

FUNDAMENTAL ONE-DIMENSIONAL VARIABLES

A. Gaussian

The PDF, CDF, and CF of a Gaussian RV $\mathbf{X} \in N_n(\bar{\mathbf{X}}, \sigma^2)$ are given in (1.1), (1.2), and (1.5) respectively with σ_x replaced by σ . For $\bar{\mathbf{X}} = 0$, the even moments of the components of $\mathbf{X}$ are given by

$$E\{X_i^{2k}\} = \overline{X_i^{2k}} = \frac{(2k)!}{k!} \left(\frac{1}{2}\sigma^2\right)^k, k \text{ integer} \quad (2.1)$$

All odd moments are equal to zero.

B. Rayleigh

1. $n = 1$

Since the square root always yields a positive quantity, then by definition $R = \sqrt{X^2} = |X|$, i.e., a single-sided Gaussian RV with PDF and CDF given by

$$p_R(r) = \frac{2}{\sqrt{2\pi}\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right), r \geq 0 \quad (2.2)$$

$$P_R(r) = 1 - 2Q\left(\frac{r}{\sigma}\right), r \geq 0 \quad (2.3)$$

Also, the moments of R are given by

$$E\{R^k\} = \frac{(2\sigma^2)^{k/2}}{\sqrt{\pi}} \Gamma\left(\frac{k+1}{2}\right), k \text{ integer} \quad (2.4)$$

2. $n = 2$

Here R corresponds to a conventional Rayleigh RV with PDF and CDF

$$p_R(r) = \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right), r \geq 0 \quad (2.5)$$

$$P_R(r) = 1 - \exp\left(-\frac{r^2}{2\sigma^2}\right), r \geq 0 \quad (2.6)$$

Also, the moments of R are given by

$$E\{R^k\} = (2\sigma^2)^{k/2} \Gamma\left(1 + \frac{k}{2}\right), k \text{ integer} \quad (2.7)$$

3. $n = 2m$

$$p_R(r) = \frac{2r^{2m-1}}{(2\sigma^2)^m (m-1)!} \exp\left(-\frac{r^2}{2\sigma^2}\right), r \geq 0 \quad (2.8)$$

$$P_R(r) = 1 - \exp\left(-\frac{r^2}{2\sigma^2}\right) \sum_{i=0}^{m-1} \frac{1}{i!} \left(\frac{r^2}{2\sigma^2}\right)^i, r \geq 0 \quad (2.9)$$

$$E\{R^k\} = (2\sigma^2)^{k/2} \frac{\Gamma\left(m + \frac{k}{2}\right)}{(m-1)!}, k \text{ integer} \quad (2.10)$$

4. $n = 2m + 1$

$$p_R(r) = \frac{2r^{2m}}{(2\sigma^2)^{m+1/2} \Gamma(m+1/2)} \exp\left(-\frac{r^2}{2\sigma^2}\right), r \geq 0 \quad (2.11)$$

$$P_R(r) = \text{-----} \quad (2.12)$$

$$E\{R^k\} = (2\sigma^2)^{k/2} \frac{\Gamma\left(m + \frac{k+1}{2}\right)}{\Gamma(m+1/2)}, k \text{ integer} \quad (2.13)$$

C. Rician

1. $n = 1$

$$p_R(r) = \frac{1}{\sqrt{2\pi}\sigma^2} \exp\left(-\frac{r^2 + a^2}{2\sigma^2}\right) \left[\exp\left(\frac{ar}{\sigma^2}\right) + \exp\left(-\frac{ar}{\sigma^2}\right) \right], r \geq 0 \quad (2.14)$$

$$P_R(r) = Q\left(\frac{a-r}{\sigma}\right) - Q\left(\frac{a+r}{\sigma}\right), r \geq 0 \quad (2.15)$$

$$E\{R^k\} = (2\sigma^2)^{k/2} \exp\left(-\frac{a^2}{2\sigma^2}\right) \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi}} {}_1F_1\left(\frac{k+1}{2}; \frac{1}{2}; \frac{a^2}{2\sigma^2}\right), k \text{ integer} \quad (2.16)$$

where ${}_1F_1(\alpha; \beta; \gamma)$ is the confluent hypergeometric function [2] and $a = |\bar{X}|$.

2. $n = 2$

Here R corresponds to a conventional Rician RV with parameter $a = \|\bar{\mathbf{X}}\|$.

$$p_R(r) = \frac{r}{\sigma^2} \exp\left(-\frac{r^2 + a^2}{2\sigma^2}\right) I_0\left(\frac{ra}{\sigma^2}\right), r \geq 0 \quad (2.17)$$

$$P_R(r) = 1 - Q_1\left(\frac{a}{\sigma}, \frac{r}{\sigma}\right), r \geq 0 \quad (2.18)$$

$$E\{R^k\} = (2\sigma^2)^{k/2} \exp\left(-\frac{a^2}{2\sigma^2}\right) \Gamma\left(1 + \frac{k}{2}\right) {}_1F_1\left(1 + \frac{k}{2}, 1; \frac{a^2}{2\sigma^2}\right), k \text{ integer} \quad (2.19)$$

where

$$Q_1(\alpha, \beta) = \int_{\beta}^{\infty} x \exp\left(-\frac{x^2 + \alpha^2}{2}\right) I_0(\alpha x) dx \quad (2.20)$$

is the first-order Marcum Q -function [8].²

3. $n = 2m$

$$p_R(r) = \frac{r^m}{\sigma^2 a^{m-1}} \exp\left(-\frac{r^2 + a^2}{2\sigma^2}\right) I_{m-1}\left(\frac{ra}{\sigma^2}\right), r \geq 0 \quad (2.21)$$

$$P_R(r) = 1 - Q_m\left(\frac{a}{\sigma}, \frac{r}{\sigma}\right), r \geq 0 \quad (2.22)$$

$$E\{R^k\} = (2\sigma^2)^{k/2} \exp\left(-\frac{a^2}{2\sigma^2}\right) \frac{\Gamma\left(m + \frac{k}{2}\right)}{(m-1)!} {}_1F_1\left(m + \frac{k}{2}; m; \frac{a^2}{2\sigma^2}\right), k \text{ integer} \quad (2.23)$$

where

$$Q_m(\alpha, \beta) = \frac{1}{\alpha^{m-1}} \int_{\beta}^{\infty} x^m \exp\left(-\frac{x^2 + \alpha^2}{2}\right) I_m(\alpha x) dx \quad (2.24)$$

is the generalized (m th-order) Marcum Q -function [8].

4. $n = 2m + 1$

$$\begin{aligned} p_R(r) = & \frac{1}{\sqrt{2\pi\sigma^2}} \left(\frac{r}{a}\right)^m \exp\left(-\frac{r^2 + a^2}{2\sigma^2}\right) \left[\exp\left(\frac{ra}{\sigma^2}\right) \sum_{i=0}^{m-1} \frac{(-1)^i (m+i-1)!}{i! (m-i-1)!} \left(\frac{\sigma^2}{2ra}\right)^i \right. \\ & \left. + (-1)^m \exp\left(-\frac{ra}{\sigma^2}\right) \sum_{i=0}^{m-1} \frac{(m+i-1)!}{i! (m-i-1)!} \left(\frac{\sigma^2}{2ra}\right)^i \right], r \geq 0 \end{aligned} \quad (2.25)$$

$$P_R(r) = \text{-----} \quad (2.26)$$

² More often than not in the literature, the subscript "1" identifying the order of the first-order Marcum Q -function is dropped from the notation. We shall maintain its identity in this text to avoid possible ambiguity with the two-dimensional Gaussian Q -function defined in Eq. (A.37) of Appendix A.

$$E\{R^k\} = (2\sigma^2)^{k/2} \exp\left(-\frac{a^2}{2\sigma^2}\right) \frac{\Gamma\left(m + \frac{k+1}{2}\right)}{\Gamma\left(m + \frac{1}{2}\right)} {}_1F_1\left(m + \frac{k+1}{2}; m + \frac{1}{2}; \frac{a^2}{2\sigma^2}\right), \quad (2.27)$$

k integer

D. Central Chi-Square

1. $n = 1$

$$p_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}y} \exp\left(-\frac{y}{2\sigma^2}\right), y \geq 0 \quad (2.28)$$

$$P_Y(y) = \text{-----} \quad (2.29)$$

$$\Psi_Y(\omega) = \left(\frac{1}{1 - 2j\omega\sigma^2} \right)^{1/2} \quad (2.30)$$

$$E\{Y^k\} = (2\sigma^2)^k \frac{\Gamma(k+1/2)}{\sqrt{\pi}}, k \text{ integer} \quad (2.31)$$

2. $n = 2m$

$$p_Y(y) = \frac{1}{2\sigma^2 \Gamma(m)} \left(\frac{y}{2\sigma^2} \right)^{m-1} \exp\left(-\frac{y}{2\sigma^2}\right), y \geq 0 \quad (2.32)$$

$$P_Y(y) = 1 - \exp\left(-\frac{y}{2\sigma^2}\right) \sum_{i=0}^{m-1} \frac{1}{i!} \left(\frac{y}{2\sigma^2} \right)^i, y \geq 0 \quad (2.33)$$

$$\Psi_Y(\omega) = \left(\frac{1}{1 - 2j\omega\sigma^2} \right)^m \quad (2.34)$$

Since the k th moment of a central chi-square RV with $2m$ degrees of freedom is equal to the $2k$ th moment of a Rayleigh RV of order $2m$, then it is straight-forward to obtain the moments of Y from (2.10) as

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$$E\{Y^k\} = (2\sigma^2)^k \frac{\Gamma(m+k)}{(m-1)!}, k \text{ integer} \quad (2.35)$$

3. $n = 2m + 1$

$$p_Y(y) = \frac{1}{2\sigma^2 \Gamma(m+1/2)} \left(\frac{y}{2\sigma^2}\right)^{m-1/2} \exp\left(-\frac{y}{2\sigma^2}\right), y \geq 0 \quad (2.36)$$

$$P_Y(y) = \text{-----} \quad (2.37)$$

$$\Psi_Y(s) = \left(\frac{1}{1-2js\sigma^2}\right)^{m+1/2} \quad (2.38)$$

$$E\{Y^k\} = (2\sigma^2)^k \frac{\Gamma\left(m+k+\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)}, k \text{ integer} \quad (2.39)$$

E. Noncentral Chi-Square

1. $n = 1$

$$p_Y(y) = \frac{1}{2\sqrt{2\pi\sigma^2 y}} \exp\left(-\frac{y+a^2}{2\sigma^2}\right) \left[\exp\left(\sqrt{\frac{a^2 y}{\sigma^4}}\right) + \exp\left(-\sqrt{\frac{a^2 y}{\sigma^4}}\right) \right], y \geq 0 \quad (2.40)$$

$$P_Y(y) = \text{-----} \quad (2.41)$$

$$\Psi_Y(\omega) = \left(\frac{1}{1-2j\omega\sigma^2}\right)^{1/2} \exp\left(\frac{j\omega a^2}{1-2j\omega\sigma^2}\right) \quad (2.42)$$

$$E\{Y^k\} = (2\sigma^2)^k \exp\left(-\frac{a^2}{2\sigma^2}\right) \frac{\Gamma(k+1/2)}{\sqrt{\pi}} {}_1F_1\left(k+\frac{1}{2}; \frac{1}{2}; \frac{a^2}{2\sigma^2}\right), k \text{ integer} \quad (2.43)$$

2. $n = 2m$

$$p_Y(y) = \frac{1}{2\sigma^2} \left(\frac{y}{a^2} \right)^{(m-1)/2} \exp\left(-\frac{y+a^2}{2\sigma^2}\right) I_{m-1} \left(\sqrt{\frac{a^2 y}{\sigma^4}} \right), y \geq 0 \quad (2.44)$$

$$P_Y(y) = 1 - Q_m \left(\frac{a}{\sigma}, \frac{\sqrt{y}}{\sigma} \right), y \geq 0 \quad (2.45)$$

$$\Psi_Y(\omega) = \left(\frac{1}{1-2j\omega\sigma^2} \right)^m \exp\left(\frac{j\omega a^2}{1-2j\omega\sigma^2} \right) \quad (2.46)$$

Since the k th moment of a noncentral chi-square RV with $2m$ degrees of freedom is equal to the $2k$ th moment of a Rician RV of order $2m$, then it is straightforward to obtain the moments of Y from (2.23) as

$$E\{Y^k\} = (2\sigma^2)^k \exp\left(-\frac{a^2}{2\sigma^2}\right) \frac{(m+k-1)!}{(m-1)!} {}_1F_1\left(m+k; m; \frac{a^2}{2\sigma^2}\right), k \text{ integer} \quad (2.47)$$

3. $n = 2m + 1$

$$p_Y(y) = \frac{1}{2\sigma^2} \left(\frac{y}{a^2} \right)^{(m-1/2)/2} \exp\left(-\frac{y+a^2}{2\sigma^2}\right) I_{m-1/2} \left(\sqrt{\frac{a^2 y}{\sigma^4}} \right), y \geq 0 \quad (2.48)$$

$$P_Y(y) = \text{-----} \quad (2.49)$$

$$\Psi_Y(\omega) = \left(\frac{1}{1-2j\omega\sigma^2} \right)^{m+1/2} \exp\left(\frac{j\omega a^2}{1-2j\omega\sigma^2} \right) \quad (2.50)$$

$$E\{Y^k\} = (2\sigma^2)^k \exp\left(-\frac{a^2}{2\sigma^2}\right) \frac{\Gamma\left(m+k+\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)} {}_1F_1\left(m+k+\frac{1}{2}; m+\frac{1}{2}; \frac{a^2}{2\sigma^2}\right), \quad (2.51)$$

k integer

F. Log-Normal

Let $X \in N_1(\bar{X}, \sigma^2)$. Then the PDF of $\gamma = 10^{X/10}$ is given by

$$p_{\gamma}(\gamma) = \frac{\xi}{\sqrt{2\pi}\sigma\gamma} \exp\left[-\frac{(10\log_{10}\gamma - \bar{X})^2}{2\sigma^2}\right], \gamma \geq 0 \quad (2.52)$$

$$P_{\gamma}(\gamma) = 1 - Q\left(\frac{10\log_{10}\gamma - \bar{X}}{\sigma}\right), \gamma \geq 0 \quad (2.53)$$

where $\xi = 10/\ln 10$ and $\bar{X}$ (dB) and σ^2 (dB) correspond to the mean and variance of $10\log_{10}\gamma$. The CF of γ is not obtainable in closed form but can be approximated by a Gauss-Hermite expansion as

$$\Psi_{\gamma}(\omega) \cong \frac{1}{\sqrt{\pi}} \sum_{n=1}^{N_p} H_{x_n} \exp\left(10^{(\sqrt{2}\alpha x_n + \bar{X})/10} j\omega\right) \quad (2.54)$$

where x_n are the zeros and H_{x_n} are the weight factors of the N_p -order Hermite polynomial and can be found in Table 25.10 of [2]. In addition, the moments of γ are given by

$$E\{\gamma^k\} = \exp\left[\frac{k}{\xi} \bar{X} + \frac{1}{2} \left(\frac{k}{\xi}\right)^2 \sigma^2\right], k \text{ integer} \quad (2.55)$$

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