i) $\Leftrightarrow$ ii).

ii) $\Rightarrow$ iii) Let $\phi(\overline{v})$ be an $\mathcal{L}$-formula such that $[\phi(\overline{v})]$ is a nonempty open set in $S_n(T)$. We must show that $[\phi(\overline{v})]$ contains an isolated type.

Let $\mathcal{M} \models T$ be atomic. Because T is complete and $T \cup \{\phi(\overline{v})\}$ is satisfiable, $T \models \exists \overline{v}\ \phi(\overline{v})$. Thus, there is $\overline{a} \in M^n$ such that $\mathcal{M} \models \phi(\overline{a})$. Then, $\mathrm{tp}^{\mathcal{M}}(\overline{a}) \in [\phi]$ and, because $\mathcal{M}$ is atomic, $\mathrm{tp}^{\mathcal{M}}(\overline{a})$ is isolated. Therefore, the isolated types are dense.

iii) $\Rightarrow$ ii) Suppose that the isolated types in T are dense. We will build an atomic model of T by a Henkin argument. Let $C = \{c_0, \ldots, c_n, \ldots\}$ be a new set of constant symbols, and let $\mathcal{L}^* = \mathcal{L} \cup C$. Let $\phi_0, \phi_1, \ldots$ list all $\mathcal{L}^*$-sentences. We build $\theta_0, \theta_1, \ldots$ a sequence of $\mathcal{L}^*$-sentences such that $T^* = \{\theta_i : i = 0, 1, \ldots\} \cup T$ is a complete satisfiable theory with the witness property. We do this so that the canonical model of T^* is atomic. We assume inductively that $T \cup \{\theta_s\}$ is satisfiable and $\theta_{s+1} \models \theta_s$.

<u>stage 0</u>: $\theta_0 = \exists x\ x = x$.

<u>stage $s+1 = 3i+1$</u>: (completeness) If $T + \theta_s \wedge \phi_i$ is satisfiable, let $\theta_{s+1} = \theta_s \wedge \phi_i$; otherwise, $\theta_{s+1} = \theta_s \wedge \neg\phi_i$.

<u>stage $s+1 = 3i+2$</u>: (witness property) If ϕ_i is $\exists v\ \psi(v)$ and $\theta_s \models \phi_i$, let $c \in C$ be a constant symbol not occurring in θ_s, and let $\theta_{s+1} = \theta_s \wedge \psi(c)$. Otherwise, let $\theta_{s+1} = \theta_s$. As in Theorem 4.2.3, $T \cup \{\theta_{s+1}\}$ is satisfiable.

<u>stage $s+1 = 3i+3$</u>: Let n be minimal such that all of the constants in C occurring in θ_s are from $\{c_0, \ldots, c_n\}$. Let $\psi(v_0, \ldots, v_n)$ be an $\mathcal{L}$-formula such that $\theta_s = \psi(c_0, \ldots, c_n)$. Clearly, $T \cup \{\psi(v_0, \ldots, v_n)\}$ is satisfiable. Because the isolated types in $S_n(T)$ are dense, there is an isolated type $p \in [\psi(\overline{v})]$. Let $\chi(\overline{v})$ be an $\mathcal{L}$-formula isolating p; in particular, $[\chi(\overline{v})] = \{p\}$ and $T \cup \{\chi(\overline{v})\}$ is satisfiable. Let $\theta_{s+1} = \chi(\overline{c})$. Then, $T \cup \{\theta_{s+1}\}$ is satisfiable. Because $\psi(\overline{v}) \in p$, $\theta_{s+1} \models \theta_s$.

As in Theorem 4.2.3, the theory $T^* = T \cup \{\theta_1, \theta_2, \ldots\}$ is a complete theory with the witness property. Let $\mathcal{M}$ be the canonical model of T^*. We must show $\mathcal{M}$ is atomic. Let $\overline{d} \in \mathcal{M}$. We can find an n and an $s = 3_i + 2$ such that each d_i is in $\{c_0, \ldots, c_n\}$ and n is minimal such that $\{c_0, \ldots, c_n\}$ contains all the constants occuring in θ_s. At stage $s+1$ we make sure that $(c_0^{\mathcal{M}}, \ldots, c_n^{\mathcal{M}})$ realizes an isolated type. By 4.2.9, $\overline{d}$ realizes an isolated type.

In Exercise 4.5.16 we use Theorem 4.2.10 to give an example of a theory with no prime models. We now give one important case where the isolated types are dense. Note that if $\mathcal{L}$ is countable and A is countable, then $|S_n^{\mathcal{M}}(A)| \leq 2^{\aleph_0}$ because there are only $2^{\aleph_0}$ sets of $\mathcal{L}_A$-formulas. We will show that if there are fewer than the maximal possible number of types, then there are prime models.

Theorem 4.2.11 *Suppose that T is a complete theory in a countable language and $A \subseteq \mathcal{M} \models T$ is countable. If $|S_n^{\mathcal{M}}(A)| < 2^{\aleph_0}$, then*
i) the isolated types in $S_n^{\mathcal{M}}(A)$ are dense and
ii) $|S_n^{\mathcal{M}}(A)| \leq \aleph_0$.
In particular, if $|S_n(T)| < 2^{\aleph_0}$, then T has a prime model.

Proof
i) We first prove that the isolated types are dense. Suppose that there is a formula ϕ such that $[\phi]$ contains no isolated types. Because ϕ does not isolate a type, we can find ψ such that $[\phi \wedge \psi] \neq \emptyset$ and $[\phi \wedge \neg\psi] \neq \emptyset$. Because $[\phi]$ does not contain an isolated type, neither does $[\phi \wedge \pm\psi]$.

We build a binary tree of formulas $(\phi_\sigma : \sigma \in 2^{<\omega})$ such that:
i) each $[\phi_\sigma]$ is nonempty but contains no isolated types;
ii) if $\sigma \subset \tau$, then $\phi_\tau \models \phi_\sigma$;
iii) $\phi_{\sigma,i} \models \neg\phi_{\sigma,1-i}$.

Let $\phi_\emptyset = \phi$ for some formula ϕ where $[\phi]$ contains no isolated types. Suppose that $[\phi_\sigma]$ is nonempty but contains no isolated types. As above, we can find ψ such that $[\phi_\sigma \wedge \psi]$ and $[\phi_\sigma \wedge \neg\psi]$ are both nonempty and neither contains an isolated type. Let $\phi_{\sigma,0} = \phi \wedge \psi$ and $\phi_{\sigma,1} = \phi \wedge \neg\psi$.

Let $f : \omega \to 2$. Because

$$[\phi_{f|0}] \supseteq [\phi(f|1)] \supseteq [\phi(f|2)] \supseteq \ldots$$

and $S_n^{\mathcal{M}}(A)$ is compact, there is

$$p_f \in \bigcup_{n=0}^{\infty} [\phi_{f|n}].$$

If $g \neq f$, we can find m such that $f|m = g|m$ but $f(m) \neq g(m)$. By construction, $\phi_{f|m+1} \models \neg\phi_{g|m+1}$; thus $p_f \neq p_g$. Because $f \mapsto p_f$ is a one-to-one function from 2^ω into $S_n^{\mathcal{M}}(A)$, $|S_n^{\mathcal{M}}(A)| = 2^{\aleph_0}$.

ii) Suppose that $|S_n^{\mathcal{M}}(A)| > \aleph_0$. We claim that $|S_n^{\mathcal{M}}(A)| = 2^{\aleph_0}$. Because $|S_n^{\mathcal{M}}(A)| > \aleph_0$ and there are only countably many $\mathcal{L}_A$-formulas, there is a formula ϕ such that $|[\phi]| > \aleph_0$.

Claim If $|[\phi]| > \aleph_0$, there is an $\mathcal{L}_A$-formula ψ such that $|[\phi \wedge \psi]| > \aleph_0$ and $|[\phi \wedge \neg\psi] > \aleph_0$.

Suppose not. Let $p = \{\psi(\overline{v}) : |[\phi \wedge \psi]| > \aleph_0\}$. Clearly, for each ψ either $\psi \in p$ or $\neg\psi \in p$ but not both. We claim that p is satisfiable. Suppose that

$\psi_1, \ldots, \psi_m \in p$. Either $\psi_1 \wedge \ldots \wedge \psi_m \in p$, in which case $\{\psi_1, \ldots, \psi_m\} \cup \mathrm{Th}_A(\mathcal{M})$ is satisfiable, or $\neg\psi_1 \vee \ldots \vee \neg\psi_m \in p$. Because

$$[\neg\psi_1 \vee \ldots \vee \neg\psi_m] = [\neg\psi_1] \cup \ldots \cup [\neg\psi_m],$$

we must have $|[\neg\psi_i]| > \aleph_0$ for some $\aleph_0$, a contradiction. Thus $p \in S_n^{\mathcal{M}}(A)$. Moreover, if $\psi \notin p$, then $|[\phi \wedge \psi]| \leq \aleph_0$. But

$$[\phi] = \bigcup_{\psi \notin p} [\phi \wedge \psi] \cup \{p\}.$$

Because $[\phi]$ is the union of at most $\aleph_0$ sets each of size at most $\aleph_0$, we have $|[\phi]| \leq \aleph_0$, a contradiction.

We build a binary tree of formulas $(\phi_\sigma : \sigma \in 2^{<\omega})$ such that:

i) if $\sigma \subset \tau$ then $\phi_\tau \models \phi_\sigma$;

ii) $\phi_{\sigma,i} \models \neg\phi_{\sigma,1-i}$;

iii) $|[\phi_\sigma]| > \aleph_0$.

Let $\phi_\emptyset = \phi$ for some formula ϕ with $|[\phi]| > \aleph_0$. Given ϕ_σ where $|[\phi_\sigma]| > \aleph_0$, by the claim we can find ψ such that $|[\phi_\sigma \wedge \psi]| > \aleph_0$ and $|[\phi_\sigma \wedge \neg\psi]| > \aleph_0$. Let $\phi_{\sigma,0} = \phi_\sigma \wedge \psi$ and $\phi_{\sigma,1} = \phi_\sigma \wedge \neg\psi$.

As in i), for each $f \in 2^\omega$ there is a

$$p_f \in \bigcap_{m=0}^{\infty} [\phi_{f|m}],$$

and if $f \neq g$, then $p_f \neq p_g$. Thus $|S_n^{\mathcal{M}}(A)| = 2^{\aleph_0}$.

We note that it is possible for there to be prime models even if $|S_n(T)| = 2^{\aleph_0}$. For example, $\mathrm{Th}(\mathbb{N}, +, \cdot, <, 0, 1)$ and RCF have prime models.

Countable Homogeneous Models

Our next goal is to show that prime models are unique up to isomorphism. This will follow from work on homogeneous models.

Definition 4.2.12 Let κ be an infinite cardinal. We say that $\mathcal{M} \models T$ is *κ-homogeneous* if whenever $A \subset M$ with $|A| < \kappa$, $f : A \to M$ is a partial elementary map, and $a \in M$, there is $f^* \supseteq f$ such that $f^* : A \cup \{a\} \to M$ is partial elementary.

We say that $\mathcal{M}$ is *homogeneous* if it is $|M|$-homogeneous.

In homogeneous models, partial elementary maps are just restrictions of automorphisms.

Proposition 4.2.13 *Suppose that $\mathcal{M}$ is homogeneous, $A \subset M$, $|A| < |M|$, and $f : A \to M$ is a partial elementary map. Then, there is an automorphism σ of $\mathcal{M}$ with $\sigma \supseteq f$.*

In particular, if $\mathcal{M}$ is homogeneous and $\overline{a}, \overline{b} \in M^n$ realize the same n-type, then there is an automorphism σ of $\mathcal{M}$ with $\sigma(\overline{a}) = \overline{b}$.

Proof Let $|M| = \kappa$, and let $(a_\alpha : \alpha < \kappa)$ be an enumeration of M. We build a sequence of partial elementary maps $(f_\alpha : \alpha < \kappa)$ extending f with $f_\alpha \subseteq f_\beta$ for $\alpha < \beta$ such that a_α is in the domain and image of $f_{\alpha+1}$ and $|f_{\alpha+1}| \leq |f_\alpha| + 2 < \kappa$. Then, $\sigma = \bigcup_{\alpha<\kappa} f_\alpha$ is the desired automorphism. Let $f_0 = f$.

If α is a limit ordinal and f_β is partial elementary with

$$|f_\beta| \leq |A| + |\beta| + \aleph_0 < \kappa$$

for all $\beta < \alpha$, let $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$. Then, f_α is partial elementary and

$$|f_\alpha| \leq |\alpha|(|A| + |\alpha| + \aleph_0) \leq |A| + |\alpha| + \aleph_0 < \kappa.$$

Given f_α with $|f_\alpha| < \kappa$, because $\mathcal{M}$ is homogeneous, there is $b \in M$ such that if $g_\alpha = f_\alpha \cup \{(a_\alpha, b)\}$, then g_α is partial elementary. Note that g_α^{-1} is also partial elementary. Thus, because $\mathcal{M}$ is homogeneous there is $c \in M$ such that $g_\alpha^{-1} \cup \{(a_\alpha, c)\}$ is partial elementary. Thus, $f_{\alpha+1} = g_\alpha \cup \{(c, a_\alpha)\}$ is partial elementary, $|f_{\alpha+1}| \leq |f_\alpha| + 2 \leq |A| + |\alpha| + \aleph_0$, and a_α is in the domain and range of $f_{\alpha+1}$.

If $\mathcal{M}$ is homogeneous and $\mathrm{tp}^{\mathcal{M}}(\overline{a}) = \mathrm{tp}^{\mathcal{M}}(\overline{b})$, then $\overline{a} \mapsto \overline{b}$ is a partial elementary map that must extend to an automorphism.

Lemma 4.2.14 *If $\mathcal{M}$ is atomic, then $\mathcal{M}$ is $\aleph_0$-homogeneous. In particular, countable atomic models are homogeneous.*

Proof Suppose that $\overline{a} \mapsto \overline{b}$ is elementary and $c \in M$. Let $\phi(\overline{v}, w)$ isolate $\mathrm{tp}^{\mathcal{M}}(\overline{a}, c)$. Because $\mathcal{M} \models \exists w\ \phi(\overline{a}, w)$ and $\overline{a} \mapsto \overline{b}$ is elementary, $\mathcal{M} \models \exists w\ \phi(\overline{b}, w)$. Suppose that $\mathcal{M} \models \phi(\overline{b}, d)$. Because $\phi(\overline{v}, w)$ isolates a type, $\mathrm{tp}^{\mathcal{M}}(\overline{a}, c) = \mathrm{tp}^{\mathcal{M}}(\overline{b}, d)$. Thus, $\overline{a}, c \mapsto \overline{b}, d$ is elementary.

For countable homogeneous models, there is a simple test for isomorphism. Clearly, if $\mathcal{M} \cong \mathcal{N}$, then $\mathcal{M}$ and $\mathcal{N}$ realize the same types from $S_n(T)$. For countable homogeneous models, this condition is also sufficient.

Theorem 4.2.15 *Let T be a complete theory in a countable language. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are countable homogeneous models of T and $\mathcal{M}$ and $\mathcal{N}$ realize the same types in $S_n(T)$ for $n \geq 1$. Then $\mathcal{M} \cong \mathcal{N}$.*

Proof We build an isomorphism $f : \mathcal{M} \to \mathcal{N}$ by a back-and-forth argument. We will build $f_0 \subset f_1 \subset \ldots$, a sequence of partial elementary maps with finite domain, and let $f = \bigcup_{i=0}^{\infty} f_i$. Let $a_0, a_1, \ldots$ enumerate M and $b_0, b_1, \ldots$ enumerate N. We will ensure that $a_i \in \mathrm{dom}(f_{2i+1})$ and $b_i \in \mathrm{img}(f_{2i+2})$. Thus, we will have dom(f)=M and $f : M \to N$, a surjective elementary map, as desired.

stage 0: Let $f_0 = \emptyset$. Because T is complete f_0 is partial elementary.

We inductively assume that f_s is partial elementary. Let $\overline{a}$ be the domain of f_s and $\overline{b} = f_s(\overline{a})$.

stage $s+1=2i+1$: Let $p=\mathrm{tp}^{\mathcal{M}}(\overline{a},a_i)$. Because $\mathcal{M}$ and $\mathcal{N}$ realize the same types, we can find $\overline{c},d\in N$ such that $\mathrm{tp}^{\mathcal{N}}(\overline{c},d)=p$. Note that $\mathrm{tp}^{\mathcal{N}}(\overline{c})=\mathrm{tp}^{\mathcal{M}}(\overline{a})$, by choice of $\overline{c}$, and $\mathrm{tp}^{\mathcal{M}}(\overline{a})=\mathrm{tp}^{\mathcal{N}}(\overline{b})$ because f_s is partial elementary. Thus, $\mathrm{tp}^{\mathcal{N}}(\overline{c})=\mathrm{tp}^{\mathcal{N}}(\overline{b})$. Because $\mathcal{N}$ is homogeneous, there is $e\in N$ such that $\mathrm{tp}^{\mathcal{N}}(\overline{b},e)=\mathrm{tp}^{\mathcal{N}}(\overline{c},d)=p$. Thus, $f_{s+1}=f_s\cup\{(a_i,e)\}$ is partial elementary with a_i in the domain.

stage $s+1=2i+2$: As in the previous case, we can find $\overline{c},d\in M$ such that $\mathrm{tp}^{\mathcal{M}}(\overline{c},d)=\mathrm{tp}^{\mathcal{N}}(\overline{b},b_i)$. Because $\mathcal{M}$ is homogeneous, there is $e\in M$ such that $\mathrm{tp}^{\mathcal{M}}(\overline{c},d)=\mathrm{tp}^{\mathcal{M}}(\overline{a},e)$. Then, $f_{s+1}=f_s\cup\{(e,b_i)\}$ with b_i in the range.

Corollary 4.2.16 *Let T be a complete theory in a countable language. If $\mathcal{M}$ and $\mathcal{N}$ are prime models of T, then $\mathcal{M}\cong\mathcal{N}$.*

Proof By Theorem 4.2.8, $\mathcal{M}$ and $\mathcal{N}$ are atomic. Because the types in $S_n(T)$ realized in an atomic model are exactly the isolated types, $\mathcal{M}$ and $\mathcal{N}$ realize the same types. By Lemma 4.2.14, countable atomic models are homogeneous. Thus, by Theorem 4.2.15, $\mathcal{M}\cong\mathcal{N}$.

Prime Model Extensions of ω-Stable Theories

We conclude this section by looking at a relative notion of prime models. Suppose that $\mathcal{M}\models T$ and $A\subseteq M$. We say that $\mathcal{M}$ is *prime over A* if whenever $\mathcal{N}\models T$ and $f:A\to\mathcal{N}$ is partial elementary, there is an elementary $f^*:\mathcal{M}\to\mathcal{N}$ extending f.

We give three examples. Let L be any linear order. We build $L^*\models$ DLO prime over L as follows. If L has a least element a, add a copy of $\mathbb{Q}$ below a. If L has a greatest element b, add a copy of $\mathbb{Q}$ above b. If $c,d\in L$ with $c<d$ but there are no elements of L between c and d, add a copy of $\mathbb{Q}$ between c and d. We add no other new elements. It is easy to see that $L^*\models$ DLO and that if $f:L\to\mathcal{M}\models$ DLO, then f extends to $f^*:L^*\to\mathcal{M}$. Because DLO has quantifier elimination, it is model-complete and f^* is elementary.

For ACF, if R is any integral domain and F is the algebraic closure of the fraction field of R, then F is prime over R and any embedding of R into an algebraically closed field K extends to F. Because ACF is model-complete, this map is elementary. Similarly, if R is an ordered integral domain, then the real closure of the fraction field of R is a model of RCF prime over R. In Exercise 4.5.26, we will give examples of theories without prime model extensions.

There is one very natural class of theories with prime model extensions. This class will play a very important role later in the book.

Definition 4.2.17 Let T be a complete theory in a countable language, and let κ be an infinite cardinal. We say that T is *κ-stable* if whenever $\mathcal{M}\models T$, $A\subseteq M$, and $|A|=\kappa$, then $|S_n^{\mathcal{M}}(A)|=\kappa$.

We say that $\mathcal{M}$ is κ-stable if $\mathrm{Th}(\mathcal{M})$ is κ-stable.

For historical reasons, we will refer to $\aleph_0$-stable theories as being "ω-stable." By Corollary 4.1.18, ACF is ω-stable. On the other hand, $|S_1^{\mathbb{Q}}(\mathbb{Q})| = 2^{\aleph_0}$ so DLO is not ω-stable.

We will show that ω-stable theories have prime model extensions. An important first step is to show that if there are few types over countable sets, then there are few types over arbitrary sets.

Theorem 4.2.18 *Let T be a complete theory in a countable language. If T is ω-stable, then T is κ-stable for all infinite cardinals κ.*

Proof Suppose that $\mathcal{M} \models T$, $A \subseteq M$, $|A| = \kappa$ and $|S_n^{\mathcal{M}}(A)| > \kappa$. Because there are only κ formulas with parameters from A, there is some $\mathcal{L}_A$-formula $\phi_\emptyset(\overline{v})$ such that $|[\phi_\emptyset]| > \kappa$. The argument from Theorem 4.2.11 ii) can be extended to show that if $|[\phi]| > \kappa$ there is an $\mathcal{L}_A$-formula ψ such that $|[\phi \wedge \psi]| > \kappa$ and $|[\phi \wedge \neg\psi]| > \kappa$.

As in Theorem 4.2.11 ii), we build a binary tree of formulas $(\phi_\sigma : \sigma \in 2^{<\omega})$ such that:

i) if $\sigma \subset \tau$, then $\phi_\tau \models \phi_\sigma$;

ii) $\phi_{\sigma,i} \models \neg\phi_{\sigma,1-i}$;

iii) $|[\phi_\sigma]| > \kappa$.

Let A_0 be the set of all parameters from A occurring in any formula ϕ_σ. Clearly A_0 is a countable set. Arguing as in Theorem 4.2.11 ii), $|S_n^{\mathcal{M}}(A_0)| = 2^{\aleph_0}$, contradicting the ω-stability of T.

Proposition 4.2.19 *Let T be a complete theory in a countable language. If T is ω-stable, then for all $\mathcal{M} \models T$ and $A \subseteq M$, the isolated types in $S_n^{\mathcal{M}}(A)$ are dense.*

Proof Suppose not. We can build a binary tree of formulas as in Theorem 4.2.11 i). As in Theorem 4.2.18, we can find a countable $A_0 \subseteq A$ such that all parameters come from A_0. But then $|S_n^{\mathcal{M}}(A_0)| = 2^{\aleph_0}$, contradicting the ω-stability of T.

Theorem 4.2.20 *Suppose that T is ω-stable. Let $\mathcal{M} \models T$ and $A \subseteq M$. There is $\mathcal{M}_0 \prec \mathcal{M}$, a prime model extension of A. Moreover, we can choose $\mathcal{M}_0$ such that every element of $\mathcal{M}_0$ realizes an isolated type over A.*

Proof We will find an ordinal δ and build a sequence of sets $(A_\alpha : \alpha \leq \delta)$ where $A_\alpha \subseteq M$ and

i) $A_0 = A$;

ii) if α is a limit ordinal, then $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$;

iii) if no element of $M \setminus A_\alpha$ realizes an isolated type over A_α, we stop and let $\delta = \alpha$; otherwise, pick a_α realizing an isolated type over A_α, and let $A_{\alpha+1} = A_\alpha \cup \{a_\alpha\}$. Let $\mathcal{M}_0$ be the substructure of $\mathcal{M}$ with universe A_δ.

Claim 1 $\mathcal{M}_0 \prec \mathcal{M}$.

We apply the Tarski–Vaught test. Suppose that $\mathcal{M} \models \phi(v, \overline{a})$, where $\overline{a} \in A_\delta$. By Proposition 4.2.19, the isolated types in $S^{\mathcal{M}}(A_\delta)$ are dense. Thus, there is $b \in M$ such that $\mathcal{M} \models \phi(b, \overline{a})$ and $\mathrm{tp}^{\mathcal{M}}(b/A_\delta)$ is isolated. By choice of δ, $b \in A_\delta$. Thus, by Proposition 2.3.5, $\mathcal{M}_0 \prec \mathcal{M}$.

Claim 2 $\mathcal{M}_0$ is a prime model extension of A.

Suppose that $\mathcal{N} \models T$ and $f : A \to \mathcal{N}$ is partial elementary. We show by induction that there are $f = f_0 \subset \ldots \subset f_\alpha \subset \ldots \subset f_\delta$,where $f_\alpha : A_\alpha \to \mathcal{N}$ is elementary.

If α is a limit ordinal, we let $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$.

Given $f_\alpha : A_\alpha \to \mathcal{N}$ partial elementary, l $\phi(v, \overline{a})$ isolate $\mathrm{tp}^{\mathcal{M}_0}(a_\alpha/A_\alpha)$. Because f_α is partial elementary, by Lemma 4.1.9 iii), $\phi(v, f_\alpha(\overline{a}))$ isolates $f_\alpha(\mathrm{tp}^{\mathcal{M}_0}(a_\alpha/A_\alpha))$ in $S_1^{\mathcal{N}}(f_\alpha(A))$. Also, because f_α is partial elementary, there is $b \in N$ with $\mathcal{N} \models \phi(b, f_\alpha(\overline{a}))$. Thus,$f_{\alpha+1} = f_\alpha \cup \{(a_\alpha, \overline{b})\}$ is elementary.

In particular,$f_\delta : \mathcal{M}_0 \to \mathcal{N}$ is elementary. Thus, $\mathcal{M}_0$ is a prime model extension of A.

To see that every element of $\mathcal{M}_0$ realizes an isolated type over A, we must show that $\overline{a}$ realizes an isolated type over A for all $\overline{a} \in A_\alpha$, $\alpha < \delta$. We argue by induction on α. For α a limit ordinal,this is clear. For successor ordinals,it follows from the following lemma.

Lemma 4.2.21 *Suppose that $A \subseteq B \subseteq \mathcal{M} \models T$ and every $\overline{b} \in B^m$ realizes an isolated type in $S_m^{\mathcal{M}}(A)$. Suppose that $\overline{a} \in M^n$ realizes an isolated type in $S_n^{\mathcal{M}}(B)$. Then, $\overline{a}$ realizes an isolated type in $S_n^{\mathcal{M}}(A)$.*

Proof Let $\phi(\overline{v}, \overline{w})$ be an $\mathcal{L}$-formula and $\overline{b} \in B^m$ such that $\phi(\overline{v}, \overline{b})$ isolates $\mathrm{tp}^{\mathcal{M}}(\overline{a}/B)$. Let $\theta(\overline{w})$ be an $\mathcal{L}_A$-formula isolating $\mathrm{tp}^{\mathcal{M}}(\overline{b}/A)$. We first claim that $\phi(\overline{v}, \overline{w}) \wedge \theta(\overline{w})$ isolates $\mathrm{tp}^{\mathcal{M}}(\overline{a}, \overline{b}/A)$.

Suppose that $\mathcal{M} \models \psi(\overline{a}, \overline{b})$. Because $\phi(\overline{v}, \overline{b})$ isolates $\mathrm{tp}^{\mathcal{M}}(\overline{a}/B)$,

$$\mathrm{Th}_A(\mathcal{M}) \models \phi(\overline{v}, \overline{b}) \to \psi(\overline{v}, \overline{b}).$$

Thus, because $\theta(\overline{w})$ isolates $\mathrm{tp}^{\mathcal{M}}(\overline{b}/A)$,

$$\mathrm{Th}_A(\mathcal{M}) \models \theta(\overline{w}) \to (\phi(\overline{v}, \overline{w}) \to \psi(\overline{v}, \overline{w}))$$

and

$$\mathrm{Th}_A(\mathcal{M}) \models (\theta(\overline{w}) \wedge \phi(\overline{v}, \overline{w})) \to \psi(\overline{v}, \overline{w}),$$

as desired.

Because $\mathrm{tp}^{\mathcal{M}}(\overline{a}, \overline{b}/A)$ is isolated, so is $\mathrm{tp}^{\mathcal{M}}(\overline{a}/A)$ by Lemma 4.2.9.

For ω-stable theories (indeed, for theories that are κ-stable for some κ), prime model extensions are unique, although we postpone the proof to Chapter 6.

Theorem 4.2.22 *Let T be ω-stable. Suppose that $\mathcal{M} \models T$ and $\mathcal{N} \models T$ are prime model extensions of A and $\mathrm{Th}_A(\mathcal{M}) = \mathrm{Th}_A(\mathcal{N})$. Then, there is $f : \mathcal{M} \to \mathcal{N}$, an isomorphism fixing A.*

4.3 Saturated and Homogeneous Models

In Section 4.2 we concentrated on models that realize very few types. In this section, we will study models realizing many types. Throughout this section, we will assume that T is a complete theory with infinite models in a countable language $\mathcal{L}$.

Definition 4.3.1 Let κ be an infinite cardinal. We say that $\mathcal{M} \models T$ is *κ-saturated* if, for all $A \subseteq M$, if $|A| < \kappa$ and $p \in S_n^{\mathcal{M}}(A)$, then p is realized in $\mathcal{M}$.

We say that $\mathcal{M}$ is *saturated* if it is $|M|$-saturated.

Proposition 4.3.2 *Let $\kappa \geq \aleph_0$. The following are equivalent:*
i) $\mathcal{M}$ is κ-saturated.
ii) If $A \subseteq M$ with $|A| < \kappa$ and p is a (possibly incomplete) n-type over A, then p is realized in $\mathcal{M}$.
iii) If $A \subseteq M$ with $|A| < \kappa$ and $p \in S_1^{\mathcal{M}}(A)$, then p is realized in $\mathcal{M}$.

Proof
i)$\Rightarrow$ ii) If $\mathcal{M}$ is κ-saturated and p is an incomplete n-type over A where $|A| < \kappa$, then there is a complete type $p^* \in S_n^{\mathcal{M}}(A)$ with $p^* \supseteq p$. Because p^* is realized in $\mathcal{M}$ so is p.

ii) $\Rightarrow$ iii) Clear.

iii) $\Rightarrow$ i) We prove this by induction on n. Let $p \in S_n^{\mathcal{M}}(A)$. Let $q \in S_{n-1}^M$ be the type $\{\phi(v_1, \ldots, v_{n-1}) : \phi \in p\}$. By induction, q is realized by some $\overline{a}$ in $\mathcal{M}$. Let $r \in S_1^{\mathcal{M}}(A \cup \{a_1, \ldots, a_{n-1}\})$ be the type $\{\psi(\overline{a}, w) : \psi(v_1, \ldots, v_n) \in p\}$. By iii), we can realize r by some b in $\mathcal{M}$. Then, $(\overline{a}, b)$ realizes p.

Homogeneity is a weak form of saturation.

Proposition 4.3.3 *If $\mathcal{M}$ is κ-saturated, then $\mathcal{M}$ is κ-homogeneous.*

Proof Suppose that $A \subseteq \mathcal{M}$, $|A| < \kappa$, and $f : A \to M$ is partial elementary. Let $b \in M \setminus A$. Let

$$\Gamma = \{\phi(v, f(\overline{a})) : \overline{a} \in A^m \text{ and } \mathcal{M} \models \phi(b, \overline{a})\}.$$

If $\phi(v, f(\overline{a})) \in \Gamma$, then $\mathcal{M} \models \exists v\ \phi(v, \overline{a})$ and hence, because f is partial elementary, $\mathcal{M} \models \exists v\ \phi(v, f(\overline{a}))$. Thus, because Γ is closed under conjunctions, Γ is satisfiable. Because $\mathcal{M}$ is saturated, there is $c \in M$ realizing Γ. Thus, $f \cup \{(b, c)\}$ is elementary and $\mathcal{M}$ is κ-homogeneous.

Countably Saturated Models

We will begin by examining $\aleph_0$-saturated models. If $\mathcal{M}$ is $\aleph_0$-saturated, then $\mathcal{M}$ realizes every type in $S_n(T)$. We will show that for $\aleph_0$-homogeneous models this condition is also sufficient.

Proposition 4.3.4 *If $\mathcal{M} \models T$, then $\mathcal{M}$ is $\aleph_0$-saturated if and only if $\mathcal{M}$ is $\aleph_0$-homogeneous and $\mathcal{M}$ realizes all types in $S_n(T)$.*

Proof

($\Rightarrow$) Clear.

($\Leftarrow$) Let $\overline{a} \in M^m$ and let $p \in S_n^{\mathcal{M}}(\overline{a})$. Let $q \in S_{n+m}(T)$ be the type $\{\phi(\overline{v}, \overline{w}) : \phi(\overline{v}, \overline{a}) \in p\}$. By assumption, there is $(\overline{b}, \overline{c}) \in M^{n+m}$ realizing q. Because $\mathrm{tp}^{\mathcal{M}}(\overline{c}) = \mathrm{tp}^{\mathcal{M}}(\overline{a})$ and $\mathcal{M}$ is $\aleph_0$-homogeneous, there is $\overline{d} \in \mathcal{M}$ such that $\mathrm{tp}^{\mathcal{M}}(\overline{a}, \overline{d}) = \mathrm{tp}^{\mathcal{M}}(\overline{c}, \overline{b})$. Hence, $\overline{d}$ realizes p and $\mathcal{M}$ is $\aleph_0$-saturated.

Countable saturated models are unique up to isomorphism.

Corollary 4.3.5 *If $\mathcal{M}, \mathcal{N} \models T$ are countable saturated models, then $\mathcal{M} \cong \mathcal{N}$.*

Proof Because $\mathcal{M}$ and $\mathcal{N}$ are $\aleph_0$-homogeneous and both realize all types in $S_n(T)$ for all $n < \omega$, by Theorem 4.2.15, $\mathcal{M} \cong \mathcal{N}$.

The next proposition shows that we can extend models to $\aleph_0$-homogeneous models without increasing the cardinality.

Proposition 4.3.6 *Let $\mathcal{M} \models T$. There is $\mathcal{M} \prec \mathcal{N}$ such that $\mathcal{N}$ is $\aleph_0$-homogeneous and $|N| = |M|$.*

Proof We first argue that we can find $\mathcal{M} \prec \mathcal{N}_1$ such that $|M| = |N_1|$, and if $\overline{a}, \overline{b}, c \in M$ and $\mathrm{tp}^{\mathcal{M}}(\overline{a}) = \mathrm{tp}^{\mathcal{M}}(\overline{b})$, then there is $d \in N_1$ such that $\mathrm{tp}^{\mathcal{N}_1}(\overline{a}, c) = \mathrm{tp}^{\mathcal{N}_1}(\overline{b}, d)$.

Let $((\overline{a}_\alpha, \overline{b}_\alpha, c_\alpha) : \alpha < |M|)$ list all tuples $(\overline{a}, \overline{b}, c)$ where $\overline{a}, \overline{b}, c \in M$ and $\mathrm{tp}^{\mathcal{M}}(\overline{a}) = \mathrm{tp}^{\mathcal{M}}(\overline{b})$. We build an elementary chain $\mathcal{M}_0 \prec \mathcal{M}_1 \ldots \prec \mathcal{M}_\alpha \prec \ldots$ for $\alpha < |M|$.

Let $\mathcal{M}_0 = \mathcal{M}$.

If α is a limit ordinal, let $\mathcal{M}_\alpha = \bigcup_{\beta<\alpha} \mathcal{M}_\beta$.

Given $\mathcal{M}_\alpha$, let $\mathcal{M}_\alpha \prec \mathcal{M}_{\alpha+1}$ with $|M_\alpha| = |M_{\alpha+1}|$ such that there is $d \in \mathcal{M}_\alpha$ with $\mathrm{tp}^{\mathcal{M}_{\alpha+1}}(\overline{b}, d) = \mathrm{tp}^{\mathcal{M}_{\alpha+1}}(\overline{a}, c)$. Let $\mathcal{N}_1 = \bigcup_{\alpha<|M|} \mathcal{M}_\alpha$. Because $\mathcal{N}_1$ is a union of $|M|$ models of size $|M|$, $|N_1| = |M|$.

We now build $\mathcal{N}_0 \prec \mathcal{N}_1 \prec \mathcal{N}_2 \ldots$ such that $|N_i| = |M|$ and if $\overline{a}, \overline{b}, c \in N_i$ and $\mathrm{tp}^{\mathcal{N}_i}(\overline{a}) = \mathrm{tp}^{\mathcal{N}_i}(\overline{b})$, then there is $d \in N_{i+1}$ such that $\mathrm{tp}^{\mathcal{N}_{i+1}}(\overline{a}, c) = \mathrm{tp}^{\mathcal{N}_{i+1}}(\overline{b}, d)$.

Let $\mathcal{N} = \bigcup_{i<\omega} \mathcal{N}_i$. Clearly, $|\mathcal{N}| = |\mathcal{M}|$ and $\mathcal{N}$ is $\aleph_0$-homogeneous.

Propositions 4.3.5 and 4.3.6 allow us to characterize theories with countable saturated models.

Theorem 4.3.7 *T has a countable saturated model if and only if $|S_n(T)| \leq \aleph_0$ for all n.*

Proof We need only show that if $|S_n(T)| \leq \aleph_0$ for all n then T has a countable saturated model. Let $p_0, p_1, \ldots$ list all elements of $\bigcup_{n\in\omega} S_n(T)$. Let $\mathcal{M}_0 \models T$. Iterating Lemma 4.1.3, we build $\mathcal{M}_0 \prec \mathcal{M}_1 \prec \ldots$ such that $\mathcal{M}_i$ is countable and $\mathcal{M}_{i+1}$ realizes p_i. Thus, $\mathcal{M} = \bigcup_{i\in\omega} \mathcal{M}_i$ is countable and contains realizations of all types in $S_n(T)$ for $n < \omega$. By Proposition 4.3.6, there is $\mathcal{M} \prec \mathcal{N}$ such that $\mathcal{N}$ is countable and $\aleph_0$-homogeneous. By Corollary 4.3.5, $\mathcal{N}$ is $\aleph_0$-saturated.

Curiously, theories with large countable models also have small countable models.

Corollary 4.3.8 *i) If T has a countable saturated model, then T has a prime model.*

ii) If T has fewer than $2^{\aleph_0}$ countable models, then T has a countable saturated model and a prime model.

Proof

i) If T has a saturated model, then $|S_n(T)|$ is countable for all n. By Theorem 4.2.11, the isolated types are dense in $S_n(T)$ for all n. Thus, by Theorem 4.2.10, T has a prime model.

ii) It suffices to show that $S_n(T)$ is countable for all $n < \omega$. Suppose not. By Theorem 4.2.11, if $|S_n(T)| > \aleph_0$, then $|S_n(T)| = 2^{\aleph_0}$. Each n-type must be realized in some countable model. Because each countable model realizes only countably many n-types, if there are $2^{\aleph_0}$ n-types, then there must be $2^{\aleph_0}$ nonisomorphic countable models.

We consider several examples.

Example 4.3.9 *Dense Linear Orders*

We will show that $(\mathbb{Q}, <)$ is saturated. Suppose $A \subset \mathbb{Q}$ is finite. Suppose that $A = \{a_1, \ldots, a_m\}$ where $a_1 < \ldots < a_m$. By the analysis of types in DLO given in Section 4.1, there are exactly $2m+1$ types in $S_1(A)$. Each type is isolated by one of the formulas $v = a_i$, $v < a_0$, $a_i < v < a_{i+1}$, or $a_m < v$. Clearly, all of these types are realized in $\mathbb{Q}$. Note that in this case $\mathbb{Q}$ is both saturated and prime! In Section 4.4, we will see that this always happens in $\aleph_0$-categorical theories.

Example 4.3.10 *Algebraically Closed Fields*

Fix p prime or 0. Let k be $\mathbb{F}_p$ if $p > 0$ and $\mathbb{Q}$ if $p = 0$. Because $S_n(\mathrm{ACF}_p)$ is in bijection with $\mathrm{Spec}(k[X_1, \ldots, X_n])$, by Corollary 4.1.18, $|S_n(\mathrm{ACF}_p)| = \aleph_0$. Thus, there is a countable saturated model of ACF_p.

Let q_n be the type corresponding to the 0 ideal in $k[X_1, \ldots, X_n]$. If $a_1, \ldots, a_n$ realizes q_n, then $a_1, \ldots, a_n$ are algebraically independent over k. Thus, any saturated model has infinite transcendence degree. It follows that the countable saturated model of ACF_p is the unique algebraically closed field of characteristic p and transcendence degree $\aleph_0$.

Example 4.3.11 *Real Closed Fields*

Let $r \in \mathbb{R} \setminus \mathbb{Q}$. Let p_r be the set of formulas $\Big\{\underbrace{v + \ldots + v}_{m-\text{times}} < \underbrace{1 + \ldots + 1}_{n-\text{times}} : m, n \in \mathbb{N}, r < \dfrac{n}{m}\Big\} \cup \Big\{\underbrace{v + \ldots + v}_{m-\text{times}} > \underbrace{1 + \ldots + 1}_{n-\text{times}} : m, n \in \mathcal{N}, r > \dfrac{n}{m}\Big\}$. Clearly, p_r is satisfiable. Let $p_r^* \in S_1(\mathrm{RCF})$ with $p_r^* \supseteq p_r$. If $r \neq s$, then $p_r^* \neq p_s^*$. Thus, $|S_1(\mathrm{RCF})| = 2^{\aleph_0}$ and RCF has no saturated model.

Existence of Saturated Models

Next we think about the existence of κ-saturated models for $\kappa > \aleph_0$.

Theorem 4.3.12 *For all $\mathcal{M}$, there is a κ^+-saturated $\mathcal{M} \prec \mathcal{N}$ with $|N| \leq |M|^\kappa$.*

Proof

Claim For any $\mathcal{M}$ there is $\mathcal{M} \prec \mathcal{M}'$ such that $|M'| \leq |M|^\kappa$, and if $A \subseteq M$, $|A| \leq \kappa$ and $p \in S_1^{\mathcal{M}}(A)$, then p is realized in $\mathcal{M}'$.

We first note that

$$|\{A \subseteq M : |A| \leq \kappa\}| \leq |M|^\kappa$$

because for each such A there is f mapping κ onto A. Also, for each such A, $|S_1^{\mathcal{M}}(A)| \leq 2^\kappa$. Let $(p_\alpha : \alpha < |M|^\kappa)$ list all types in $S_1^{\mathcal{M}}(A)$ for $n < \omega$, $A \subseteq M$ with $|A| \leq \kappa$. We build an elementary chain $(\mathcal{M}_\alpha : \alpha < |M|^\kappa)$ as follows:

i) $\mathcal{M}_0 = \mathcal{M}$;

ii) $\mathcal{M}_\alpha = \bigcup_{\beta<\alpha} \mathcal{M}_\beta$ for α a limit ordinal;

iii) $\mathcal{M}_\alpha \prec \mathcal{M}_{\alpha+1}$ with $|M_{\alpha+1}| = |M_\alpha|$, and $\mathcal{M}_{\alpha+1}$ realizes p_α. By induction, we see that $|M_\alpha| \leq |M|^\kappa$ for all α. Let $\mathcal{M}' = \bigcup_{\alpha<|M|^\kappa} \mathcal{M}_\alpha$. Then, $|M'| \leq |M|^\kappa$ and $\mathcal{M}'$ is the desired model. This proves the claim.

We build an elementary chain $(\mathcal{N}_\alpha : \alpha < \kappa^+)$ such that each $|N_\alpha| \leq |M|^\kappa$ and

i) $\mathcal{N}_0 = \mathcal{M}$;

ii) $\mathcal{N}_\alpha = \bigcup_{\beta<\alpha} \mathcal{N}_\beta$ for α a limit ordinal;

iii) $\mathcal{N}_\alpha \prec \mathcal{N}_{\alpha+1}$, $|\mathcal{N}_\alpha| \leq |M|^\kappa$, and if $A \subseteq N_\alpha$ with $|A| \leq \kappa$ and $p \in S_n^{\mathcal{N}_\alpha}(A)$, then p is realized in $\mathcal{N}_{\alpha+1}$. This is possible by the claim because, by induction,

$$|N_\alpha|^\kappa \leq (|M|^\kappa)^\kappa = |M|^\kappa.$$

Let $\mathcal{N} = \bigcup_{\alpha<\kappa^+} \mathcal{N}_\alpha$. Because $\kappa^+ \leq |M|^\kappa$, N is the union of at most $|M|^\kappa$ sets of size $|M|^\kappa$ so $|N| \leq |M|^\kappa$. Suppose that $|A| \subseteq N$, $|A| \leq \kappa$, and

$p \in S_n^{\mathcal{N}}(A)$. Because κ^+ is a regular cardinal, there is $\alpha < \kappa^+$ such that $A \subset N_\alpha$ and p is realized in $\mathcal{N}_{\alpha+1} \prec \mathcal{N}$. Thus, $\mathcal{N}$ is κ^+-saturated.

Theorem 4.3.12 guarantees the existence of saturated models under suitable set-theoretic assumptions.

Corollary 4.3.13 *Suppose that $2^\kappa = \kappa^+$. Then, there is a saturated model of T of size κ^+. In particular, if the Generalized Continuum Hypothesis is true, there are saturated models of size κ^+ for all κ.*

For arbitrary T, some set-theoretic assumption is necessary. For example, if $|S_n(T)| = 2^{\aleph_0}$, then any $\aleph_0$-saturated model has size $2^{\aleph_0}$. If $\aleph_1 < 2^{\aleph_0}$, then there is no saturated model of size $\aleph_1$.

We can extend this a bit further.

Corollary 4.3.14 *Suppose that $\kappa \geq \aleph_1$ is regular and $2^\lambda \leq \kappa$ for $\lambda < \kappa$. Then, there is a saturated model of size κ. In particular, if $\kappa \geq \aleph_1$ is strongly inaccessible, then there is a saturated model of size κ.*

Proof Let $\mathcal{M} \models T$ with $|M| = \kappa$. If $\kappa = \lambda^+$ for $\lambda < \kappa$, then the corollary follows from Corollary 4.3.13. Thus, we may assume that κ is a limit cardinal. We build an elementary chain $(\mathcal{M}_\lambda : \lambda < \kappa, \lambda \text{ a cardinal})$. Each $\mathcal{M}_\lambda$ will have cardinality κ. Let $\mathcal{M}_0 = \mathcal{M}$.

Let $\mathcal{M}_\lambda = \bigcup_{\mu<\lambda} \mathcal{M}_\mu$ for λ a limit cardinal. Because $\mathcal{M}_\alpha$ is the union of fewer than κ models of size κ, $|M_\alpha| = \kappa$.

Given $\mathcal{M}_\lambda$, by Theorem 4.3.12 there is $\mathcal{M}_\lambda \prec \mathcal{M}_{\lambda^+}$ such that $\mathcal{M}$ is λ^+-saturated and $|M_{\lambda^+}| \leq \kappa^\lambda = \kappa$ (see Corollary A.17).

Let $\mathcal{N} = \bigcup \mathcal{M}_\lambda$. Because κ is a regular limit cardinal, $\kappa = \aleph_\kappa$ (see Proposition A.13). Thus, because κ is regular, if $A \subset N$ and $|A| < \kappa$, then there is $\lambda < \kappa$ such that $A \subset M_\lambda$. Thus, if $p \in S_n^{\mathcal{N}}(A)$, then p is realized in $\mathcal{M}_{\lambda^+} \prec \mathcal{N}$.

The assumption of regularity is necessary for some T. For example, suppose that $\mathcal{M} \models$ DLO with $|M| = \aleph_\omega$. We claim that $\mathcal{M}$ is not saturated. Let $M = \bigcup_{n<\omega} M_n$ where $|M_n| = \aleph_n$. If $\mathcal{M}$ is saturated, then for each $n < \omega$ we can find $a_n \in M$ such that $a_n > b$ for all $b \in M_n$. One more use of saturation allows us to find $c \in M$ such that $c > a_n$ for $n < \omega$. This is impossible. Similar arguments show that all saturated dense linear orders must have regular cardinality.

If T is κ-stable, then we can eliminate all assumptions about cardinal exponentiation.

Theorem 4.3.15 *Let κ be a regular cardinal. If T is κ-stable, then there is a saturated $\mathcal{M} \models T$ with $|M| = \kappa$. Indeed, if $\mathcal{M}_0 \models T$ with $|M_0| = \kappa$, then there is a saturated elementary extension $\mathcal{M}$ of $\mathcal{M}_0$ with $|M| = \kappa$.*

In particular, if T is ω-stable, then there are saturated models of size κ for all regular cardinals κ.

Proof We build an elementary chain $(\mathcal{M}_\alpha : \alpha < \kappa)$ where $|\mathcal{M}_\alpha| = \kappa$ such that:

i) $\mathcal{M}_0 \models T$ with $|\mathcal{M}_0| = \kappa$;

ii) $\mathcal{M}_\alpha = \bigcup_{\beta<\alpha} \mathcal{M}_\beta$ for α a limit ordinal;

iii) $\mathcal{M}_\alpha \prec \mathcal{M}_{\alpha+1}$ and if $p \in S_1^{\mathcal{M}_\alpha}(M_\alpha)$, then p is realized in $\mathcal{M}_{\alpha+1}$.

Because T is κ-stable, if $|\mathcal{M}_\alpha| = \kappa$, then $|S_1^{\mathcal{M}_\alpha}(M_\alpha)| = \kappa$. Thus, as in Theorem 4.3.12, we can find $\mathcal{M}_\alpha \prec \mathcal{M}_{\alpha+1}$ such that $|M_{\alpha+1}| = \kappa$ and $\mathcal{M}_{\alpha+1}$ realizes all types in $S_1^{\mathcal{M}_\alpha}(M_\alpha)$.

Let $\mathcal{M} = \bigcup \mathcal{M}_\alpha$. Because $\mathcal{M}$ is the union of κ models of size κ, $|M| = \kappa$. We claim that $\mathcal{M}$ is saturated. Let $A \subset M$ with $|A| < \kappa$. Because κ is regular, there is an $\alpha < \kappa$ such that $A \subseteq M_\alpha$. If $p \in S_1^{\mathcal{M}}(A)$, then there is $q \in S_1^{\mathcal{M}}(M_\alpha) = S_1^{\mathcal{M}_\alpha}(M_\alpha)$ with $p \subseteq q$. Because q is realized in $\mathcal{M}_{\alpha+1}$, p is realized in $\mathcal{M}$. Thus, $\mathcal{M}$ is saturated.

Saturated models of singular cardinality exist for ω-stable theories, but the proof is much more subtle. We prove this in Theorem 6.5.4.

Homogeneous and Universal Models

Although prime models elementarily embed into all models of T, saturated models embed all small models.

Definition 4.3.16 We say that $\mathcal{M} \models T$ is *κ-universal* if for all $\mathcal{N} \models T$ with $|N| < \kappa$ there is an elementary embedding of $\mathcal{N}$ into $\mathcal{M}$.

We say that $\mathcal{M}$ is *universal* if it is $|M|^+$-universal.

Lemma 4.3.17 *Let $\kappa \geq \aleph_0$. If $\mathcal{M}$ is κ-saturated, then $\mathcal{M}$ is κ^+-universal.*

Proof Let $\mathcal{N} \models T$ with $|N| \leq \kappa$. Let $(n_\alpha : \alpha < \kappa)$ enumerate N. Let $A_\alpha = \{n_\beta : \beta < \alpha\}$. We build a sequence of partial elementary maps $f_0 \subset f_1 \subset \ldots \subset f_\alpha \subset \ldots$ for $\alpha < \kappa$ with $f_\alpha : A_\alpha \to \mathcal{M}$.

Let $f_0 = \emptyset$ and, if α is a limit ordinal, let $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$.

Given $f_\alpha : A_\alpha \to \mathcal{M}$ partial elementary, let

$$\Gamma(v) = \{\phi(v, f_\alpha(\overline{a})) : \mathcal{M} \models \phi(n_\alpha, \overline{a})\}.$$

Because f_α is partial elementary and $|A_\alpha| < \kappa$, Γ is satisfiable and, by κ-saturation, realized by some b in $\mathcal{M}$. The $f_{\alpha+1} = f_\alpha \cup \{(n_\alpha, b)\}$ is the desired partial elementary map.

We have constructed $f = \bigcup f_\alpha$, an elementary embedding of $\mathcal{N}$ into $\mathcal{M}$.

Theorem 4.3.18 *Let $\kappa \geq \aleph_0$. The following are equivalent.*

i) $\mathcal{M}$ is κ-saturated.

ii) $\mathcal{M}$ is κ-homogeneous and κ^+-universal.

If $\kappa \geq \aleph_1$ i) and ii) are also equivalent to:

iii) $\mathcal{M}$ is κ-homogeneous and κ-universal.

Proof By Proposition 4.3.2 and Lemma 4.3.17, i) $\Rightarrow$ ii). Clearly, ii) $\Rightarrow$ iii). We argue that ii) $\Rightarrow$ i) and, if κ is uncountable, iii) $\Rightarrow$ i).

Let $A \subseteq M$ with $|A| < \kappa$, and let $p \in S_1^{\mathcal{M}}(A)$. We can find $\mathcal{N} \models \mathrm{Th}_A(\mathcal{M})$ such that $A \subseteq N$ and there is $a \in N$ realizing p. If $\kappa = \aleph_0$, then we can choose $\mathcal{N}$ with $|N| = \aleph_0$. If $\kappa \geq \aleph_1$, then we can choose $\mathcal{N}$ with $|N| < \kappa$. By assumption, there is an elementary embedding $f : \mathcal{N} \to \mathcal{M}$. Because $f|A$ is partial elementary, by κ-homogeneity, there is $b \in M$ such that

$$\mathrm{tp}^{\mathcal{M}}(b/A) = \mathrm{tp}^{\mathcal{M}}(f((a))/f(A)) = \mathrm{tp}^{\mathcal{N}}(a/A) = p.$$

Thus, $\mathcal{M}$ is κ-saturated.

Corollary 4.3.19 *$\mathcal{M}$ is saturated if and only if it is homogeneous and universal.*

Similar arguments can be used to show that there is at most one saturated model of any particular cardinality.

Theorem 4.3.20 *If $\mathcal{M}$ and $\mathcal{N}$ are saturated models of T of cardinality κ, then $\mathcal{M} \cong \mathcal{N}$.*

Proof By Corollary 4.3.5, we may assume that $\kappa \geq \aleph_1$. Let $(m_\alpha : \alpha < \kappa)$ enumerate $\mathcal{M}$ and $(n_\alpha : \alpha < \kappa)$ enumerate $\mathcal{N}$. We build a sequence of partial embeddings $f_0 \subset \ldots \subset f_\alpha \ldots$ for $\alpha < \kappa$ such that $m_\alpha \in \mathrm{dom}(f_{\alpha+1})$ and $n_\alpha \in \mathrm{img}(f_{\alpha+1})$. Let A_α denote the domain of f_α. We will have $|A_\alpha| \leq |\alpha| + \aleph_0 < \kappa$ for all α.

Let $f_0 = \emptyset$, and let $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$ for β a limit ordinal.

Suppose that f_α is partial elementary. By saturation, we can find $b \in N$ such that

$$\mathcal{N} \models \phi(b, f_\alpha(\overline{a})) \Leftrightarrow \mathcal{M} \models \phi(m_\alpha, \overline{a})$$

for all ϕ and all $\overline{a} \in A_\alpha$. Then $g_\alpha = f_\alpha \cup \{(m_\alpha, b)\}$ is partial elementary. Again by saturation, we can find $a \in M$ such that

$$\mathcal{N} \models \phi(n_\alpha, g(\overline{a})) \Leftrightarrow \mathcal{M} \models \phi(a, \overline{a})$$

for all ϕ and all $\overline{a} \in A_\alpha \cup \{m_\alpha\}$. Then, $f_{\alpha+1} = g_\alpha \cup \{(a, n_\alpha)\}$ is partial elementary and $f = \bigcup f_\alpha$ is an isomorphism from $\mathcal{M}$ to $\mathcal{N}$.

Lemma 4.3.17 and Theorem 4.3.20 are special cases of embedding and uniqueness results on homogeneous models generalizing Theorem 4.2.15.

Lemma 4.3.21 *Suppose that $\mathcal{N} \models T$ is κ-homogeneous where $\kappa \leq |N|$ and $\mathcal{M} \equiv \mathcal{N}$ such that every type in $S_n(T)$ realized in $\mathcal{M}$ is realized in $\mathcal{N}$ for $n < \omega$. If $A \subseteq M$ and $|A| \leq \kappa$, then there is a partial elementary map $f : A \to \mathcal{N}$.*

Proof We prove the claim by induction on $|A|$. Suppose that $|A|$ is finite. Let $A = \{a_1, \ldots, a_n\}$. Because every type realized in $\mathcal{M}$ is realized in

$\mathcal{N}$, there is $\overline{b} \in N^n$ such that $\mathrm{tp}^{\mathcal{M}}(\overline{a}) = \mathrm{tp}^{\mathcal{N}}(\overline{b})$. Then, $\overline{a} \mapsto \overline{b}$ is partial elementary.

Suppose that $|A| = \lambda \leq \kappa$ and the claim is true for sets of size $\mu < \lambda$. Let $(a_\alpha : \alpha < \lambda)$ enumerate A. For $\alpha < \lambda$, let $A_\alpha = \{a_\beta : \beta < \alpha\}$. We build a sequence of partial elementary maps $f_0 \subseteq \ldots \subseteq f_\alpha \subseteq \ldots$ where A_α is the domain of f_α for $\alpha < \lambda$.

Let $f_0 = \emptyset$. If α is a limit ordinal, let $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$.

Suppose that we are given f_α. Because $|A_{\alpha+1}| < \lambda$, by the induction assumption, there is a partial elementary $g : A_{\alpha+1} \to \mathcal{N}$. Let B be the image of A_α under f_α and let C be the image of A_α under g. Let $h = f_\alpha \circ g^{-1} : C \to B$. Because f_α and g are partial elementary, $h : C \to B$ is partial elementary. Because $\mathcal{N}$ is homogeneous, we can extend h to a partial elementary $h^* : C \cup \{g(a_\alpha)\} \to \mathcal{N}$. Let $b = h^*(g(a_\alpha))$, and let $f_{\alpha+1} = f_\alpha \cup \{(a_\alpha, b)\}$. Then, $f_{\alpha+1} = h^* \circ g$ is partial elementary.

Clearly, $f = \bigcup_{\alpha<\lambda} f_\alpha : A \to \mathcal{N}$ is partial elementary.

Corollary 4.3.22 *If $\mathcal{M} \models T$ is κ-homogeneous and realizes all types in $S_n(T)$ for all $n < \omega$, then $\mathcal{M}$ is κ-saturated.*

Proof By Lemma 4.3.21, $\mathcal{M}$ is κ^+-universal. Thus, by Theorem 4.3.18, $\mathcal{M}$ is saturated.

Theorem 4.3.23 *If $\mathcal{M} \equiv \mathcal{N}$ are homogeneous models of T of the same cardinality realizing the same types in $S_n(T)$ for all $n < \omega$, then $\mathcal{M} \cong \mathcal{N}$.*

Proof If $\mathcal{M}$ and $\mathcal{N}$ are countable, this is Theorem 4.2.15 so we assume that $\kappa = |M| = |N|$ is uncountable. We build an isomorphism $f : \mathcal{M} \to \mathcal{N}$ by a back-and-forth argument. Let $(a_\alpha : \alpha < \kappa)$ enumerate M. Let $(b_\alpha : \alpha < \kappa)$ enumerate N. We build a sequence of partial elementary maps $f_0 \subset \ldots \subset f_\alpha \subset \ldots$ such that the domain of f_α has cardinality at most $|\alpha| + \aleph_0 < \kappa$, a_α is in the domain of $f_{\alpha+1}$, and b_α is in the image of $f_{\alpha+1}$. Then, $f = \bigcup_{\alpha<\kappa} f_\alpha$ is the desired isomorphism.

Let $f_0 = \emptyset$. If α is a limit ordinal, then $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$. Let A be the domain of f_α, and let B be its image. By Lemma 4.3.21, there is a partial elementary $h : A \cup \{a_\alpha\} \to \mathcal{N}$. Let C be the image of A under h, and let $c = h(a_\alpha)$. As in the proof of Lemma 4.3.21, $f_\alpha \circ h^{-1} : C \to B$ is partial elementary and, because $\mathcal{N}$ is homogeneous, we can extend this map to $C \cup \{c\}$. Let b be the image of c under this extension. Then, $g_\alpha = f_\alpha \cup \{(a_\alpha, b)\}$ is partial elementary and a_α is in the domain.

Let D be the image of g_α. Then, $g_\alpha^{-1} : D \to \mathcal{M}$ is partial elementary. By a symmetric argument, we can find $a \in M$ such that $g_\alpha^{-1} \cup \{(b_\alpha, a)\}$ is partial elementary. Let $f_{\alpha+1} = g_\alpha \cup \{(a, b_\alpha)\}$.

Corollary 4.3.24 *i) The number of nonisomorphic homogeneous models of T of size κ is at most $2^{2^{\aleph_0}}$.*

ii) If T has a countable saturated model, then the number of homogeneous models of T of size κ is at most $2^{\aleph_0}$.

Proof Homogeneous models of cardinality κ are determined by the set of types realized. Because $|S_n(T)| \leq 2^{\aleph_0}$, the number of possible sets of types realized in a model is at most $2^{2^{\aleph_0}}$. If T has a saturated model, then $|S_n(T)| \leq \aleph_0$ for all $n < \omega$ and there are at most $2^{\aleph_0}$ possible sets of types.

Applications of Saturated Models

We conclude this section with several applications of saturated and homogeneous models. Saturated models are useful because we can do things in the model that we usually could only do in an elementary extension.

Proposition 4.3.25 *Let $\mathcal{M}$ be saturated. Let $A \subset M$ with $|A| < |M|$. Let $X \subset M^n$ be definable with parameters from M. Then, X is A-definable if and only if every automorphism of $\mathcal{M}$ that fixes A pointwise fixes the X setwise.*

Proof
($\Rightarrow$) If $\overline{a} \in A$, $X = \{\overline{b} \in M^n : \mathcal{M} \models \phi(\overline{b}, \overline{a})\}$ and σ is an automorphism of $\mathcal{M}$, then

$$\begin{aligned} \sigma(X) &= \{\overline{c} \in M^n : \mathcal{M} \models \phi(\sigma^{-1}(\overline{c}), \overline{a})\} \\ &= \{\overline{c} \in M^n : \mathcal{M} \models \phi(\overline{c}, \sigma(\overline{a}))\} \quad \text{because } \sigma \text{ is an automorphism} \\ &= \{\overline{c} \in M^n : \mathcal{M} \models \phi(\overline{c}, \overline{a})\} \qquad \text{because } \sigma(\overline{a}) = \overline{a} \\ &= X. \end{aligned}$$

($\Leftarrow$) Let $\psi(\overline{v}, \overline{m})$ define X, where $\overline{m} \in M^k$. Consider the type $\Gamma(\overline{v}, \overline{w}) =$

$$\{\psi(\overline{v}, \overline{m}), \neg\psi(\overline{w}, \overline{m})\} \cup \{\phi(\overline{v}) \Leftrightarrow \phi(\overline{w}) : \phi \text{ an } \mathcal{L}_A\text{-formula}\}.$$

Suppose that $\Gamma \cup \mathrm{Diag_{el}}(\mathcal{M})$ is satisfiable. Then, by saturation, we can find $(\overline{a}, \overline{b})$ realizing Γ in $\mathcal{M}$. Let f be the map that is the identity on A and sends $\overline{a}$ to $\overline{b}$. By choice of Γ, f is elementary. Because $\mathcal{M}$ is homogeneous, f extends to an automorphism σ of $\mathcal{M}$. But $\mathcal{M} \models \psi(\overline{a}, \overline{m}) \wedge \neg\psi(\overline{b}, \overline{m})$, thus $\overline{a} \in X$ and $\sigma(\overline{a}) = \overline{b} \notin X$, a contradiction. Thus, $\Gamma \cup \mathrm{Diag_{el}}(\mathcal{M})$ is not satisfiable.

Therefore, there are $\mathcal{L}_A$-formulas $\phi_1, \ldots \phi_m$ such that

$$\mathcal{M} \models \forall \overline{v} \forall \overline{w} \left(\bigwedge_{i=1}^{n} (\phi_i(\overline{v}) \leftrightarrow \phi_i(\overline{w})) \rightarrow (\psi(\overline{v}, \overline{m}) \leftrightarrow \psi(\overline{w}, \overline{m})) \right). \qquad (*)$$

For $\tau : \{1, \ldots, n\} \to 2$, let $\theta_\tau(\overline{v})$ be the formula

$$\bigwedge_{\tau(i)=1} \phi_i(\overline{v}) \wedge \bigwedge_{\tau(i)=0} \neg\phi_i(\overline{v}).$$

If $\theta_\tau(\overline{a})$ and $\theta_\tau(\overline{b})$, then, by $(*)$, $\overline{a} \in X$ if and only if $\overline{b} \in X$. Let $S = \{\tau : \{1, \ldots, m\} \to 2 : \mathcal{M} \models \theta_\tau(\overline{a})$ for some $\overline{a}$ in $M^n\}$. Then,

$$\overline{a} \in X \quad \text{if and only if} \quad \mathcal{M} \models \bigvee_{\tau \in S} \theta_\tau(\overline{v}).$$

Hence, X is definable with parameters from A.

Recall that $b \in M$ is *definable from* A if $\{b\}$ is A-definable. The next corollary is a simple consequence of Proposition 4.3.25.

Corollary 4.3.26 *Let $\mathcal{M}$ be saturated, and let $A \subset M$ with $|A| < |M|$. Then, b is definable from A if and only if b is fixed by all automorphisms of $\mathcal{M}$ that fix A pointwise.*

Proof By Proposition 4.3.25, $\{b\}$ is A-definable if and only if every automorphism that fixes A pointwise fixes the set $\{b\}$.

Recall that b is *algebraic* over A if there is a finite A-definable set X such that $b \in X$.

Proposition 4.3.27 *Let $\mathcal{M}$ be saturated. Let $A \subset M$ with $|A| < |M|$ and $b \in M$. The following are equivalent:*

i) b is algebraic over A;

ii) b has only finitely many images under automorphisms of $\mathcal{M}$ fixing A pointwise;

iii) $\mathrm{tp}^M(b/A)$ *has finitely many realizations.*

Proof

i)$\Rightarrow$ ii) Let X be a finite A-definable set with $b \in A$. By Proposition 4.3.25, any automorphism of $\mathcal{M}$ that fixes A pointwise permutes the elements of the finite set X.

ii)$\Rightarrow$ iii) If c realizes $\mathrm{tp}^{\mathcal{M}}(b/A)$, then, because $\mathcal{M}$ is homogeneous, there is an automorphism of $\mathcal{M}$ fixing A pointwise and mapping b to c. Thus, if b has only finitely many images under automorphisms fixing A, then $\mathrm{tp}^{\mathcal{M}}(b/A)$ has only finitely many realizations.

iii) $\Rightarrow$ i) Suppose that $p = \mathrm{tp}^{\mathcal{M}}(b/A)$ has exactly n realizations. Let

$$\Gamma = \mathrm{Th}_A(\mathcal{M}) \cup \{\phi(v_i) : \phi \in p, i = 0, \ldots, n\} \cup \{\bigwedge_{0 \le i < j \le n} v_i \neq v_j\}.$$

Because p has only n realizations in $\mathcal{M}$ and $\mathcal{M}$ is saturated, Γ is not satisfiable. Thus there are $\phi_1, \ldots, \phi_m \in p$ such that

$$\mathcal{M} \models \left(\bigwedge_{k=1}^{m} \bigwedge_{i=0}^{n} \phi_k(v_i)\right) \to \bigvee_{i \neq j} v_i = v_j.$$

In particular $\{c \in M : \mathcal{M} \models \bigwedge_{j=1}^{m} \phi_j(c)\}$ is an A-definable set of size n containing b, so b is algebraic over A.

Saturated models can be used to give a new test for quantifier elimination.

Proposition 4.3.28 *If $\mathcal{L}$ is a language containing a constant symbol and T is an $\mathcal{L}$-theory, then T has quantifier elimination if and only if whenever $\mathcal{M} \models T$, $A \subseteq M$, $\mathcal{N} \models T$ is $|M|^+$-saturated, and $f : A \to \mathcal{N}$ is a partial embedding, f extends to an embedding of $\mathcal{M}$ into $\mathcal{N}$.*

Proof

($\Rightarrow$) By quantifier elimination f is a partial elementary embedding. As in the proof of Lemma 4.3.17, we can extend f to an elementary embedding of $\mathcal{M}$ into $\mathcal{N}$.

($\Leftarrow$) We use the quantifier elimination criterion from Corollary 3.1.6. Suppose that $\mathcal{M}, \mathcal{N} \models T$, $\mathcal{A} \subseteq \mathcal{M} \cap \mathcal{N}$, and $\mathcal{M} \models \phi(b, \overline{a})$, where ϕ is quantifier-free, $\overline{a} \in A$, and $b \in M$. Let $\mathcal{N} \prec \mathcal{N}'$ be an $|M|^+$-saturated model of T. By assumption the identity map on A extends to an embedding $f : \mathcal{M} \to \mathcal{N}'$. Because f is the identity on A, $\mathcal{N}' \models \phi(f(b), \overline{a})$. Because $\mathcal{N} \prec \mathcal{N}'$, $\mathcal{N} \models \exists v\ \phi(v, \overline{a})$, as desired.

Quantifier Elimination for Differentially Closed Fields

We will show how to apply Proposition 4.3.28 in one very interesting case. A *derivation* on a commutative ring R is a map $\delta : R \to R$ such that

$$\delta(x + y) = \delta(x) + \delta(y)$$

and

$$\delta(xy) = x\delta(y) + y\delta(x).$$

We often write $a', a'', \ldots, a^{(n)}$ for $\delta(a), \delta(\delta(a)), \ldots$.

If (R, δ) is a differential ring, we form the ring of differential polynomials $R\{X\} = R[X, X', X'', \ldots, X^{(n)}, \ldots]$. There is a natural extension of the derivation δ to $R\{X\}$ where $\delta(X^{(n)}) = X^{(n+1)}$. For f in $R\{X\} \setminus R$, the order of f is the least n such that $f \in R[X, \ldots, X^{(n)}]$, whereas if $f \in R$ we say that f has order $-\infty$.

We will consider differential fields, which we always assume have characteristic zero.

Definition 4.3.29 We say that K is a *differentially closed field* if K is a differential field of characteristic zero such that if $f, g \in K\{X\} \setminus \{0\}$ and the order of f is less than the order of g, then there is $x \in K$ such that $f(x) = 0$ and $g(x) \neq 0$.

In particular, if f has order 0, there is $x \in K$ with $f(x) = 0$, so K is algebraically closed. We can give axioms for DCF, the theory of differentially closed fields, in the language $\mathcal{L} = \{+, -, \cdot, \delta, 0, 1\}$, where δ is a unary function symbol for the derivation. Our goal is to show that DCF has quantifier elimination.

Let $k \subseteq K$ be differential fields, we say that $a \in K$ is *differentially algebraic over* k if $f(a) = 0$ for some nonzero $f \in k\{X\}$. Otherwise, we say that a is *differentially transcendental* over k.

The next proposition summarizes some basic algebra of differential fields that we will need. We assume that all of our fields have characteristic zero. If $k \subset K$ are differential fields and $a \in K$, we let $k\langle a\rangle$ be the differential subfield of K generated by a over k.

Proposition 4.3.30 *Let $k \subset K$ be differential fields of characteristic zero.*

i) Suppose that $f(X, X', \ldots, X^{(n)}) \in k\{X\} \setminus 0$ and $a,b \in K$ such that $f(a) = f(b) = 0$, $a, \ldots, a^{(n-1)}$ are algebraically independent over k, $b, \ldots, b^{(n-1)}$ are algebraically independent over k, and $g(a) \neq 0$, $g(b) \neq 0$ for any g of order n of lower degree in $X^{(n)}$. Then, $k\langle a\rangle$ and $k\langle b\rangle$ are isomorphic over k.

ii) If $a \in K$ is differentially algebraic over k, then there is $f \in k\{X\} \setminus \{0\}$ such that $f(a) = 0$ and if $g \in k\{X\} \setminus \{0\}$ has lower order, then $g(a) \neq 0$. Moreover, we can choose f such that if $f(b) = 0$ and $g(b) \neq 0$ for any lower order g, then $k\langle a\rangle$ and $k\langle b\rangle$ are isomorphic over k.

iii) If $f \in k\{X\}$, there is a differential field $F \supset k$ and $a \in F$ such that $f(a) = 0$ and $g(a) \neq 0$ for all $g \in k\{X\} \setminus \{0\}$ where the order of g is less than the order of f.

Proof

i) Certainly, $k(a, \ldots, a^{(n)})$ and $k(b, \ldots, b^{(n)})$ are isomorphic as fields. We need only show that the isomorphism preserves the derivation. For $i < n$ we have $\delta(a^{(i)}) = a^{(i+1)}$ and $\delta(b^{(i)}) = b^{(i+1)}$. Because $f(a, \ldots, a^n) = 0$, we must have $\delta(f(a, \ldots, a^n)) = 0$, but an easy calculation shows that

$$\delta(f(a, \ldots, a^{(n)}) = f^\delta(a, \ldots, a^{(n)}) + \sum_{i=0}^{n} \frac{\partial f}{\partial X^{(i)}}(a, \ldots, a^n)\delta(a^{(i)}),$$

where f^δ is the polynomial obtained by differentiating the coefficients of f. Because $f(a, \ldots, a^{(n-1)}, Y)$ is irreducible,

$$\frac{\partial f}{\partial X^{(n)}}(a, \ldots, a^{(n)}) \neq 0.$$

Thus

$$\delta(a^{(n)}) = \frac{-f^\delta(a, \ldots, a^{(n)}) - \sum_{i=0}^{n-1} \frac{\partial f}{\partial X^{(i)}}(a, \ldots, a^{(n)})\delta(a^{(i)})}{\frac{\partial f}{\partial X^{(n)}}(a, \ldots, a^n)}.$$

Similarly,

$$\delta(b^{(n)}) = \frac{-f^\delta(b, \ldots, b^{(n)}) - \sum_{i=0}^{n-1} \frac{\partial f}{\partial X^{(i)}}(b, \ldots, b^{(n)})\delta(b^{(i)})}{\frac{\partial f}{\partial X^{(n)}}(b, \ldots, b^n)}.$$

Thus, the natural field isomorphism is a differential field isomorphism.

ii) Let n be minimal such that $a, a', \ldots, a^{(n)}$ are algebraically dependent over k and let $f(X, \ldots, X^n) \in k[X, \ldots, X^{(n)}]$ be of minimal degree such that $f(a, a', \ldots, a^{(n)}) = 0$. Clearly, $g(a) \neq 0$ for any $g \in k\{X\} \setminus \{0\}$ of order less than n.

Suppose that $f(b) = 0$ and $g(b) \neq 0$ for any lower order g. Then, $b, \ldots, b^{(n-1)}$ are algebraically independent over k and b_n is a solution to the irreducible polynomial $f(b, \ldots, b_{n-1}, Y)$. Thus, by i), $k\langle a \rangle$ and $k\langle b \rangle$ are isomorphic over k.

iii) Let n be the order of f. By taking an irreducible factor of f of maximal order, we may assume that f is irreducible. Let K_0 be the field obtained from k by first adding elements $a, a', \ldots, a^{(n-1)}$ algebraically independent over k. Let K be the algebraic extension of K_0 obtained by adding a solution $a^{(n)}$ to the irreducible algebraic equation $f(a, a', \ldots, a^{(n-1)}, Y) = 0$. We must extend the derivation δ from k to K. For $i < n$, let $\delta(a^{(i)}) = a^{(i+1)}$. As in i), we let

$$\delta(a^{(n)}) = \frac{-f^{\delta}(a, \ldots, a^{(n)}) - \sum_{i=0}^{n-1} \frac{\partial f}{\partial X^{(i)}}(a, \ldots, a^{(n)})\delta(a^{(i)})}{\frac{\partial f}{\partial X^{(n)}}(a, \ldots, a^{(n)})}.$$

Because $a, \ldots, a^{(n-1)}$ are algebraically independent over k, a satisfies no differential polynomial over k of order less than n.

Corollary 4.3.31 *If k is a differential field of characteristic zero, then there is $K \supseteq k$ with $K \models$ DCF.*

Proof If $f, g \in k\{X\} \setminus \{0\}$ with g of lower order than f, then by Proposition 4.3.30 iii) we can find $k_1 \supset k$ with $a \in k_1$ where $f(a) = 0$ and $g(a) \neq 0$. Iterating this process, we build $K \supset k$ differentially closed.

We can now prove quantifier elimination.

Theorem 4.3.32 DCF *has quantifier elimination.*

Proof Let K, L be differential closed fields where L is $|K|^+$-saturated. Let R be a differential subring of K, and let $f : R \to L$ be a differential ring embedding. We must show that f extends to an embedding of K into L. Because there is a unique extension of the derivation from R to its fraction field k, we may as well assume that $R = k$ is a field. By induction, it suffices to show that if $f : k \to L$ is a differential field embedding and $a \in K \setminus k$, there is a differential field embedding of $k\langle a \rangle$ into L extending f. Identifying k with $f(k)$, we may assume that $k \subset L$ and f is the identity on k. There are two cases to consider.

case 1: a is differentially algebraic over k.

Let f be as in Proposition 4.3.30 ii). Let n be the order of f. Let p be the type $\{f(v) = 0\} \cup \{g(v) \neq 0 : g$ is nonzero of order less than $n\}$. If

$g_1, \dots, g_m$ are nonzero differential polynomials of order less than n, then there is $x \in L$ such that $f(x) = 0$ and $\prod g_i(x) \neq 0$. Thus p is satisfiable. If $b \in L$ realizes p, then $b, b', \dots, b_{(n-1)}$ are algebraically independent over k; thus, by i), we can extend the embedding by sending a to b.

case 2: a is differentially transcendental over k.

We claim that there is $b \in L$ differentially transcendental over k. Let p be the type $\{f(v) \neq 0 : f \in k\{X\} \setminus 0\}$. Let $f_1, \dots, f_n \in k\{X\} \setminus \{0\}$. Let N be greater than the order of f_i for $i = 1, \dots, N$. Because L is differentially closed, there is $x \in L$ such that $x^{(N)} = 0$ and $\prod f_i(x) \neq 0$. Thus p is consistent and must be realized in L by some element b differentially transcendental over k. Because a and b are differentially transcendental over k, $k\langle a\rangle$ and $k\langle b\rangle$ are isomorphic to the fraction field of the differential polynomial ring $k\{X\}$ over k. In particular, we can extend the embedding by sending a to b.

Vaught's Two-Cardinal Theorem

We conclude this section with an application of homogeneous models that will be useful in Chapter 6. If $\mathcal{M}$ is an $\mathcal{L}$-structure and $\phi(v_1, \dots, v_n)$ is an $\mathcal{L}$-formula, we let $\phi(\mathcal{M}) = \{\overline{x} \in M^n : \mathcal{M} \models \phi(\overline{x})\}$.

Definition 4.3.33 Let $\kappa > \lambda \geq \aleph_0$. We say that an $\mathcal{L}$-theory T has a (κ, λ)-*model* if there is $\mathcal{M} \models T$ and $\phi(\overline{v})$ an $\mathcal{L}$-formula such that $|M| = \kappa$ and $|\phi(\mathcal{M})| = \lambda$.

(κ, λ)-models are an obstruction to κ-categoricity. If T is a theory in a countable language with infinite models, then an easy compactness argument shows that there is $\mathcal{M} \models T$ of cardinality κ where every $\emptyset$-definable subset of $\mathcal{M}$ has cardinality κ. If T also has a (κ, λ)-model, then T is not κ-categorical. Our main goal is the following theorem of Vaught.

Theorem 4.3.34 *If T has a (κ, λ)-model where $\kappa > \lambda \geq \aleph_0$, then T has an $(\aleph_1, \aleph_0)$-model.*

We will prove Theorem 4.3.34 by first showing that the existence of a (κ, λ)-model has interesting implications for the countable models of T.

Definition 4.3.35 We say that $(\mathcal{N}, \mathcal{M})$ is a *Vaughtian pair* of models of T if $\mathcal{M} \prec \mathcal{N}$, $M \neq N$, and there is an $\mathcal{L}_M$-formula ϕ such that $\phi(\mathcal{M})$ is infinite and if $\phi(\mathcal{M}) = \phi(\mathcal{N})$.

For example, if $\mathcal{M}$ and $\mathcal{N}$ are nonstandard models of Peano arithmetic and $\mathcal{N}$ is a proper elementary end extension of $\mathcal{M}$, then $(\mathcal{N}, \mathcal{M})$ is a Vaughtian pair. If a is any infinite element of $\mathcal{M}$, then the formula $v < a$ defines an infinite set containing no elements of $N \setminus M$.

Lemma 4.3.36 *If T has a (κ, λ)-model where $\kappa > \lambda \geq \aleph_0$, then there is $(\mathcal{N}, \mathcal{M})$ a Vaughtian pair of models of T.*

Proof Let $\mathcal{N}$ be a (κ, λ)-model. Suppose that $X = \phi(\mathcal{N})$ has cardinality λ. By the Löwenheim–Skolem Theorem, there is $\mathcal{M} \prec \mathcal{N}$ such that $X \subseteq M$ and $|M| = \lambda$. Because $X \subseteq M$, $(\mathcal{N}, \mathcal{M})$ is a Vaughtian pair.

We would like to show that if there is a Vaughtian pair, then there is a Vaughtian pair of countable models. In the right context, this is a simple Löwenheim–Skolem argument.

Let $\mathcal{L}^* = \mathcal{L} \cup \{U\}$, where U is a unary predicate symbol. If $\mathcal{M} \subseteq \mathcal{N}$ are $\mathcal{L}$-structures, we consider the pair $(\mathcal{N}, \mathcal{M})$ as an $\mathcal{L}^*$-structure by interpreting U as M.

If $\phi(v_1, \ldots, v_n)$ is an $\mathcal{L}$-formula, we define $\phi^U(\overline{v})$, the restriction of ϕ to U, inductively as follows:

i) if ϕ is atomic, then ϕ^U is $U(v_1) \wedge \ldots \wedge U(v_n) \wedge \phi$;
ii) if ϕ is $\neg\psi$, then ϕ^U is $\neg\psi^U$;
iii) if ϕ is $\psi \wedge \theta$, then ϕ^U is $\psi^U \wedge \theta^U$;
iv) if ϕ is $\exists v\ \psi$, then ϕ^U is $\exists v\ U(v) \wedge \psi^U$.

An easy induction shows that if $\mathcal{M} \subset \mathcal{N}$, $\overline{a} \in M^k$ and we view $(\mathcal{N}, \mathcal{M})$ as an $\mathcal{L}^*$-structure, then $\mathcal{M} \models \phi(\overline{a})$ if and only if $(\mathcal{N}, \mathcal{M}) \models \phi^U(\overline{a})$.

Lemma 4.3.37 *If $(\mathcal{N}, \mathcal{M})$ is a Vaughtian pair for T, then there is a Vaughtian pair $(\mathcal{N}_0, \mathcal{M}_0)$ where $\mathcal{N}_0$ is countable.*

Proof Let ϕ be an $\mathcal{L}_M$-formula such that $\phi(\mathcal{M})$ is infinite and $\phi(\mathcal{M}) = \phi(\mathcal{N})$. Let $\overline{m}_0$ be the parameters from M occurring in ϕ. By the Löwenheim–Skolem Theorem, there is $(\mathcal{N}_0, \mathcal{M}_0)$ a countable $\mathcal{L}^*$-structure such that $\overline{m} \in M_0$ and $(\mathcal{N}_0, \mathcal{M}_0) \prec (\mathcal{N}, \mathcal{M})$. Because $\mathcal{M} \prec \mathcal{N}$, for any formula $\psi(v_1, \ldots, v_k)$

$$(\mathcal{N}, \mathcal{M}) \models \forall \overline{v} \ \left(\left(\bigwedge_{i=1}^{k} U(v_i) \wedge \psi(\overline{v}) \right) \rightarrow \psi^U(\overline{v}) \right).$$

Because $(\mathcal{N}_0, \mathcal{M}_0) \prec (\mathcal{N}, \mathcal{M})$, these sentences are also true in $(\mathcal{N}_0, \mathcal{M}_0)$, so $\mathcal{N}_0 \prec \mathcal{M}_0$.

Let $\phi(\overline{v})$ be an $\mathcal{L}_M$-formula with infinitely many realizations in $\mathcal{M}$ and none in $\mathcal{N} \setminus \mathcal{M}$, witnessing that $(\mathcal{N}, \mathcal{M})$ is a Vaughtian pair. For each k, the sentences

$$\exists \overline{v}_1 \ldots \exists \overline{v}_k \ \left(\bigwedge_{i<j} \overline{v}_i \neq \overline{v}_j \wedge \bigwedge_{i=1}^{k} \phi(v_i) \right)$$

hold in $(\mathcal{N}, \mathcal{M})$, as do the sentences $\exists x\ \neg U(x)$ and

$$\forall \overline{v}\ (\phi(\overline{v}) \rightarrow \bigwedge U(v_i)).$$

Because these sentences also hold in $(\mathcal{N}_0, \mathcal{M}_0)$, this structure is also a Vaughtian pair.

We need one more lemma before proving Vaught's Theorem.

Lemma 4.3.38 *Suppose that $\mathcal{M}_0 \prec \mathcal{N}_0$ are countable models of T. We can find $(\mathcal{N}_0, \mathcal{M}_0) \prec (\mathcal{N}, \mathcal{M})$ such that $\mathcal{N}$ and $\mathcal{M}$ are countable, homogeneous, and realize the same types in $S_n(T)$. By Theorem 4.2.15 $\mathcal{M} \cong \mathcal{N}$.*

Proof

Claim 1 If $\overline{a} \in M_0$ and $p \in S_n(\overline{a})$ is realized in $\mathcal{N}_0$, then there is $(\mathcal{N}_0, \mathcal{M}_0) \prec (\mathcal{N}', \mathcal{M}')$ such that p is realized in $\mathcal{M}'$.

Let $\Gamma(\overline{v}) = \{\phi^U(\overline{v}, \overline{a}) : \phi(\overline{v}, \overline{a}) \in p\} \cup \mathrm{Diag_{el}}(\mathcal{N}_0, \mathcal{M}_0)$. If $\phi_1, \dots, \phi_m \in p$, then $\mathcal{N}_0 \models \exists \overline{v} \bigwedge \phi_i(\overline{v}, \overline{a})$, thus $\mathcal{M}_0 \models \exists \overline{v} \bigwedge \phi_i(\overline{v}, \overline{a})$ and $(\mathcal{N}_0, \mathcal{M}_0) \models \exists \overline{v} \bigwedge \phi_i^U(\overline{v}, \overline{a})$. Thus, $\Gamma(\overline{v})$ is satisfiable. Let $(\mathcal{N}', \mathcal{M}')$ be a countable elementary extension realizing Γ.

By iterating Claim 1, we can find $(\mathcal{N}_0, \mathcal{M}_0) \prec (\mathcal{N}^*, \mathcal{M}^*)$ countable such that if $\overline{a} \in M_0$ and $p \in S_n(\overline{a})$ is realized in $\mathcal{N}_0$, then p is realized in $\mathcal{M}^*$.

Claim 2 If $\overline{b} \in N_0$ and $p \in S_n(\overline{b})$, then there is $(\mathcal{M}_0, \mathcal{N}_0) \prec (\mathcal{N}', \mathcal{M}')$ such that p is realized in $\mathcal{N}'$.

Let $\Gamma(\overline{v}) = p \cup \mathrm{Diag_{el}}(\mathcal{N}_0, \mathcal{M}_0)$. If $\phi_1, \dots, \phi_m \in p$, then $\mathcal{N}_0 \models \exists \overline{v} \bigwedge \phi_i(\overline{v}, \overline{b})$; thus, we can find a countable elementary extension of $(\mathcal{N}_0, \mathcal{M}_0)$ realizing p.

We build an elementary chain of countable models

$$(\mathcal{N}_0, \mathcal{M}_0) \prec (\mathcal{N}_1, \mathcal{M}_1) \prec \dots$$

such that

i) if $p \in S_n(T)$ is realized in $\mathcal{N}_{3i}$, then p is realized in $\mathcal{M}_{3i+1}$;

ii) if $\overline{a}, \overline{b}, c \in \mathcal{M}_{3i+1}$ and $\mathrm{tp}^{\mathcal{M}_{3i+1}}(\overline{a}) = \mathrm{tp}^{\mathcal{M}_{3i+1}}(\overline{b})$, then there is $d \in M_{3i+2}$ such that $\mathrm{tp}^{\mathcal{M}_{3i+2}}(\overline{a}, c) = \mathrm{tp}^{\mathcal{M}_{3i+2}}(\overline{b}, d)$;

iii) if $\overline{a}, \overline{b}, c \in \mathcal{N}_{3i+2}$ and $\mathrm{tp}^{\mathcal{N}_{3i+2}}(\overline{a}) = \mathrm{tp}^{\mathcal{N}_{3i+2}}(\overline{b})$, then there is $d \in N_{3i+3}$ such that $\mathrm{tp}^{\mathcal{N}_{3i+3}}(\overline{a}, c) = \mathrm{tp}^{\mathcal{N}_{3i+3}}(\overline{b}, d)$.

i) and ii) are done by using the first claim to build elementary chains, iii) is done by using the second claim to build an elementary chain.

Let $(\mathcal{N}, \mathcal{M}) = \bigcup_{i<\omega} (\mathcal{N}_i, \mathcal{M}_i)$. Then, $(\mathcal{N}, \mathcal{M})$ is a countable Vaughtian pair. By i), $\mathcal{M}$ and $\mathcal{N}$ realize the same types. By ii) and iii), $\mathcal{M}$ and $\mathcal{N}$ are homogeneous and hence isomorphic by Theorem 4.2.15.

Proof of 4.3.34 Suppose that T has a (κ, λ)-model. By the lemmas above, we can find $(\mathcal{N}, \mathcal{M})$ a countable Vaughtian pair such that $\mathcal{M}$ and $\mathcal{N}$ are homogeneous models realizing the same types. Let $\phi(\overline{v})$ be an $\mathcal{L}_M$-formula with infinitely many realizations in M and none in $N \setminus M$.

We build an elementary chain $(\mathcal{N}_\alpha : \alpha < \omega_1)$, each $\mathcal{N}_\alpha$ is isomorphic to $\mathcal{N}$, and $(\mathcal{N}_{\alpha+1}, \mathcal{N}_\alpha) \cong (\mathcal{N}, \mathcal{M})$. In particular, $N_{\alpha+1} \setminus N_\alpha$ contains no elements satisfying ϕ.

Let $\mathcal{N}_0 = \mathcal{N}$. For α a limit ordinal, let $\mathcal{N}_\alpha = \bigcup_{\beta<\alpha} \mathcal{N}_\beta$. Because $\mathcal{N}_\alpha$ is a union of models isomorphic to $\mathcal{N}$, $\mathcal{N}_\alpha$ is homogeneous and realizes the same types as $\mathcal{N}$ so $\mathcal{N}_\alpha \cong \mathcal{N}$ by Theorem 4.2.15.

Given $\mathcal{N}_\alpha \cong \mathcal{N}$, because $\mathcal{N} \cong \mathcal{M}$ there is $\mathcal{N}_{\alpha+1}$ an elementary extension of $\mathcal{N}_\alpha$ such that $(\mathcal{N}, \mathcal{M}) \cong (\mathcal{N}_{\alpha+1}, \mathcal{N}_\alpha)$. Clearly, $\mathcal{N}_{\alpha+1} \cong \mathcal{N}$.

Let $\mathcal{N}^* = \bigcup_{\alpha<\omega_1} \mathcal{N}_\alpha$. Then, $|N^*| = \aleph_1$ and if $\mathcal{N}^* \models \phi(\overline{a})$, then $\overline{a} \in M$; thus, $\mathcal{N}^*$ is an $(\aleph_1, \aleph_0)$-model.

Corollary 4.3.39 *If T is $\aleph_1$-categorical, then T has no Vaughtian pairs and hence no (κ, λ) models for $\kappa > \lambda \geq \aleph_0$.*

If T is ω-stable, we can prove a partial converse to Vaught's Theorem.

Lemma 4.3.40 *Suppose that T is ω-stable, $\mathcal{M} \models T$, and $|M| \geq \aleph_1$. There is a proper elementary extension $\mathcal{N}$ of $\mathcal{M}$ such that if $\Gamma(\overline{w})$ is a countable type over M realized in $\mathcal{N}$, then $\Gamma(\overline{w})$ is realized in $\mathcal{M}$.*

Proof

Claim There is an $\mathcal{L}_M$-formula $\phi(v)$ such that $|[\phi(v)]| \geq \aleph_1$ and for all $\psi(v) \in \mathcal{L}_M$ either $|[\phi(v) \wedge \psi(v)]| \leq \aleph_0$ or $|[\phi(v) \wedge \neg\psi(v)]| \leq \aleph_0$.

Suppose not. Then for any $\mathcal{L}_M$-formula $\phi(v)$ with $|[\phi(v)]| \geq \aleph_1$, there is a formula $\psi(v)$ such that $[\phi(v) \wedge \psi(v)]$ and $[\phi(v) \wedge \neg\psi(v)]$ are both uncountable. Let $\phi_\emptyset$ be the formula $v = v$. Then $[\phi_\emptyset] = |M| \geq \aleph_1$. We can build an infinite tree of formulas $(\phi_\sigma : \sigma \in 2^{<\omega})$ such that for all $\sigma \in 2^{<\omega}$:

i) $|[\phi_\sigma]| \geq \aleph_1$;

ii) $[\phi_{\sigma,0}] \cap [\phi_{\sigma,1}] = \emptyset$.

As in Theorem 4.2.18 we can find a countable $A \subset M$ such that $|S_1^M(A)| = 2^{\aleph_0}$, contradicting ω-stability.

Let $\phi(v)$ be as above. We construct the type p of formulas that are true for "almost all" elements satisfying $\phi(v)$. Let $p = \{\psi(v) : \psi$ an $\mathcal{L}_M$-formula and $|[\phi(v) \wedge \psi(v)]| \geq \aleph_1\}$. If $\psi_1, \ldots, \psi_m \in p$, then $|[\phi(v) \wedge \bigvee \neg\psi_i(v)]| \leq \aleph_0$. Thus, $\bigwedge_{i=1}^m \psi(v) \in p$ and p is finitely satisfiable. Because $|[\phi(v)]| \geq \aleph_1$, for each $\mathcal{L}_M$-formula $\psi(v)$ exactly one of $\psi(v)$ and $\neg\psi(v)$ is in p. Thus, p is a complete type over M.

Let $\mathcal{M}'$ be an elementary extension of $\mathcal{M}$ containing c, a realization of p. By Theorem 4.2.20, there is $\mathcal{N} \prec \mathcal{M}'$ prime over $M \cup \{c\}$ such that every $\overline{a} \in N$ realizes an isolated type over $M \cup \{c\}$.

Let $\Gamma(\overline{w})$ be a countable type over $\mathcal{M}$ realized by $\overline{b} \in \mathcal{N}$. There is an $\mathcal{L}_M$-formula $\theta(\overline{w}, v)$ such that $\theta(\overline{w}, c)$ isolates $\mathrm{tp}^{\mathcal{N}}(\overline{b}/M \cup \{c\})$. Note that $\exists \overline{w}\ \theta(\overline{w}, v) \in p$ and

$$\forall \overline{w}\ (\theta(\overline{w}, v) \rightarrow \gamma(\overline{w})) \in p$$

for all $\gamma(\overline{w}) \in \Gamma$. Let

$$\Delta = \{\exists \overline{w}\ \theta(\overline{w}, v)\} \cup \{\forall \overline{w}\ (\theta(\overline{w}, v) \rightarrow \gamma(\overline{w})) : \gamma \in \Gamma\}.$$

Then, $\Delta \subset p$ is countable and, if c' realizes Δ, then $\exists \overline{w}\ \theta(\overline{w}, c')$, and if $\theta(\overline{b}', c')$, then $\overline{b}'$ realizes Γ.

Let $\delta_0(v), \delta_1(v), \ldots$ enumerate Δ. By choice of p, $|\{x \in M : \phi(x)\}| \geq \aleph_1$ and $|\{x \in M : \phi(x) \wedge \neg(\delta_0(x) \wedge \ldots \wedge \delta_n(x))\}| \leq \aleph_0$ for all $n < \omega$. Thus $|\{x \in M : \phi(x)$ and x realizes $\Delta\}| \geq \aleph_1$. Let $c' \in M$ realize Δ and choose $\overline{b}'$ such that $\mathcal{M} \models \theta(\overline{b}', c')$. Then, $\overline{b}'$ is a realization of Γ in $\mathcal{M}$.

Theorem 4.3.41 *Suppose that T is ω-stable and there is an $(\aleph_1, \aleph_0)$-model of T. If $\kappa > \aleph_1$, then there is a $(\kappa, \aleph_0)$-model of T.*

Proof Let $\mathcal{M} \models T$ with $|M| \geq \aleph_1$ such that $|\phi(\mathcal{M})| = \aleph_0$ and let $\mathcal{M} \prec \mathcal{N}$ be as in Lemma 4.3.40. The type $\Gamma(v) = \{\phi(v)\} \cup \{v \neq m : m \in M$ and $\mathcal{M} \models \phi(m)\}$ is a countable type omitted in $\mathcal{M}$ and hence in $\mathcal{N}$. Thus $\phi(\mathcal{N}) = \phi(\mathcal{M})$.

Iterating this construction, we build an elementary chain $(\mathcal{M}_\alpha : \alpha < \kappa)$ such that $\mathcal{M}_0 = \mathcal{M}$ and $\mathcal{M}_{\alpha+1} \neq \mathcal{M}_\alpha$ but $\phi(\mathcal{M}_\alpha) = \phi(\mathcal{M}_0)$. If $\mathcal{N} = \bigcup_{\alpha<\kappa} \mathcal{M}_\alpha$, then $\mathcal{N}$ is a $(\kappa, \aleph_0)$-model of T.

Without the assumption of ω-stability, Theorem 4.3.41 is false (see Exercise 5.5.7).

4.4 The Number of Countable Models

Throughout this section, T will be a complete theory in a countable language with infinite models.

For any infinite cardinal κ, we let $I(T, \kappa)$ be the number of nonisomorphic models of T of cardinality κ. In this section, we will look at the possible values of $I(T, \aleph_0)$. We have already considered a number of examples.

- $I(\mathrm{DLO}, \aleph_0) = 1$.
- In Exercise 2.5.28, we gave examples of T_n where $I(T_n, \aleph_0) = n$ for $n = 3, 4, \ldots,$.
- $I(\mathrm{ACF}_p, \aleph_0) = \aleph_0$.
- $I(\mathrm{RCF}, \aleph_0) = I(\mathrm{Th}(\mathbb{N}), \aleph_0) = 2^{\aleph_0}$.

Because there are at most $2^{\aleph_0}$ nonisomorphic countable models of T, there are two natural questions:

Can we have $I(T, \aleph_0) = 2$?

Can we have $\aleph_0 < I(T, \aleph_0) < 2^{\aleph_0}$?

Surprisingly, Vaught answered the first question negatively. If the Continuum Hypothesis is true, then the second question has a trivial negative answer. Vaught conjectured that the answer is negative even when the Continuum Hypothesis fails. This remains one of the deep open questions of model theory. Although Vaught's Conjecture has been proved for some special classes of theories (for example, Shelah [93] proved Vaught's Conjecture for ω-stable theories), the best general result is Morley's theorem that if $I(T, \aleph_0) > \aleph_1$, then $I(T, \aleph_0) = 2^{\aleph_0}$.

$\aleph_0$-*categorical Theories*

We begin by taking a closer look at $\aleph_0$-categorical theories. In particular, we show how to recognize $\aleph_0$-categoricity by looking at the type space.

Theorem 4.4.1 *The following are equivalent:*
i) T is $\aleph_0$-categorical.
ii) Every type in $S_n(T)$ is isolated for $n < \omega$.
iii) $|S_n(T)| < \aleph_0$ for all $n < \omega$.
iv) For each $n < \omega$, there is a finite list of formulas

$$\phi_1(v_1, \dots, v_n), \dots, \phi_m(v_1, \dots, v_n)$$

such that for every formula $\psi(v_1, \dots, v_n)$

$$T \models \phi_i(\overline{v}) \leftrightarrow \psi(\overline{v})$$

for some $i \le m$.

Proof

i) $\Rightarrow$ ii) If $p \in S_n(T)$ is nonisolated, then there is a countable $\mathcal{M} \models T$ omitting p. There is also a countable $\mathcal{N} \models T$ realizing p. Clearly, $\mathcal{M} \not\cong \mathcal{N}$ so T is not $\aleph_0$-categorical.

ii) $\Rightarrow$ iii) Suppose that $S_n(T)$ is infinite. For each $p \in S_n(T)$, let ϕ_p isolate p. Because $\bigcup_{p \in S_n(T)} [\phi_p] = S_n(T)$ and $S_n(T)$ is compact, there are $p_1, \dots, p_m$ such that $[\phi_{p_1}] \cup \dots \cup [\phi_{p_m}] = S_n(T)$. Because $[\phi_p] = \{p\}$, $S_n(T)$ is finite.

iii) $\Rightarrow$ iv) For each i, we can find a formula θ_i such that $\theta_i \in p_i$ and $\neg\theta_i \in p_j$ for $i \neq j$. Then, θ_i isolates p_i. For any formula $\psi(v_1, \dots, v_n)$,

$$T \models \psi(\overline{v}) \leftrightarrow \bigvee_{\psi \in p_i} \theta_i.$$

Thus, each ψ with free variables $v_1, \dots, v_n$ is equivalent to $\bigvee\limits_{i \in S} \theta_i$ for some $S \subseteq \{1, \dots, m\}$. There are at most 2^m such formulas.

iv) $\Rightarrow$ i) Let $\mathcal{M}$ be a countable model of T. If $\overline{a} \in M^n$, let $S_{\overline{a}} = \{i \le m : \mathcal{M} \models \phi_i(\overline{a})\}$. Then, $\text{tp}^{\mathcal{M}}(\overline{a})$ is isolated by

$$\bigwedge_{i \in S_{\overline{a}}} \phi_i(\overline{v}) \wedge \bigwedge_{i \notin S_{\overline{a}}} \neg\phi(\overline{v}).$$

Thus, $\mathcal{M}$ is atomic and hence, by Theorem 4.2.8, prime. Because there is a unique prime model, T is $\aleph_0$-categorical.

Theorem 4.4.1 tells us a great deal about definability in $\aleph_0$-categorical theories. Recall that b is algebraic over A if there is a formula $\phi(v, \overline{w})$ and $\overline{a} \in A$ such that $\mathcal{M} \models \phi(b, \overline{a})$ and $\{x \in M : \mathcal{M} \models \phi(x, \overline{a})\}$ is finite. Also, $\text{acl}(A) = \{b \in A : b \text{ is algebraic over } A\}$.

Corollary 4.4.2 *Suppose that T is $\aleph_0$-categorical. There is a function $f : \mathbb{N} \to \mathbb{N}$ such that if $\mathcal{M} \models T$, $A \subset M$, and $|A| \le n$, then $|\text{acl}(A)| \le f(n)$.*

Proof By Theorem 4.4.1, $|S_{n+1}(T)|$ is finite. Let $q_1, \ldots, q_k$ list all $n+1$-types. Let $X = \{i : q_i$ contains a formula $\phi(v, \overline{w})$ such that $\mathcal{M} \models \forall v_0, \ldots, v_N \ \bigwedge_{i=0}^{N} \phi(v_i, \overline{w}) \rightarrow \bigvee_{i<j\leq N} v_i = v_j$ for some $N\}$. For $i \in X$, let N_i be the least N such that some formula ϕ,

$$\forall v_0, \ldots, v_N \ \bigwedge_{i=0}^{N} \phi(v_i, \overline{w}) \rightarrow \bigvee_{i<j} v_i = v_j,$$

is in q_i.

If $a, b_1, \ldots, b_n \in M$ and a is algebraic over $\overline{b}$, then $(a, \overline{b})$ realizes some $q_i \in X$ and $|\{x : (x, \overline{b}) \text{ realizes } q_i\}| \leq N_i$. Thus,

$$|\mathrm{acl}(b_1, \ldots, b_n)| \leq \sum_{i \in X} N_i.$$

Let

$$f(n) = \sum_{i \in X} N_i.$$

Corollary 4.4.2 is very useful in understanding algebraic examples.

Corollary 4.4.3 *If F is an infinite field, then the theory of F is not $\aleph_0$-categorical.*

Proof By compactness, we can find an elementary extension K of F such that K contains a transcendental element t. Because $t, t^2, t^3, \ldots$ are distinct, $\mathrm{acl}(t)$ is infinite. Thus, by Corollary 4.4.2, $\mathrm{Th}(F)$ is not $\aleph_0$-categorical.

For groups, the situation is more interesting. We study groups in the multiplicative language $\mathcal{L} = \{\cdot, 1\}$. We say that a group G is *locally finite* if, for any finite $X \subseteq G$, the subgroup generated by X is finite.

Corollary 4.4.4 *Let G be an infinite group.*

i) If $\mathrm{Th}(G)$ is $\aleph_0$-categorical, then G is locally finite. Moreover, there is a number b such that if $g \in G$, then $g^n = 1$ for some $n \leq b$ (we say that G has bounded exponent*).*

ii) If G is an infinite Abelian group of bounded exponent, then $\mathrm{Th}(G)$ is $\aleph_0$-categorical.

Proof

i) By Corollary 4.4.2, there is a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that if $|X| \leq n$, the group generated by X has size at most $f(n)$. In particular, if $g \in G$, then $g^n = 1$ for some $n \leq f(1)$.

ii) Suppose that G is a countable abelian group of bounded exponent. Then, there are $q_1, \ldots, q_m$ distinct prime powers such that

$$G \cong (\mathbb{Z}/q_1\mathbb{Z})^{n_1} \oplus \ldots \oplus (\mathbb{Z}/q_k\mathbb{Z})^{n_k} \oplus \bigoplus_{i=1}^{\infty} \mathbb{Z}/q_{k+1}\mathbb{Z} \oplus \ldots \oplus \bigoplus_{i=1}^{\infty} \mathbb{Z}/q_m\mathbb{Z}$$

where $n_i \in \mathbb{N}$ for $i \le k$. Because G is infinite, we must have $k < m$.

Let $q_i = p_i^{l_i}$, where p_i is a prime. The group $(\mathbb{Z}/q_i\mathbb{Z})^{n_i}$ has $p_i^{n_i l_i} - p_i^{n_i(l_i-1)}$ elements of order exactly q_i. If $g \in (\mathbb{Z}/q_i\mathbb{Z})^{n_i}$ has order less than q_i, then there is $h \in (\mathbb{Z}/q_i\mathbb{Z})^{n_i}$ with $p_i h = g$ (i.e., g is p_i-divisible).

Let T be the theory with the following axioms:

i) the axioms for Abelian groups;

ii) $\forall x\ x^{\prod q_i} = 1$;

iii) there are $p_i^{n_i l_i} - p_i^{n_i(l_i-1)}$ elements of order exactly q_i that are not p_i-divisible for $i \le k$;

iv) there are infinitely many elements of order exactly q_i that are not p_i-divisible for $i > k$.

By the remarks above $G \models T$. If H is a countable model of T, then $H \cong G$. Thus, T is $\aleph_0$-categorical.

We now move on to Vaught's result that $I(T, \aleph_0) \neq 2$. We will use the next lemma, although we leave the proof for the exercises.

Lemma 4.4.5 *Let $\kappa \ge \aleph_0$. Let $A \subset M$ with $|A| < \kappa$. Let $\mathcal{M}_A$ be the $\mathcal{L}_A$-structure obtained from $\mathcal{M}$ by interpreting the new constant symbols in the natural way. If $\mathcal{M}$ is κ-saturated, then so is $\mathcal{M}_A$.*

Theorem 4.4.6 $I(T, \aleph_0) \neq 2$.

Proof Suppose that $I(T, \aleph_0) = 2$. By Corollary 4.3.8 ii), there is $\mathcal{N}$ a prime model of T and $\mathcal{M}$ a countable saturated model of T. Because T is not $\aleph_0$-categorical, by Theorem 4.4.1, there is a nonisolated type $p \in S_n(T)$ for some n. The type p is realized in $\mathcal{M}$ and omitted in $\mathcal{N}$. Let $\overline{a} \in M$ realize p. Let T^* be the $\mathcal{L}_{\overline{a}}$-theory of $\mathcal{M}_{\overline{a}}$ (in the notation of the previous lemma).

By Theorem 4.4.1, there are infinitely many T-inequivalent formulas in the free variables $v_1, \ldots, v_n$. As they are still T^*-inequivalent, T^* is not $\aleph_0$-categorical. By Lemma 4.4.5, $\mathcal{M}_{\overline{a}}$ is a saturated $\mathcal{L}_{\overline{a}}$-structure. Thus, by Corollary 4.3.8 i), T^* has a countable atomic model $\mathcal{A}$. Let $\mathcal{B}$ denote the $\mathcal{L}$-reduct of $\mathcal{B}$. Because $\mathcal{A} \models T^*$, $\mathcal{B}$ contains a realization of p, thus $\mathcal{B} \not\cong \mathcal{N}$. Because T^* is not $\aleph_0$-categorical, there is a nonisolated $\mathcal{L}_{\overline{a}}$-type. This type is not realized in $\mathcal{A}$. Thus $\mathcal{A}$ is not saturated. If $\mathcal{B}$ were saturated, then, by Lemma 4.4.5, $\mathcal{A}$ would be saturated. Thus, $\mathcal{B} \not\cong \mathcal{M}$ and $I(T, \aleph_0) \ge 3$.

Morley's Analysis of Countable Models

Next we prove Morley's theorem that if $I(T, \aleph_0) > \aleph_1$, then $I(T, \aleph_0) = 2^{\aleph_0}$. As in the proof of Theorem 2.4.15 , we will use infinitary logic to analyze countable models.

Definition 4.4.7 A *fragment* of $\mathcal{L}_{\omega_1,\omega}$ is a set of $\mathcal{L}_{\omega_1,\omega}$-formulas containing all first-order formulas and closed under subformulas, finite Boolean combinations, quantification, and change of free variables.

If F is a fragment of $\mathcal{L}_{\omega_1,\omega}$, we say that $\mathcal{M} \equiv_F \mathcal{N}$ if

$$\mathcal{M} \models \phi \text{ if and only if } \mathcal{N} \models \phi$$

for all sentences $\phi \in F$.

If F is a fragment of $\mathcal{L}_{\omega_1,\omega}$, we say that $p \subset F$ is an F*-type* if there is a countable $\mathcal{L}$-structure $\mathcal{M}$ and $a_1, \ldots, a_n \in M$ such that $p = \{\phi(v_1, \ldots, v_n) \in F : \mathcal{M} \models \phi(\overline{a})\}$. Let $S_n(F, T)$ be the set of all F-types realized by some n-tuple in some countable model of T.

We will count models by counting types for various fragments. If $|S_n(F,T)| = 2^{\aleph_0}$ for some countable fragment F, then, because a countable model can realize only countably many types, we must have $I(T, \aleph_0) = 2^{\aleph_0}$.

Next, we look at a case where we have the minimal number of types for all countable fragments.

Definition 4.4.8 We say that an $\mathcal{L}$-theory T is *scattered* if $|S_n(F,T)|$ is countable for all countable fragments F of $\mathcal{L}_{\omega_1,\omega}$ and all $n < \omega$.

In particular, if T is scattered, then for countable fragments F, there are only countably many $\equiv_F$-classes of countable models of T. We will show that if T is scattered, then $I(T, \aleph_0) \leq \aleph_1$.

Suppose that T is scattered. We build a sequence of countable fragments $(L_\alpha : \alpha < \omega_1)$ as follows. Let L_0 be all first-order $\mathcal{L}$-formulas. If α is a limit ordinal, then $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$.

Suppose that L_α is a countable fragment. For $p \in S_n(L_\alpha, T)$, let $\Phi_p(v_1, \ldots, v_n)$ be the $\mathcal{L}_{\omega_1,\omega}$-formula $\bigwedge\limits_{\phi \in p} \phi$. This is an $\mathcal{L}_{\omega_1,\omega}$-formula because L_α is countable. Let $L_{\alpha+1}$ be the smallest fragment containing Φ_p for $p \in S_n(L_\alpha, T)$, $n < \omega$. Because T is scattered, $L_{\alpha+1}$ is a countable fragment.

If $\mathcal{M}$ is a countable model of T and $a_1, \ldots, a_n \in M$, let $\mathrm{tp}_\alpha^{\mathcal{M}}(\overline{a}) \in S_n(L_\alpha, T)$ be the L_α-type realized by $\overline{a}$ in M.

Lemma 4.4.9 *For each countable $\mathcal{M} \models T$, there is an ordinal $\gamma < \omega_1$ such that if $\overline{a}, \overline{b} \in M^n$ and $\mathrm{tp}_\gamma^{\mathcal{M}}(\overline{a}) = \mathrm{tp}_\gamma^{\mathcal{M}}(\overline{b})$, then $\mathrm{tp}_\alpha^{\mathcal{M}}(\overline{a}) = \mathrm{tp}_\alpha^{\mathcal{M}}(\overline{b})$ for all $\alpha < \omega_1$.*

We call the least such γ the height *of $\mathcal{M}$.*

Proof Note first that if $\mathrm{tp}_\alpha^{\mathcal{M}}(\overline{a}) \neq \mathrm{tp}_\alpha^{\mathcal{M}}(\overline{b})$, then $\mathrm{tp}_\beta^{\mathcal{M}}(\overline{a}) \neq \mathrm{tp}_\beta^{\mathcal{M}}(\overline{b})$ for all $\beta > \alpha$. For $\overline{a}, \overline{b}$ in M^n, let

$$f(\overline{a}, \overline{b}) = \begin{cases} -1 & \mathrm{tp}_\alpha^{\mathcal{M}}(\overline{a}) = \mathrm{tp}_\alpha^{\mathcal{M}}(\overline{b}) \text{ for all } \alpha < \omega_1 \\ \alpha & \text{if } \alpha \text{ is least } \mathrm{tp}_\alpha^{\mathcal{M}}(\overline{a}) \neq \mathrm{tp}_\alpha^{\mathcal{M}}(\overline{b}). \end{cases}$$

Because $\mathcal{M}$ is countable, we can find $\gamma < \omega_1$ such that $\gamma > f(\overline{a}, \overline{b})$ for all $\overline{a}, \overline{b} \in M^n$ and $n < \omega$.

Lemma 4.4.10 *Suppose that $\mathcal{M}$ and $\mathcal{N}$ are countable models of T such that $\mathcal{M}$ has height γ and $\mathcal{M} \equiv_{L_{\gamma+1}} \mathcal{N}$. If $\overline{a}, \overline{b} \in N^n$ and $\mathrm{tp}_\gamma^{\mathcal{N}}(\overline{a}) = \mathrm{tp}_\gamma^{\mathcal{N}}(\overline{b})$, then $\mathrm{tp}_{\gamma+1}^{\mathcal{N}}(\overline{a}) = \mathrm{tp}_{\gamma+1}^{\mathcal{N}}(\overline{b})$.*

Proof Let $p = \mathrm{tp}_\gamma^{\mathcal{N}}(\overline{a}) = \mathrm{tp}_\gamma^{\mathcal{N}}(\overline{b})$ and let $\psi(\overline{v})$ be an $\mathcal{L}_{\gamma+1}$-formula. Let Θ be the $\mathcal{L}_{\gamma+1}$-formula

$$\forall \overline{v} \forall \overline{w} \ ((\Phi_p(\overline{v}) \wedge \Phi_p(\overline{w})) \to (\psi(\overline{v}) \leftrightarrow \psi(\overline{w}))).$$

Because γ is the height of $\mathcal{M}$, $\mathcal{M} \models \Theta$. Because $\mathcal{N} \equiv_{L_{\gamma+1}} \mathcal{M}$, $\mathcal{N} \models \Theta$. Thus $\mathrm{tp}_{\gamma+1}^{\mathcal{N}}(\overline{a}) = \mathrm{tp}_{\gamma+1}^{\mathcal{N}}(\overline{b})$.

Lemma 4.4.11 *If $\mathcal{M}$ and $\mathcal{N}$ are countable models of T such that $\mathcal{M}$ has height γ and $\mathcal{M} \equiv_{L_{\gamma+1}} \mathcal{N}$, then $\mathcal{M} \cong \mathcal{N}$.*

Proof Let $a_0, a_1, \ldots$ list M and let $b_0, b_1, \ldots$ list N. We build a sequence of finite partial embeddings $f_0 \subseteq f_1 \subseteq \ldots$ such that if $\overline{a}$ is the domain of f_n, then $\mathrm{tp}_\gamma^{\mathcal{M}}(\overline{a}) = \mathrm{tp}_\gamma^{\mathcal{N}}(f_n(\overline{a}))$. We will ensure that a_n is in the domain of f_{n+1} and b_n is in the image of f_{n+1}. Then $f = \bigcup f_n$ is the desired isomorphism.

Let $f_0 = \emptyset$. Suppose that $\overline{a}$ is the domain of f_n and $f_n(\overline{a}) = \overline{b}$. Let $p = \mathrm{tp}_\gamma^{\mathcal{M}}(\overline{a}, a_n)$. We must find $e \in N$ such that $\mathrm{tp}_\gamma^{\mathcal{N}}(\overline{b}, e) = p$. Because

$$\mathcal{M} \models \exists \overline{v} \exists w \bigwedge_{\phi \in p} \phi(\overline{v}, w)$$

and this is an $L_{\gamma+1}$-sentence,

$$\mathcal{N} \models \exists \overline{v} \exists w \bigwedge_{\phi \in p} \phi(\overline{v}, w).$$

Let $(\overline{c}, d) \in N$ realize p. Because $\overline{a}$ and $\overline{c}$ realize the same L_γ-type, $\overline{c}$ and $\overline{b}$ realize the same L_γ-type. By Lemma 4.4.10, $\overline{c}$ and $\overline{b}$ realize the same $L_{\gamma+1}$-type. Because

$$\mathcal{N} \models \exists w \bigwedge_{\phi \in p} \phi(\overline{c}, w)$$

and this is an $L_{\gamma+1}$-formula,

$$\mathcal{N} \models \exists w \bigwedge_{\phi \in p} \phi(\overline{b}, w).$$

Thus, there is $e \in N$ such that $p = \mathrm{tp}_\gamma^{\mathcal{N}}(\overline{b}, e)$.

By a symmetric argument, we can find $s \in M$ such that $\mathrm{tp}_\gamma^{\mathcal{M}}(\overline{a}, a_n, s) = \mathrm{tp}_\gamma^{\mathcal{N}}(\overline{b}, e, b_n)$. Let $f_{n+1} = f_n \cup \{(a_n, e), (s, b_n)\}$.

Theorem 4.4.12 *If T is scattered, then $I(T, \aleph_0) \leq \aleph_1$.*

Proof For each countable $\mathcal{M} \models T$, let $i(\mathcal{M}) = (\gamma, \mathrm{tp}^{\mathcal{M}}_{\gamma+1}(\emptyset))$, where γ is the height of $\mathcal{M}$. Note that $\mathcal{M} \equiv_{L_\alpha} \mathcal{N}$ if and only if $\mathrm{tp}^{\mathcal{M}}_{\alpha}(\emptyset) = \mathrm{tp}^{\mathcal{N}}_{\alpha}(\emptyset)$. By Lemma 4.4.11, if $\mathcal{M}$ and $\mathcal{N}$ are countable models of T, then $\mathcal{M} \cong \mathcal{N}$ if and only $i(\mathcal{M}) = i(\mathcal{N})$. There are only $\aleph_1$ possible heights and, for any given α, there are only $\aleph_0$ possibilities for $\mathrm{tp}^{\mathcal{M}}_{\alpha}(\emptyset)$. Thus $I(T, \aleph_0) \leq \aleph_1$.

To finish the proof of Morley's theorem, we will show that if T is not scattered, then $|S_n(F,T)| = 2^{\aleph_0}$ for some countable fragment F. Although this is a generalization of Theorem 4.2.11 i), complications arise because we do not have the Compactness Theorem in $\mathcal{L}_{\omega_1,\omega}$. The proof requires some ideas from descriptive set theory.

Suppose F is a fragment of $\mathcal{L}_{\omega_1,\omega}$. We will consider $\mathcal{L}$-structures where the universe of the models is ω. If $\mathcal{M} = (\omega, \ldots)$ is an $\mathcal{L}$-structure, the *F-diagram* of $\mathcal{M}$ is $\{\phi(v_0, \ldots, v_n) \in F : \mathcal{M} \models \phi(0, 1, \ldots, n)\}$.

We consider $D(F,T)$ the set of all possible F-diagrams of models of T. There is a natural bijection between the power set $\mathcal{P}(F)$ and 2^F, the set of all functions from F to $\{0,1\}$ (identifying a set with its characteristic function). Because $D(F,T)$ is a set of subsets of F, we can view $D(F,T)$ as a subset of 2^F. If we think of $\{0,1\}$ as the two-element space with the discrete topology, then we can give 2^F the product topology. The topology on 2^F has a basis of clopen sets of the form $\{f \in 2^F : \forall x \in F_0\ f(x) = \sigma(x)\}$ where $F_0 \subseteq F$ is finite and $\sigma : F_0 \to 2$. If F is countable, then 2^F is homeomorphic to 2^ω.

Lemma 4.4.13 *If F is a countable fragment of $\mathcal{L}_{\omega_1,\omega}$, then $D(F,T)$ is a Borel subset of 2^F.*[2]

Proof Let

$$\begin{aligned} E_0 &= \{f \in 2^F : f(\phi) = 1 \Leftrightarrow f(\neg\phi) = 0 \text{ for all } \phi \in F\} \\ &= \bigcap_{\phi \in F} \{f \in 2^F : (f(\phi) = 0 \wedge f(\neg\phi) = 1) \vee (f(\phi) = 1 \wedge f(\neg\phi) = 0)\}. \end{aligned}$$

Because E_0 is an intersection of clopen sets, E_0 is closed.

Let $E_1 = \{f \in 2^F : f(\exists v \phi(v)) = 1$ if and only if $f(\phi(v_i)) = 1$ for some i for all $\phi \in F$ with one free variable$\}$. If

$$E^+_{1,\phi} = \{f \in 2^F : f(\exists v\phi(v)) = 1\} \cap \bigcup_{i=0}^{\infty} \{f \in 2^F : f(\phi(v_i)) = 1\}$$

and

$$E^-_{1,\phi} = \{f \in 2^F : f(\exists v\phi(v)) = 0\} \cap \bigcap_{i=0}^{\infty} \{f \in 2^F : f(\phi(v_i)) = 0\},$$

[2] Recall that the collection of Borel subsets of 2^F is the smallest collection of sets containing the open sets and closed under complement and countable unions and intersections.

then

$$E_1 = \bigcap_{\phi \in F} (E^+_{1,\phi} \cup E^-_{1,\phi})$$

and E_1 is Borel.

If $\psi = \bigwedge_{i \in I} \phi_i$ and $\psi \in F$, let

$$E_{2,\psi} = \{f \in 2^F : f(\psi) = 1 \text{ if and only if } f(\phi_i) = 1 \text{ for all } i \in I\}.$$

Because I is countable, we argue as above that $E_{2,\psi}$ is Borel. Thus

$$E_2 = \bigcap \Big\{ E_{2,\psi} : \psi = \bigwedge_{i \in I} \phi_i \text{ and } \psi \in F \Big\}$$

is Borel. Similarly the following sets are Borel:

$E_3 = \{f \in 2^F : f(v_i = v_j) = 0 \text{ for all } i \neq j\}$,
$E_4 = \{f \in 2^F : f(v_i = v_i) = 1 \text{ for all } i\}$,
$E_5 = \{f \in 2^F : f(v_i = v_j \to v_j = v_i) = 1 \text{ for all } i, j\}$,
$E_6 = \{f \in 2^F : f((v_i = v_j \wedge v_j = v_k) \to v_i = v_k) = 1 \text{ for all } i, j, k\}$, and
$E_7 = \{f \in 2^F : f(\phi) = 1 \text{ for all } \phi \in T\}$.

Let $D = E_0 \cap \ldots \cap E_7$. Clearly, D is Borel. We claim that $D = D(F, T)$. It is easy to see that if $\mathcal{M} \models T$ with universe ω, then the F-diagram of $\mathcal{M}$ is in D.

Suppose that $f \in D$. We build an $\mathcal{L}$-structure $\mathcal{M}_f$ with universe ω. If R is an n-ary relation symbol of $\mathcal{L}$, then $(i_1, \ldots, i_n) \in R^{\mathcal{M}_f}$ if and only if $f(R(v_{i_1}, \ldots, v_{i_n})) = 1$. Let g be an n-ary function symbol of $\mathcal{L}$. Because $f \in E_7$, $f(\exists v g(v_{i_1}, \ldots, v_{i_n}) = v) = 1$. Because $f \in E_1$, $f(g(v_{i_1}, \ldots, i_{i_n}) = v_j) = 1$ for some j. Let $g^{M_f}(i_1, \ldots, i_n) = j$. Because $f \in D$, $f(g(v_{i_1}, \ldots, i_{i_n}) = v_k) = 0$ for $j \neq k$ and g^{M_f} is well-defined. Now, using the fact that $f \in D$, we can do an induction on formulas to show that

$$\mathcal{M}_f \models \phi(i_1, \ldots, i_n) \Leftrightarrow f(\phi(v_{i_1}, \ldots, v_{i_n})) = 1$$

for all $\phi \in F$. Thus, f is in $D(F, T)$.

We may also view $S_n(F, T)$ as a subset of 2^F. Although this set may not be Borel, it is not much more complicated.

We construct a continuous map Ψ such that $S_n(F, T)$ is the image of $D(F, T)$ under this map. For $f \in 2^F$, let $\Psi(f) \in 2^F$, where

$$\Psi(f)(\phi) = \begin{cases} 1 & \phi \text{ has free variable } v_0, \ldots, v_{n-1} \text{ and } f(\phi) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Because $\Psi(f)(\phi) = \Psi(g)(\phi)$ if $f(\phi) = g(\phi)$, Ψ is continuous. If $p \in S_n(F, T)$, then there is $\mathcal{M} \models T$ with universe ω such that $(0, 1, \ldots, n-1)$ realizes p in $\mathcal{M}$. Thus, the space of F-types $S_n(F, T)$ is the image of $D(F, T)$ under Ψ.

We now need a classical result from descriptive set theory.

Definition 4.4.14 If $|X| = \aleph_0$, we say that $Y \subseteq 2^X$ is *analytic* if there is a continuous map $\tau : 2^X \to 2^X$ and a Borel set $B \subseteq 2^X$ such that Y is the image of B under τ.

By the remarks above $S_n(F, T)$ is an analytic subset of 2^F for any countable fragment F.

Theorem 4.4.15 *Suppose that X is countable and $Y \subseteq 2^X$ is analytic. If $|Y| > \aleph_0$, then $|Y| = 2^{\aleph_0}$.*

Proof See [52] 14.13.

Theorem 4.4.16 *Let T be a complete theory in a countable language. If $I(T, \aleph_0) > \aleph_1$, then $I(T, \aleph_0) = 2^{\aleph_0}$.*

Proof For any countable fragment F, $S_n(F, T)$ is analytic. Thus, by Theorem 4.4.15, we either have $|S_n(F, T)| \leq \aleph_0$ or $|S_n(F, T)| = 2^{\aleph_0}$. If there is any countable fragment F, where $|S_n(F, T)| = 2^{\aleph_0}$, then $I(T, \aleph_0) = 2^{\aleph_0}$. If not, then T is scattered and, by Theorem 4.4.12, $I(T, \aleph_0) \leq \aleph_1$.

4.5 Exercises and Remarks

We assume throughout that $\mathcal{L}$ is a countable language and that T is an $\mathcal{L}$-theory with only infinite models.

Exercise 4.5.1 a) Let $\mathcal{M} = (X, <)$ be a dense linear order, let $A \subset M$ and $\overline{b}, \overline{c} \in M^n$ with $b_1 < \ldots < b_n$ and $c_1 < \ldots < c_n$. Show that $\mathrm{tp}^{\mathcal{M}}(\overline{a}/A) = \mathrm{tp}^{\mathcal{M}}(\overline{b}/A)$ if and only if $b_i < a \Leftrightarrow c_i < a$ and $b_i > a \Leftrightarrow c_i < a$ for all $i = 1, \ldots, n$ and $a \in A$. In particular, show that any two elements of X realize the same 1-type over $\emptyset$.

b) If $a, b \in \mathbb{Q}$, then $\mathrm{tp}^{\mathbb{Q}}(a/\mathbb{N}) = \mathrm{tp}^{\mathbb{Q}}(b/\mathbb{N})$ if and only if there is an automorphism σ of $\mathbb{Q}$ fixing $\mathbb{N}$ pointwise with $\sigma(a) = b$.

c) Let $A = \{1 - \frac{1}{n} : n = 1, 2, \ldots\} \cup \{2 + \frac{1}{n} : n = 1, 2, \ldots\}$. Show that 1 and 2 realize the same type over A, but there is no automorphism of $\mathbb{Q}$ fixing A pointwise sending 1 to 2.

Exercise 4.5.2 Let T be the theory of $(\mathbb{Z}, s)$ where $s(x) = x+1$. Determine the types in $S_n(T)$ for each n. Which types are isolated? Do the same for $(\mathbb{Z}, <, s)$.

Exercise 4.5.3 Recall that for $A \subset M$, $\mathrm{dcl}(A)$ denotes the definable closure of A (see Exercise 1.4.10). Show that if $\overline{a}, \overline{b} \in M^n$ and $\mathrm{tp}^{\mathcal{M}}(\overline{a}/A) = \mathrm{tp}^{\mathcal{M}}(\overline{b}/A)$, then $\mathrm{tp}^{\mathcal{M}}(\overline{a}/\mathrm{dcl}(A)) = \mathrm{tp}^{\mathcal{M}}(\overline{b}/\mathrm{dcl}(A))$.

Exercise 4.5.4 Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure, $A \subseteq M$, $b \in M$, and b is algebraic over A (see Exercise 1.4.11). Show that $\mathrm{tp}^{\mathcal{M}}(b/A)$ is isolated.

Exercise 4.5.5 Let K be an algebraically closed field and k be a subfield of K. What are the isolated types in $S_1^K(k)$?

Exercise 4.5.6 Let R be a real closed field. Show that 1-types over R correspond to cuts in the ordering $(R,<)$.

Exercise 4.5.7 Let T be a complete extension of Peano arithmetic. Show that $|S_1(T)| = 2^{\aleph_0}$. [Hint: Let p_n be the nth prime number. For $X \subseteq \mathbb{N}$, let $\Gamma_X(v) = \{p_n \text{ divides } v : n \in X\} \cup \{p_n \text{ does not divide } v : n \notin X\}$.]

Exercise 4.5.8 † If A is a commutative ring, then an ideal $P \subset A$ is *real* if whenever $a_1^2 + \ldots + a_n^2 \in P$, then $a_1, \ldots, a_n \in P$.

a) Show that a prime ideal P is real if and only if A/P is orderable.

Let $\mathrm{Spec}_r(A) = \{(P,<) : P \subset A$ is a real prime ideal and $<$ is an ordering of $A/P\}$. We call $\mathrm{Spec}_r(A)$ the *real spectrum* of A. If $a \in A$, let $X_a = \{(P,<) \in \mathrm{Spec}_r(A) : a/P > 0 \text{ in } A/P\}$. We topologize $\mathrm{Spec}_r(A)$ by taking the weakest topology in which the sets X_a are open.

If R is a real closed field, k is a subfield of R, and $p \in S_n^R(k)$, let $P_p = \{f \in k[X_1, \ldots, X_n] : f(v_1, \ldots, v_n) = 0 \in p\}$.

b) Show that P_p is a real prime ideal.

c) Show that we can order $k[\overline{X}]/P_p$ by $f(\overline{X})/P_p <_p g(\overline{X})/P_p$ if and only if $f(\overline{v}) < g(\overline{v}) \in p$. Thus $(P_p, <_p) \in \mathrm{Spec}_r(k[\overline{X}])$.

d) Show that $p \mapsto (P_p, <_p)$ is a continuous bijection between $S_n^R(k)$ and $\mathrm{Spec}_r(k[\overline{X}])$.

e) Show that $\mathrm{Spec}_r(k[\overline{X}])$ is compact.

f) What are the isolated types in $S_1^R(k)$?

Exercise 4.5.9 Let x and y be algebraically independent over $\mathbb{R}$. Order $\mathbb{R}(x,y)$ such that $x > r$ for all $r \in \mathbb{R}$ and $y > x^n$ for all $n > 0$. Let F be the real closure of $\mathbb{R}(x,y)$. Show that $\mathrm{tp}^F(x) = \mathrm{tp}^F(y)$, but there is no automorphism of F sending x to y.

Exercise 4.5.10 Suppose that $A \subseteq B$, $\theta(\overline{v})$ is a formula with parameters from A, and θ isolates $\mathrm{tp}^{\mathcal{M}}(\overline{a}/B)$. Then, θ isolates $\mathrm{tp}^{\mathcal{M}}(\overline{a}/A)$.

Exercise 4.5.11 Suppose that $A \subset M$, $\overline{a}, \overline{b} \in M$ such that $\mathrm{tp}^{\mathcal{M}}(\overline{a}, \overline{b}/A)$ is isolated. Show that $\mathrm{tp}^{\mathcal{M}}(\overline{a}/A, \overline{b})$ is isolated.

Combining this with Lemma s4.2.9 and 4.2.21, we have shown that $\mathrm{tp}^{\mathcal{M}}(\overline{a}, \overline{b}/A)$ is isolated if and only if $\mathrm{tp}^{\mathcal{M}}(\overline{a}/A, \overline{b})$ is isolated and $\mathrm{tp}^{\mathcal{M}}(\overline{b}/A)$ is isolated.

Exercise 4.5.12 Prove Lemma 4.1.9 iii).

Exercise 4.5.13 Let Δ be a set of $\mathcal{L}$-formulas closed under $\wedge, \vee, \neg$ and let $\mathcal{M}$ be an $\mathcal{L}$-structure. Let $S_n^{\Delta}(T) = \{\Sigma \subset \Delta : \Sigma \cup T$ is satisfiable and $\phi \in \Sigma$ or $\neg\phi \in \Sigma$ for all $\phi \in \Delta\}$.

a) Show that for all $p \in S_n^{\Delta}(T)$ there is $q \in S_n(T)$ with $p \subseteq q$.

b) Suppose that for each n and each $p \in S_n^{\Delta}(T)$ there is a unique $q \in S_n(T)$ with $p \subseteq q$. Show that for every $\mathcal{L}$-formula $\phi(\overline{v})$ there is $\psi(\overline{v}) \in \Delta$

such that $T \models \phi(\overline{v}) \leftrightarrow \psi(\overline{v})$. In particular, if every quantifier-free type has a unique extension to a complete type, then T has quantifier elimination.

Exercise 4.5.14 † We continue with the notation from Exercise 2.5.24. Suppose that p is a non-isolated n-type over $\emptyset$ and $c_1, \ldots, c_n$ are constants in $\mathcal{L}^*$. Let $D'_{p,\overline{c}} = \{\Sigma \in P : \neg\phi(\overline{c}) \in \Sigma \text{ for some } \phi(v_1, \ldots, v_n) \in p\}$.

a) Show that $D'_{p,\overline{c}}$ is dense.

b) Use a) and Exercise 2.5.24 to give another proof of Theorem 4.2.4.

c) Assume that Martin's Axiom is true (see Appendix A). Suppose that $\mathcal{L}$ is a countable language, T is an $\mathcal{L}$-theory, and X is a collection of nonisolated types over $\emptyset$ with $|X| < 2^{\aleph_0}$. Show that there is a countable $\mathcal{M} \models T$ that omits all of the types $p \in X$.

Exercise 4.5.15 We say that a linear order $(X, <)$ is $\aleph_1$-*like* if $|X| = \aleph_1$ but $|\{y : y < x\}| \leq \aleph_0$ for all $x \in X$.

Show that there is an $\aleph_1$-like model of Peano arithmetic.

Exercise 4.5.16 Let $\mathcal{L}_n = \{U_0, U_1, \ldots, U_n\}$, where $U_0, U_1, \ldots, U_n$ are unary predicates. Let T_n be the $\mathcal{L}_n$-theory that asserts that for each $X \subseteq \{0, \ldots, n\}$ there are infinitely many x such that $U_i(x)$ for $i \in X$ and $\neg U_i(x)$ for $i \notin X$.

a) For which κ is T_n κ-categorical?

b) Show that T_n is complete.

c) Show that T_n has quantifier elimination. [Remark: It is probably easiest to do this explicitly.]

Let $\mathcal{L} = \bigcup \mathcal{L}_n$. For X and Y finite subsets of $\mathbb{N}$, let $\Phi_{X,Y}$ be the sentence

$$\exists x \bigwedge_{i \in X} U_i(x) \wedge \bigwedge_{i \in Y} \neg U_i(x).$$

Let T be the $\mathcal{L}$-theory $\{\Phi_{X,Y} : X, Y \text{ disjoint finite subsets of } \mathbb{N}\}$.

d) Suppose that $\mathcal{M} \models T$. Show that $\mathcal{M} \models T_n$ for all n.

e) Show that T is complete and has quantifier elimination.

f) For $X \subset \mathbb{N}$, let $\Gamma_X = \{U_i(v) : i \in X\} \cup \{\neg U_i(v) : i \notin X\}$. Show that there is a unique 1-type p_X over $\emptyset$ with $p_X \supset \Gamma_X$.

g) Show that $X \mapsto p_X$ is a bijection between 2^ω and $S_1^{\mathcal{M}}(\emptyset)$.

h) Show that $S_1^{\mathcal{M}}(\emptyset)$ has no isolated points and hence has no prime model.

i) If 2^ω is given the product topology, then $X \mapsto p_X$ is a homeomorphism between 2^ω and $S_1^{\mathcal{M}}(\emptyset)$.

j) Describe all 2-types over $\emptyset$.

k) Show that T is κ-stable for all $\kappa \geq 2^{\aleph_0}$.

Exercise 4.5.17 Show that every algebraically closed field is homogeneous. Show that any uncountable algebraically closed field is saturated.

Exercise 4.5.18 Suppose that $\mathcal{M} = (R, +, \cdot, <, 0, 1)$ is a real closed field. Show that $\mathcal{M}$ is κ-saturated if and only if the ordering $(R, <)$ is κ-saturated.

Exercise 4.5.19 a) Show that the theory of $\mathbb{Z}$-groups is κ-stable for all $\kappa \geq 2^{\aleph_0}$.

b) Does the theory of $\mathbb{Z}$-groups have prime models over sets?

Exercise 4.5.20 Let $\mathcal{L} = \{E\}$ be the language with a single binary relation symbol. Let T be the theory of an equivalence relation where for each $n \in \omega$ there is a unique equivalence class of size n.

a) Show that T is ω-stable but not $\aleph_1$-categorical.

b) Exhibit a Vaughtian pair of models of T.

Exercise 4.5.21 Show that DLO is not κ-stable for any infinite κ.

Exercise 4.5.22 Let $\mathcal{L} = \{E_1, E_2, E_3, \ldots\}$, and let T be the theory asserting that:

i) each E_n is an equivalence relation where every equivalence class is infinite;

ii) if $xE_{i+1}y$, then xE_iy.

We say that $E_1, E_2, \ldots$ is a family of refining equivalence relations.

Let $T^2 \supset T$ be the theory that asserts that E_1 has two classes and each E_i class is the union of two infinite E_{i+1} classes.

Let $T^\infty \supset T$ be the theory that asserts that E_1 has infinitely many classes and each E_i class is the union of infinitely many infinite E_{i+1} classes.

For example, if E_n is the equivalence relation $f|n = g|n$ on ω^ω, then $(\omega^\omega, E_1, E_2, \ldots) \models T^\infty$ and $(2^\omega, E_1, E_2, \ldots) \models T^2$. Both T^2 and T^∞ are complete theories with quantifier elimination.

a) Show that T^2 is κ-stable for all $\kappa \geq 2^{\aleph_0}$.

b) Show that T^∞ is κ-stable if and only if κ such that $\kappa^{\aleph_0} = \kappa$.

Exercise 4.5.23 Suppose that $\mathcal{M}$ is interpretable in $\mathcal{N}$ and $\kappa \geq \aleph_0$.

a) Show that if $\mathcal{M}$ is κ-stable, then $\mathcal{N}$ is κ-stable.

b) Show that if $\mathcal{M}$ is κ-stable, then $\mathcal{M}^{\mathrm{eq}}$ is κ-stable.

c) Show that if $\mathcal{M}$ is κ-saturated, then $\mathcal{N}$ is κ-saturated.

Exercise 4.5.24 We say that $\mathcal{M} \models T$ is minimal if $\mathcal{M}$ has no proper elementary submodels.

a) Show that the field of algebraic numbers is a minimal model of ACF and that the field of real algebraic numbers is a minimal model of RCF.

b) Give an example of a theory with a prime model that is not minimal.

Exercise 4.5.25 Suppose that T is a theory in a countable language with a prime model $\mathcal{M}$ that is not minimal. We will show that T has an atomic model of size $\aleph_1$.

a) Show that there is an elementary embedding $j : \mathcal{M} \to \mathcal{M}$ such that $j(\mathcal{M}) \neq \mathcal{M}$.

b) Use a) to show that there is $\mathcal{M} \prec \mathcal{N}$, $\mathcal{M} \cong \mathcal{N}$ and $M \neq N$.

c) Show that if $\mathcal{M}_0 \prec \mathcal{M}_1 \prec \mathcal{M}_2 \ldots$ and each $\mathcal{M}_i \cong \mathcal{M}$, then $\bigcup \mathcal{M}_i \cong \mathcal{M}$. [Hint: Use the uniqueness of atomic models.]

d) Use b) and c) to construct an elementary chain $(\mathcal{M}_\alpha : \alpha < \omega_1)$ such that each $\mathcal{M}_\alpha \cong \mathcal{M}$ and $M_\alpha \neq M_{\alpha+1}$. Let $\mathcal{M}' = \bigcup_{\alpha<\omega_1} \mathcal{M}_\alpha$. Show that $\mathcal{M}'$ is atomic and $|M'| = \aleph_1$.

e) Show that if T is not $\aleph_0$-categorical, then T has a nonatomic model of size $\aleph_1$. Conclude that if T is $\aleph_1$-categorical, but not $\aleph_0$-categorical, then any prime model is minimal. (We will prove in Corollary 5.2.10 that $\aleph_1$-categorical theories are ω-stable, thus there always is a prime model.)

f) Give an example of a theory that is $\aleph_1$-categorical and $\aleph_0$-categorical but has a prime model that is not minimal.

Exercise 4.5.26 Let $\mathcal{L} = \{U, <\}$, where U is a unary predicate and $<$ is a binary relation symbol. Let T be the $\mathcal{L}$-theory extending DLO where U picks out a subset that is dense and has a dense complement. Let $\mathcal{M} \models T$ and let $A = U^{\mathcal{M}}$. Show that there is no prime model over A.

Exercise 4.5.27 Suppose that $A \subset \mathcal{M}$, $|A| \leq \aleph_0$, $\mathcal{M}_0$, and $\mathcal{M}_1$ are elementary submodels of $\mathcal{M}$ with $A \subseteq M_0 \cap M_1$, and $\mathcal{M}_0$ and $\mathcal{M}_1$ are prime model extensions of A. Then, $\mathcal{M}_0$ and $\mathcal{M}_1$ are isomorphic over A (i.e., there is an isomorphism $f : \mathcal{M}_0 \to \mathcal{M}_1$ that fixes A pointwise).

Exercise 4.5.28 Suppose that T is an o-minimal theory, $\mathcal{M} \models T$, and $A \subseteq M$. Show that the isolated types in $A \subseteq M$ are dense. Conclude that o-minimal theories have prime models over sets.

Exercise 4.5.29 Show that the union of an elementary chain of $\aleph_0$-homogeneous structures is $\aleph_0$-homogeneous.

Exercise 4.5.30 Show that if T is $\aleph_0$-categorical, then any homogeneous model is saturated. In particular, a dense linear order is saturated if and only if it is homogeneous.

Exercise 4.5.31 Show that if $\mathcal{M}$ is κ-saturated, then every infinite definable subset of M^k has cardinality at least κ.

Exercise 4.5.32 Prove Lemma 4.4.5.

Exercise 4.5.33 Suppose that $\mathcal{M}$ is κ-saturated, $A \subset M$, and $|A| < \kappa$. If $p \in S_n^{\mathcal{M}}(A)$ has only finitely many realizations in $\mathcal{M}$ and $\overline{a}$ realizes p, then $\overline{a} \in \mathrm{acl}(p)$.

Exercise 4.5.34 Suppose that $\mathcal{M}$ is κ-saturated, and $(\phi_i(\overline{v}) : i \in I)$ and $(\theta_j(\overline{v}) : j \in J)$ are sequences of $\mathcal{L}_M$-formulas such that $|I|, |J| < \kappa$ and

$$\mathcal{M} \models \bigvee_{i\in I} \phi_i(\overline{v}) \leftrightarrow \neg\left(\bigvee_{j\in J} \theta_j(\overline{v})\right).$$

Show that there are finite sets $I_0 \subseteq I$ and $J_0 \subseteq J$ such that

$$\mathcal{M} \models \bigvee_{i\in I} \phi_i(\overline{v}) \leftrightarrow \bigvee_{i\in I_0} \phi_i(\overline{v}).$$

Exercise 4.5.35 (Expandability of Saturated Models) Suppose that $\kappa \geq \aleph_0$ and $\mathcal{M}$ is a saturated $\mathcal{L}$-structure of cardinality κ. Let $\mathcal{L}^* \supset \mathcal{L}$ with $|\mathcal{L}^*| \leq \kappa$. Suppose that T is an $\mathcal{L}^*$-theory consistent with $\mathrm{Th}(\mathcal{M})$. We show that we can interpret the symbols in $\mathcal{L}^* \setminus \mathcal{L}$ to obtain an expansion $\mathcal{M}^*$ of $\mathcal{M}$ with $\mathcal{M}^* \models T$.

Let $\mathcal{L}^*_M$ be the language obtained by adding to $\mathcal{L}^*$ constants for every element of $\mathcal{M}$. Let $(\phi_\alpha : \alpha < \kappa)$ enumerate all $\mathcal{L}^*_M$-sentences. We build an increasing sequence of $\mathcal{L}^*$-theories $(T_\alpha : \alpha < \kappa)$ such that $T_\alpha \cup T \cup \mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$ is satisfiable and $|T_\alpha| < \kappa$ for all $\alpha < \kappa$ (indeed $|T_{\alpha+1}| \leq |T_\alpha| + 2$).

Let $T_0 = \emptyset$. For α a limit ordinal, let $T_\alpha = \bigcup_{\beta<\alpha} T_\beta$. Suppose that we have T_α such that $|T_\alpha| < \kappa$ and $T_\alpha \cup T \cup \mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$ is satisfiable.

a) Show that either $T_\alpha \cup \{\phi_\alpha\} \cup T \cup \mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$ is satisfiable or $T_\alpha \cup \{\neg\phi_\alpha\} \cup T \cup \mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$ is satisfiable.

b) Show that if ϕ_α is $\exists v\ \psi(v)$ and $T_\alpha \cup \{\phi_\alpha\} \cup T \cup \mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$ is satisfiable, then for some $a \in M$, $T_\alpha \cup \{\phi_\alpha, \psi(a)\} \cup T \cup \mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$ is satisfiable. [Hint: Let $A \subset M$ be all parameters from $\mathcal{M}$ occurring in formulas in $T_\alpha \cup \{\phi_\alpha\}$. Let $\Gamma(v)$ be all of the $\mathcal{L}_A$-consequences of $T_\alpha \cup \{\phi_\alpha, \psi(v)\} \cup T \cup \mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$. Show that $\Gamma(v)$ is satisfiable and hence, by saturation, must be realized by some a in $\mathcal{M}$. Show that $T_\alpha \cup \{\phi_\alpha, \psi(a)\} \cup T \cup \mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$ is satisfiable.]

c) Show that we can always choose $T_{\alpha+1}$ such that

i) $T_{\alpha+1} \cup T \cup \mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$ is satisfiable;

ii) either $\phi_\alpha \in T_{\alpha+1}$ or $\neg\phi_\alpha \in T_{\alpha+1}$;

iii) if $\phi_\alpha \in T_{\alpha+1}$ and ϕ_α is $\exists v\ \psi(v)$, then $\psi(a) \in T_{\alpha+1}$ for some $a \in M$;

iv) $|T_{\alpha+1}| \leq |T_\alpha| + 2 < \kappa$.

Let $T^* = \bigcup_{\alpha<\kappa} T_\alpha$.

d) Show that T^* is a complete $\mathcal{L}^*_M$-theory with the witness property and $T^* \supset T \cup \mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$. Let $\mathcal{N}$ be the canonical model of T^*. Show that as an $\mathcal{L}$-structure $\mathcal{N}$ is exactly $\mathcal{M}$. Thus, $\mathcal{N}$ is the desired expansion of $\mathcal{M}$ to a model of T.

Exercise 4.5.36 Let $\mathcal{L} = \{U_0, U_1, \ldots\} \cup \{s_0, s_1, \ldots\}$. We describe an $\mathcal{L}$-structure $\mathcal{M}$ with universe $\mathbb{N} \times \mathbb{Z}$. Let $U_i^{\mathcal{M}} = \{i\} \times \mathbb{Z}$ and

$$s_i((j,x)) = \begin{cases} (j,x) & \text{if } i \neq j \\ (j,x+1) & \text{if } i = j. \end{cases}$$

Let T be the full theory of $\mathcal{M}$. Basically, T is the theory of countably many copies of $(\mathbb{Z}, s)$.

a) Show that $|S_n(T)| = \aleph_0$ for all n. (Either show or assume that T has quantifier elimination.)

b) Show that $I(T, \aleph_0) = 2^{\aleph_0}$.

Exercise 4.5.37 ($\aleph_1$-saturation of Ultraproducts) Suppose that U is a non principal ultrafilter on ω. Let $(\mathcal{M}_0, \mathcal{M}_1, \ldots)$ be a sequence of $\mathcal{L}$-structures, and let $\mathcal{M}^* = \prod \mathcal{M}_i/U$. We will show that $\mathcal{M}^*$ is $\aleph_1$-saturated.

Let $A \subset M^*$ be countable. For each $a \in A$, choose $f_a \in \prod M_i$ such that $a = f_a/\sim$. Let $\Gamma(v) = \{\phi_i(v) : i < \omega\}$ be a set of $\mathcal{L}_A$-formulas such that $\Gamma(v) \cup \mathrm{Th}_A(\mathcal{M}^*)$ is satisfiable. By taking conjunctions, we may, without loss of generality, assume that $\phi_{i+1}(v) \to \phi_i(v)$ for $i < \omega$. Let $\phi_i(v)$ be $\theta_i(v, a_{i,1}, \ldots, a_{i,m_i})$, where θ_i is an $\mathcal{L}$-formula.

a) Let $D_i = \{n < \omega : \mathcal{M}_n \models \exists v \theta_i(v, f_{a_{i,1}}(n), \ldots, f_{a_{i,m_i}}(n))\}$. Show that $D_i \in U$.

b) Find $g \in \prod M_i$ such that if $i \leq n$ and $n \in D_i$, then

$$\mathcal{M}_n \models \theta_i(g(n), f_{a_{i,1}}(n), \ldots, f_{a_{i,m_i}}(n)).$$

c) Show that g realizes $\Gamma(v)$. Where do you use the fact that U is nonprincipal? Conclude that $\mathcal{M}^*$ is $\aleph_1$-saturated. Show that if the Continuum Hypothesis holds, then $\mathcal{M}^*$ is saturated.

Exercise 4.5.38 (Recursively Saturated Models) Let $\mathcal{L}$ be a recursive language. We say that $\mathcal{M}$ is *recursively saturated* if whenever $A \subset M$ is finite and Γ is a recursive (possibly incomplete) n-type over A, then Γ is realized in $\mathcal{M}$. In particular, every $\aleph_0$-saturated structure is recursively saturated.

a) Suppose that $\mathcal{N}$ is a countable model of T. Show that there is a countable recursively saturated $\mathcal{M}$ with $\mathcal{N} \prec \mathcal{M}$.

b) Show that if $\mathcal{M}$ is recursively saturated, then $\mathcal{M}$ is $\aleph_0$-homogeneous. [Hint: If $\mathrm{tp}(\overline{a}) = \mathrm{tp}(\overline{b})$, consider the set of formulas $\{\phi(v, \overline{b}) \leftrightarrow \phi(c, \overline{a}) : \phi$ an $\mathcal{L}$-formula$\}$.]

c) Show that if $\mathcal{M}_0 \prec \mathcal{M}_1 \prec \ldots$ is an elementary chain of recursively saturated models, then $\mathcal{M} = \bigcup_{n\in\omega} \mathcal{M}_n$ is recursively saturated.

d) Suppose $\mathcal{M}, \mathcal{N} \models T$ and such that $(\mathcal{M}, \mathcal{N})$ is a countable recursively saturated model of the theory of pairs of models of T (as in our proof of Vaught's Two-Cardinal Theorem). Show $\mathcal{M} \cong \mathcal{N}$. [Hint: Recall that countable $\aleph_0$-homogeneous models are isomorphic if and only if they realize the same types.] Use this to give a simplified proof of Vaught's Two-Cardinal Theorem.

e) Let $\mathcal{M}$ be a recursively saturated $\mathcal{L}$-structure. Suppose that $\mathcal{L}^* \supset \mathcal{L}$ is recursive and T is a recursive $\mathcal{L}^*$-theory such that $\mathrm{Diag}_{\mathrm{el}}(\mathcal{M}) \cup T$ is satisfiable. Show that there is an expansion of $\mathcal{M}^*$ of $\mathcal{M}$ such that $\mathcal{M}^* \models T$. [Hint: Follow the proof of expandability of saturated models.] Show that we can make $\mathcal{M}^*$ recursively saturated.

Exercise 4.5.39 (Robinson's Consistency Theorem) Let $\mathcal{L}_0$ and $\mathcal{L}_1$ be languages, and let $\mathcal{L} = \mathcal{L}_1 \cap \mathcal{L}_2$. Let T be a complete $\mathcal{L}$-theory and let $T_i \supset T$ be a satisfiable $\mathcal{L}_i$-theory for $i = 1, 2$.

a) Show that there is a recursively saturated structure $(\mathcal{M}_1, \mathcal{M}_2)$ where $\mathcal{M}_i \models T_i$ for $i = 1, 2$.

b) Let $\mathcal{N}_i$ be the $\mathcal{L}$-reduct of $\mathcal{M}_i$. Show that $(\mathcal{N}_1, \mathcal{N}_2)$ is still recursively saturated and that $\mathcal{N}_1 \cong \mathcal{N}_2$.

c) Conclude that we can view $\mathcal{M}_1$ and $\mathcal{M}_2$ as expansions of a single $\mathcal{L}$-structure and that $T_1 \cup T_2$ is satisfiable.

Exercise 4.5.40 † Let $\mathcal{M}$ be a nonstandard model of Peano arithmetic.

a) Let A be a finite subset of $\mathcal{M}$, and let Γ be a recursive type over A of bounded quantifier complexity (i.e., there is n such that all formulas in Γ have at most n-quantifiers). Show that Γ is realized in $\mathcal{M}$. [Hint: (see [51] §9). There is a formula $S(v,w)$ that is a truth definition for formulas with at most n quantifiers. In other words if $\ulcorner\phi\urcorner$ is the Gödel code for a formula $\phi(v_1,\dots,v_n)$ and $\ulcorner\overline{b}\urcorner$ codes a sequence $\overline{b}=(b_1,\dots,b_n)\in M^n$, then $\mathcal{M}\models S(\ulcorner\phi\urcorner,b)\leftrightarrow\phi(b_1,\dots,b_n)$. Because Γ is recursive, there is a formula $G(\ulcorner\phi\urcorner)$ if and only if $\phi(v,\overline{a})\in\Gamma$. Because Γ is satisfiable for all $n<\omega$

$$\mathcal{M}\models\exists b\forall m<n\ G(m)\rightarrow S(m,\ulcorner(b,\overline{a})\urcorner)\urcorner.$$

Apply overspill (Exercise 2.5.7).]

b) Let $(G,+,<,0)$ be the ordered additive group of $\mathcal{M}$. Use a) to show that G is a recursively saturated model of Presburger arithmetic.

Exercise 4.5.41 † (Tennenbaum's Theorem) If $\mathcal{M}$ is a nonstandard model of Peano arithmetic and $a\in M$, let $r(a)=\{n\in\mathbb{N}:p_n \text{ divides } a\}$, where p_n is the nth prime number. Let $SS(\mathcal{M})=\{r(a):a\in\mathcal{M}\}$. We call $SS(\mathcal{M})$ the *Scott set* of $\mathcal{M}$.

a) Suppose that $X\in SS(\mathcal{M})$ and Y is recursive in X, then $Y\in SS(\mathcal{M})$. [Hint: use Exercise 4.5.40.]

b) We say that $T\subseteq 2^{<\omega}$ is a *tree* if whenever $\sigma\in T$ and $\tau\subset\sigma$, then $\tau\in T$. We say that $f\in 2^\omega$ is an infinite *path* through T if $f|n\in T$ for all $n<\omega$. Show that if $X\in SS(\mathcal{M})$ and T is an infinite tree recursive in X, then there is $Y\in SS(\mathcal{M})$ and f an infinite path through T recursive in Y. [Hint: use Exercise 4.5.40.]

c) Let $\phi_0,\phi_1,\dots$ be a list of all partial recursive functions. We write $\phi_i(n)\downarrow=j$ if on input n Turing machine i halts with output j. Let $A=\{i:\phi_i(i)\downarrow=0\}$ and $B=\{i:\phi_i(i)\downarrow=1\}$. Show that there is no recursive set C such that $A\subseteq C$ and $B\cap C=\emptyset$. We call A and B *recursively inseparable.* [Hint: Suppose that ϕ_i is the characteristic function of C and ask whether $i\in C$.]

d) There is a recursive infinite tree $T\subseteq 2^{<\omega}$ with no recursive infinite paths. [Hint: Let $T=\{\sigma\in 2^{<\omega}:$ if $i<|\sigma|$ and Turing machine i on input i halts by stage $|\sigma|$ with output $j\in\{0,1\}$, then $\sigma(i)=j\}$. Show that if f is a recursive infinite path through T, then $C=\{i:f(i)=0\}$ contradicts c).]

e) We can find an isomorphic copy of $\mathcal{M}$ with universe ω. Thus, we may assume that $\mathcal{M}=(\omega,\oplus,\otimes)$. Show that $r(a)$ is recursive in $\oplus$ for all $a\in M$. Conclude that $\oplus$ is not recursive.

Exercise 4.5.42 Show that there is no Vaughtian pair of real closed fields.

Exercise 4.5.43 Let k be a differential field, $K \models \mathrm{DCF}$, and $k \subseteq K$. The ring of differential polynomials in $\overline{X} = (X_1, \ldots, X_n)$ is the ring

$$k\{X_1, \ldots, X_n\} = k[X_1, \ldots, X_n, X_1', \ldots, X_n', \ldots, X_1^{(m)}, \ldots, X_n^{(m)}, \ldots].$$

We extend the derivation from k to $k\{\overline{X}\}$ by letting $\delta(X_n^{(m)}) = X_n^{(m+1)}$. An ideal $I \subset k\{\overline{X}\}$ is called a differential ideal if $\delta(f) \in I$ whenever $f \in I$.

a) For $p \in S_n^K(k)$, let $I_p = \{f \in k\{\overline{X}\} : f(v_1, \ldots, v_n) = 0 \in p\}$. Show that I_p is a differential prime ideal.

b) Show that if $I \subset k\{\overline{X}\}$ is a differential prime ideal, then $I = I_p$ for some $p \in S_n^K(k)$. Thus, $p \mapsto I_p$ is a bijection between complete n-types over k and differential prime ideals in $k\{\overline{X}\}$.

c) The Ritt–Raudenbusch Basis Theorem (see [50]) asserts that every differential prime ideal in $k\{\overline{X}\}$ is finitely generated. Use this to show that DCF is ω-stable.

d) If $K \models \mathrm{DCF}$, we say that $X \subseteq K^n$ is *Kolchin closed* if X is a finite union of sets of the form $\{\overline{x} \in K^n : f_1(\overline{x}) = \ldots = f_m(\overline{x}) = 0\}$ where $f_1, \ldots, f_m \in K\{\overline{X}\}$. Prove that there are no infinite descending chains of Kolchin closed sets.

e) (Differential Nullstellensatz) Suppose that $K \models \mathrm{DCF}$, $P \subseteq K\{X_1, \ldots, X_n\}$ is a differential prime ideal and $g \in K\{\overline{X}\} \setminus P$. Show that there is $\overline{a} \in K^n$ such that $f(\overline{a}) = 0$ for all $f \in P$ but $g(\overline{a}) \neq 0$.

f) (Existence of Differential Closures) Suppose that k is a differentially closed field and $k \subseteq K \models \mathrm{DCF}$. We say that K is a *differential closure* of k if whenever $k \subseteq L$ and $L \models \mathrm{DCF}$ there is a differential field embedding of K into L fixing k. Show that every field has a differential closure. [Hint: Show that differential closures are prime model extensions.]

Exercise 4.5.44 Suppose that $\mathcal{L}$ is a countable language and T is an $\mathcal{L}$-theory. Let $C = \{T' \supseteq T : T' \text{ a complete } \mathcal{L}\text{-theory}\}$. Show that if $|C| \geq \aleph_1$, then $|C| = 2^{\aleph_0}$. Argue that if Vaught's Conjecture is true for complete theories, then it is also true for incomplete theories.

Exercise 4.5.45 Suppose $\mathcal{L}$ is a finite language with no function symbols and T is an $\mathcal{L}$-theory with quantifier elimination. Prove that T is $\aleph_0$-categorical.

Exercise 4.5.46 Describe all $\aleph_0$-categorical linear orders.

Remarks

The Omitting Types Theorem is due to Henkin and Orey , each of whom used it to prove the Completeness Theorem for ω-logic. Theorem 4.2.5 is due to MacDowell and Specker. Their proof uses an ultraproduct construction and works for uncountable models as well. See [51] §8.2 for further results on end extensions of models of arithmetic.

The results on prime models, atomic models, and countable saturated models are due to Vaught and appear in [99], one of the most elegant papers in model theory. Theorem 4.3.23 is due to Keisler, and the other basic results on saturated and homogeneous models are due to Morley and Vaught.

Recursively saturated models were introduced by Barwise and Schlipf. Many results that can be proved using saturated models have elegant proofs using recursively saturated models (see [22] §2.4 or [53]). Friedman showed the weak recursive saturation of nonstandard models of arithmetic and used it to prove the following result (see [51] §12.1).

Theorem 4.5.47 *If $\mathcal{M}$ is a countable nonstandard model of Peano arithmetic, then there is $\mathcal{I}$ a proper initial segment of $\mathcal{M}$ with $\mathcal{I} \cong \mathcal{M}$.*

The $\aleph_1$-saturation of ultraproducts is due to Keisler (see [22] §6.1 for generalizations).

Morley introduced ω-stable theories in his proof of Theorem 6.1.1. He also proved that ω-stable theories have prime model extensions. Shelah showed that there are three possibilities for $\{\kappa \geq \aleph_0 : T \text{ is } \kappa\text{-stable}\}$.

Theorem 4.5.48 *If T is a complete theory in a countable language, then one of the following holds:*
i) there are no cardinals κ such that T is κ-stable,
ii) T is κ-stable for all $\kappa \geq 2^{\aleph_0}$,
iii) T is κ-stable if and only if $\kappa^{\aleph_0} = \kappa$.

A proof of this theorem can be found in [7], [18] or [76]. If i) holds, we say that T is *unstable*; otherwise, we say that T is *stable*. If ii) holds, we say that T is *superstable*. By Theorem 4.2.18, every ω-stable theory is superstable. In Exercise 4.5.22, we gave an example of a superstable theory that is not ω-stable and a stable theory that is not superstable.

The saturated model test for quantifier elimination is due to Blum, who also axiomatized the theory of DCF, proved that DCF is ω-stable, and deduced from that the existence of differential closures.

In Theorem 5.2.15, we will examine another two-cardinal result. There are many interesting two cardinal questions, but most can not be answered in ZFC. The following Theorem gives several interesting examples.

Theorem 4.5.49 *Let $\mathcal{L}$ be a countable language and T an $\mathcal{L}$-theory.*
i) Assume that $V = L$.[3] *If $\kappa > \lambda \geq \aleph_0$ and T has a (κ, λ)-model, then T has a (μ^+, μ)-model for all infinite cardinals μ.*
ii) Assume that $V = L$. If T has a (κ^{++}, κ)-model for some infinite cardinal κ, then T has a (λ^{++}, λ)-model for all infinite cardinals λ.

[3] $V = L$ is Gödel's Axiom of Constructibility asserting that all sets are constructible (see [57] or [47]).

iii) If ZFC is consistent, then it is consistent with ZFC that there is a countable theory with an $(\aleph_1, \aleph_0)$*-model but no* $(\aleph_2, \aleph_1)$*-model.*

The first result was proved by Chang (see [22] 7.2.7) for regular μ under the weaker assumption that the Generalized Continuum Hypothesis holds. The general case is due to Jensen who also proved the second result (see [27] §VIII). The third result is due to Mitchell and Silver (see [47] §29).

The characterization of $\aleph_0$-categorical theories was proved independently by Ryll-Nardzewski, Engler, and Svenonius.

Although Vaught's Conjecture is open for arbitrary theories, we do know that it holds for several interesting classes of theories.

Theorem 4.5.50 *Vaught's Conjecture holds for:*
i) (Shelah [93]) *ω-stable theories;*
ii) (Buechler [19]) *superstable theories of finite U-rank;*
iii) (Mayer [69]) *o-minimal theories;*
iv) (Miller) *theories of linear orders with unary predicates;*
v) (Steel [98]) *theories of trees.*

See [100] for more on iv) and v).

Theorem 4.4.16 also follows from another powerful theorem in descriptive set theory. Consider the equivalence relation on $D(\mathcal{L}, T)$ given by fEg if and only if $\mathcal{M}_f \cong \mathcal{M}_g$. It is easy to argue that E is an analytic subset of $D(\mathcal{L}, T) \times D(\mathcal{L}, T)$. Burgess (see, for example, [98]) proved that any analytic equivalence relation on a Borel subset of 2^ω with at least $\aleph_2$ classes has $2^{\aleph_0}$ classes.

If ϕ is an $\mathcal{L}_{\omega_1,\omega}$-sentence, we can ask about the number of nonisomorphic countable models of ϕ. Burgess' Theorem shows that if there are at least $\aleph_2$ nonisomorphic models, then there are $2^{\aleph_0}$, but it is unknown whether there can be an $\mathcal{L}_{\omega_1,\omega}$-sentence with exactly $\aleph_1 < 2^{\aleph_0}$ models.

Questions around Vaught's Conjecture can be reformulated in a way that does not involve any model theory. We say that a topological space $\mathbf{X}$ is *Polish* if it is a complete separable metric space. Suppose that G is a Polish topological group and G acts continuously on a Borel subset X of a Polish space $\mathbf{X}$. For example, X could be $D(\mathcal{L}, T)$ and G could be the group of permutations of ω topologized by taking subbasic open sets $N_{n,m} = \{f : f(n) = m\}$. The Topological Vaught Conjecture asserts that if G has uncountably many orbits on X, then G has $2^{\aleph_0}$ orbits. See [9] for more on this topic.

Springer
http://www.springer.com/978-0-387-98760-6

Model Theory : An Introduction
Marker, D.
2002, VIII, 345 p., Hardcover
ISBN: 978-0-387-98760-6