

5

CHAPTER

Are Four Colors Really Enough?

5.1 INTRODUCTION: A SCHOOLBOY INVENTION

There have been few problems in mathematics over the centuries that have taken the popular imagination as much as the Four Color Problem. Probably the main reason for this is that it is something that can be explained to anybody in only a minute or two. Perhaps the most surprising thing about this problem is that it was invented by a schoolboy and not by a mathematician at all. It's a very human story — we'll mention honeymoons, school challenges, and a popular magazine later. We'll also mention its 124-year history and why some people are still working on it even after it's been solved. And then there is the famous link between this problem and Lewis Carroll's poem, "The Hunting of the Snark." We'll get to that too. After reading this chapter you might like to look at the overview [1], by Appel and Haken, of their proof of the Four Color Theorem, or, for a more complete treatment, see [6], [7], or [9].

5.2 THE FOUR COLOR PROBLEM

In 1852, a young student in England, Frederick Guthrie, was doing his geography homework. Correction: he was *supposed* to be doing his geography homework. Instead, he started doodling with a map of England which had all the counties marked on it. He started to color them in. He

decided it was worth coloring the counties so that any two counties that had a common border were given a different color. So counties like Kent and Sussex, which are next to each other in the south of England, always had to be colored differently.

Frederick was able to color his map with just *four* colors. And this gave rise to the problem: is it possible to color any map in four or fewer colors so that regions with a common boundary have different colors?

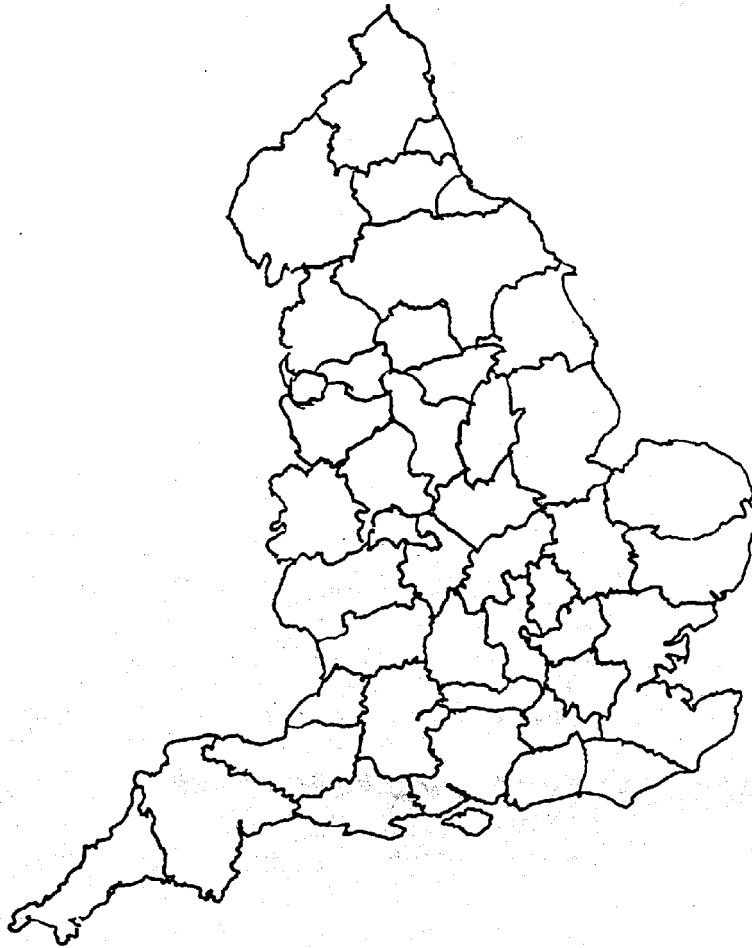
This is the famous Four Color Problem. It is an unusual problem in that it was discovered by a school student and yet has caused mathematicians so much trouble. It was a problem that was to haunt mathematicians for well over 100 years, and some of them are still worried by it today, even though it has been proved that the answer is yes.

• • • **BREAK 1**

- (1) We have reproduced a map of the counties of England for you below; repeat Fred Guthrie's experiment.
- (2) Invent your own set of countries by dividing a piece of paper into a finite set of regions (counties or countries). Will four colors suffice for your map? (Remember that no two adjacent regions can have the same color.)

Now it turns out that Fred had a brother called Frank, who was studying mathematics at University College, London, England. Fred told Frank, and Frank told his professor, the well-known Augustus de Morgan. No, de Morgan hadn't seen this problem before. And no, after doodling with it himself, he couldn't explain why you only needed four colors. However, he was pretty sure that you never needed a fifth color. All de Morgan's experimenting led him to the conclusion that four colors sufficed. But the fact that he couldn't justify it perplexed him somewhat. So he did what many a mathematician before and since has done, he got in touch with another mathematician. Unfortunately, he didn't have e-mail, so he wrote off to Sir William Rowan Hamilton. In that letter de Morgan gave an example of a map that actually *needed* four colors.

Hamilton, the Irish mathematician, who was responsible for some fine mathematics, including the invention of quaternions, was one of the few people who seemed not to get interested in the idea. He gave de Morgan a curt reply and got back to his research on trying to generalize complex numbers.



A map of the counties of England

• • • **BREAK 2**

- (1) Can you find a simple map that *has* to have four colors?
- (2) What are quaternions? Where are they written on a bridge and why?¹

¹See <http://www.maths.tcd.ie/pub/HistMath/People/Hamilton>

At this stage the Four Color Problem can really only be thought of as the Four Color Conjecture. The work on the Four Color Problem had now spread, and pretty well everyone was beginning to believe that maps only needed four colors. So the conjecture they all subscribed to was

The Four Color Conjecture *In coloring any map so that no two neighboring regions have the same color, at most four colors are needed.*

Despite his rebuff by Hamilton, de Morgan became the Four Color Conjecture's publicist. He even mentioned it in 1860 in the popular magazine, *The Athenaeum*. The fame of the problem had spread so far by 1886 that the headmaster of the English private school, Clifton College, issued the problem as a challenge to the school. He gave the boys a deadline by which time they should produce a solution of no more than one page of writing and one page of diagrams.

Now it is not clear whether the headmaster did this because he knew that the lawyer/mathematician Kempe had a proof, or because he knew that the proof was wrong. Kempe's "proof" is probably the most famous false proof in mathematical history. But we need a detour in order to be able to sneak up on it. Alfred Bray Kempe will reappear in Section 8.

5.3 GRAPHS

So what is a graph? In a way, it is unfortunate that in this chapter (and in Chapter 9, for that matter) a graph is not the thing that you are familiar with, something with x - and y -axes. We're interested in quite a different animal here, and, apparently, a somewhat simpler animal, too. The graphs we have to deal with have vertices, some or all of which may be joined by edges, which may be curved. But no vertex is joined to itself by an edge, and no pair of vertices is joined by more than one edge.

Have a look at Figure 1. This shows three graphs each with four vertices and five edges. Our convention is that these three graphs are the same graph in three different guises. In each case the vertices are shown as dots and the edges as lines joining the dots.

As we are going to consider all three graphs of Figure 1 to be the same, you can see that we don't worry about where we put the dots, or how plain or fancy the lines are that represent the edges. The way we'll decide whether two graphs are the same is if we can put the vertices on top of each other so that pairs of vertices that are joined by an edge always sit on top of corresponding pairs of vertices that are joined by an edge, and, moreover,

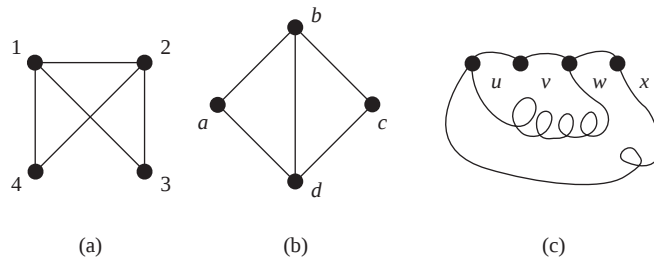


FIGURE 1 Three identical graphs

pairs of vertices that are not joined by an edge sit on top of corresponding pairs of vertices that are not joined by an edge.

Figures 1(a) and 1(b) represent the same graph because we can put the vertices 1 and b together, 2 and d together, 3 and c together, and 4 and a together. When we've done that, we can treat the edges as if they were made of elastic, and the edges 12, 13, 14, 23, and 24 sit on top of the edges bd , bc , ba , dc , and da .

In a similar way Figures 1(a) and 1(c) and Figures 1(b) and 1(c) are the same, too. We say that graphs that are the same in this way are *isomorphic*. If A and B are isomorphic graphs, we write $A \cong B$. So we are regarding isomorphic graphs as the same, or identical.

• • • BREAK 3

- (1) Show that all the graphs of Figure 1 are isomorphic.
- (2) Show that there is only one graph with one vertex, that there are two different graphs with two vertices, and four different graphs with three vertices. (Remember that you can't have an edge between a vertex and itself, and you can't have two or more edges between any two given vertices.)
- (3) Use the last problem to form a conjecture about the number of graphs with n vertices. Justify the conjecture or find a counterexample.

Now, just as sets have subsets, so graphs have *subgraphs*. If G is a graph, then H is a subgraph if (i) the vertices of H are a subset of the vertices of G and (ii) the edges of H are a subset of the edges of G . Clearly, the edges in (ii) must join vertices from (i). So the graph on vertices 1, 2, 3 with edges 12, 23 is a subgraph of the graph in Figure 1(a).

One useful concept relating to graphs is the **degree** of a vertex. This is simply the number of edges that go into that vertex. In Figure 1(b), vertex a has degree 2, vertex b has degree 3, vertex c has degree 2, and vertex d has degree 3. We write $\deg a = 2$, $\deg b = 3$, and so on. Notice that if H is a subgraph of G and a is a vertex of H , then the degree of a as a vertex of H may be smaller than its degree as a vertex of G .

It's also useful to describe a graph as **connected** if you can go between any two vertices by a sequence of vertices and edges with the edges joining consecutive vertices. Hence the graph in Figure 2(a) is connected, while the one in Figure 2(b) is not. There is no way of getting from u to v in Figure 2(b) using edges of that graph. The connected pieces of disconnected graphs are called **components**.

By a **cycle**, we mean a connected graph where each vertex has degree 2. Cycles can exist as subgraphs in which every vertex of the subgraph has degree 2. If a cycle has n vertices, then it is denoted by C_n . So in Figure 1(a), 1, 2, 3 form a cycle, but 1, 3, 4 don't. There are no cycles in Figure 2(a), but there are four cycles in Figure 2(b). Two of these cycles are isomorphic to C_3 , and two are isomorphic to C_4 . A cycle is sometimes called a **circuit** (see Chapter 7, Section 1.)

Now a connected graph which has no subgraphs which are cycles is called a **tree**. The graph in Figure 2(a) is a tree on nine vertices.

• • • BREAK 4

- (1) Show that no two of the following three graphs are isomorphic: C_8 ; the graph on eight vertices made up of two copies of C_4 ; the

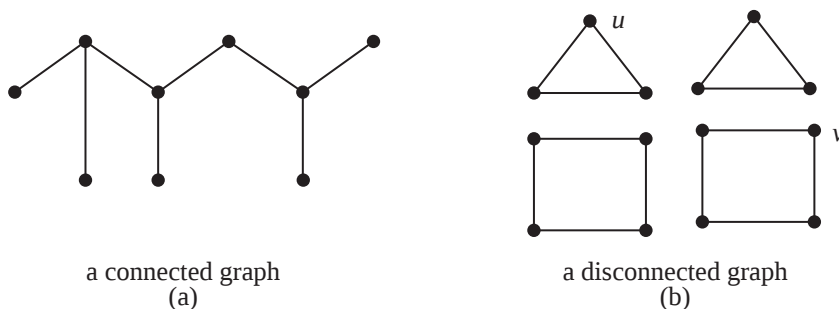


FIGURE 2

graph on eight vertices made up of one copy of C_5 and one copy of C_3 .

- (2) Find all trees with six vertices.

Showing that two graphs are isomorphic is a bit of a problem. In fact, the “isomorphism problem”, that is, finding a general method of efficiently deciding when two graphs are isomorphic, is an unsolved problem in graph theory. We can certainly formalize the idea of putting vertices on top of one another so that edges go on top of edges and non-edges go on top of non-edges. However, it’s not really worth giving a formal definition here. (See [4], [6], [9], [10], or any other graph theory book if you want to pursue this line.) It is useful to have a look at a few graphs, though, to see what makes them different. That way we’ll be able to pick out graphs that are obviously *not* isomorphic. And that will be enough for us at this stage.

So have a look at the two graphs G and H in Figure 3. Are they the same, isomorphic? Probably not. They don’t look very much alike. In what way do they differ, though?

• • • **BREAK 5**

- (1) Give a number of reasons why you think G and H might not be isomorphic.
- (2) The graphs G and H in Figure 3 are fairly easy to sort out. But how about the graphs K and L of Figure 4. Are they isomorphic or not? What do you think?

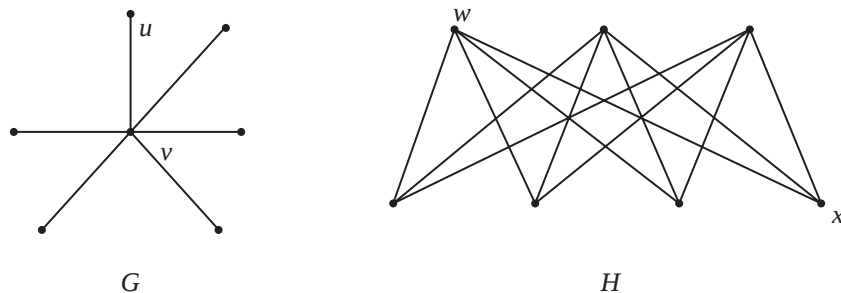


FIGURE 3

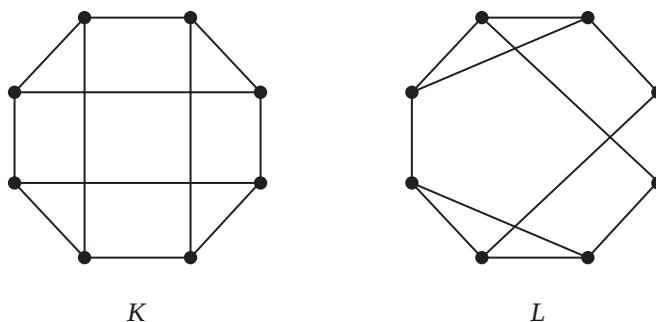


FIGURE 4

One of the things that distinguishes G from H is the number of edges. Plainly, G has only 6 edges. On the other hand, H is comparatively edge-rich with 12 edges.

In the same vein, G has some vertices of degree one. (It has six such vertices.) In fact, $\deg u = 1$ and $\deg v = 6$. It should be clear from Figure 3 and the “putting vertices on top of each other” definition of isomorphism that isomorphic graphs have corresponding vertices with the same degree.

Since G has six vertices of degree 1 and one vertex of degree 6, then any graph isomorphic to G also has to have six vertices of degree 1 and one vertex of degree 6. The graph H obviously is not one of these graphs. Hence $G \not\cong H$.

So let’s apply the degree test to the two graphs K and L of Figure 4. In this case, the test fails to distinguish between the graphs. Every vertex in K has degree 3, and so does every vertex in L . But is $K \cong L$? They don’t look alike, or do they? Is there any way to distinguish them from each other? We’ll go back to this question later, but first let’s take a break.

• • • BREAK 6

In Figure 5, is $M \cong N$?

Let’s go back to graphs K , L , M , and N for a moment. You’ve probably noticed that every vertex in these graphs has degree 3. Because of this they are usually said to be **regular of degree 3**. In fact, a graph is **regular** if every vertex has the same degree.

You may have realized by now that regular graphs of the same degree with the same number of vertices are not necessarily isomorphic. In fact,

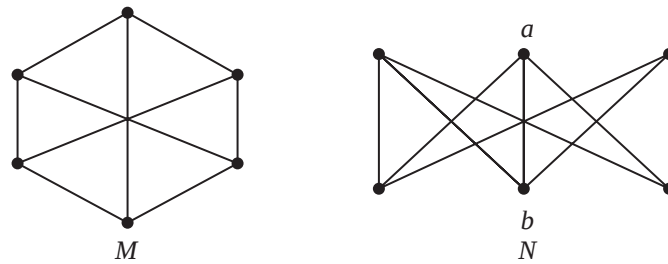


FIGURE 5

K and L are not isomorphic, although they are both regular of degree 3. One of the things that distinguishes K from L is that L has a cycle C_3 , while the smallest cycle in K is C_4 . So $K \not\cong L$.

On the other hand, this “cycles test” will not distinguish M from N . Actually, M and N each have one 6-cycle with all other cycles 4-cycles, so they can’t be distinguished that way. In fact, there is no way to distinguish them because they are isomorphic. It’s not obvious, though, just by looking at them. (But try moving vertex a of N to the bottom in Figure 5 and vertex b of N to the top.)

Among regular graphs, one important class is that of **complete** graphs, K_n . These are the graphs on n vertices in which every vertex is joined to every other vertex. We show K_2 , K_3 , K_4 , K_5 , and K_6 in Figure 6.

It’s worth noting that all of them from K_3 onwards have triangles (C_3) and that all of them from K_4 onwards have 4-cycles (C_4); and so on. The other thing worth noting is that the degree of regularity of K_2 is 1, of K_3 is 2, of K_4 is 3, of K_5 is 4, and of K_6 is 5. It ought to be reasonably clear what the degree of K_n is.

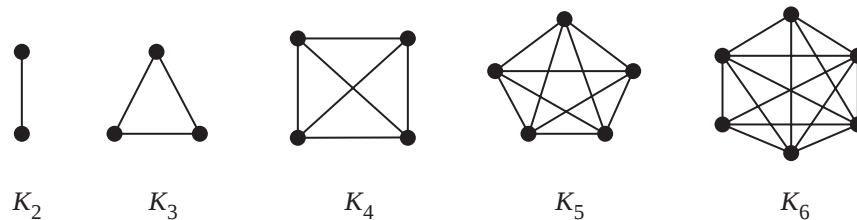


FIGURE 6

• • • **BREAK 7**

That reminds us, the notion of degree came in earlier on because of its relation to edges. How many edges has K_n ? And how is the number of edges related to the degree in K_n ? Is there any relation between degrees and the number of edges in an arbitrary graph G ?

5.4 TOURING WITH EULER

The very first problem to be tackled with the use of graphs was first discussed in the mid 1730's. That problem is the Euler Tour Problem,² and it all started with people walking over bridges in a place called Königsberg in what in the 1730's was Prussia (see [3]). The city is today in Russia, it's called Kaliningrad, and they've built another bridge since then.

Königsberg was a city on the banks of the river Pregel and one of its tributaries. Its approximate layout in the 1730's is shown in Figure 7.

The problem is this: Is it possible to plan a walk in which one goes over each bridge exactly once? Euler looked at the problem and modeled Figure 7 with a multigraph: this is a graph in which pairs of vertices may be joined by more than one edge. He constructed his model by placing a vertex on each separate land mass and joining two land masses by an edge for each bridge between them. Hence he produced the multigraph of Figure 8 and, in 1736, graph theory was born.

He then set to work to reason this way. If the multigraph of Figure 8 had a route that would take you across every edge (bridge) once and only once,

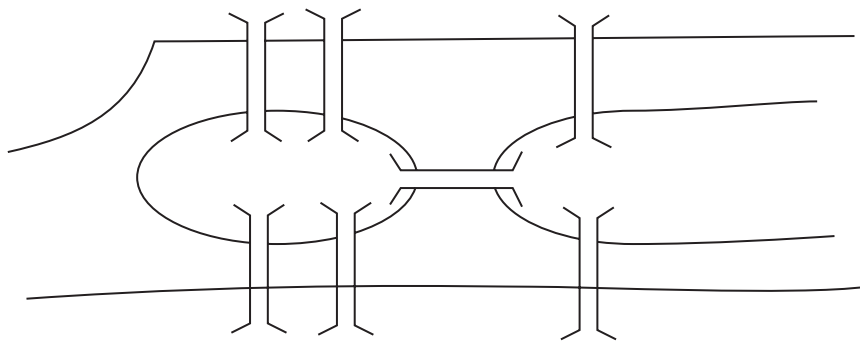


FIGURE 7

²This is popularly known, for obvious reasons, as *The Königsberg Bridge Problem*.

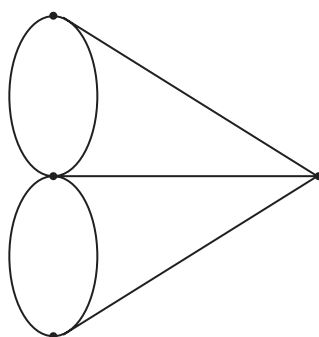


FIGURE 8

then most vertices of the graph would have to have even degree. After all, when you come to a new vertex (except possibly the last) you go into it and then out of it. So almost every time you use a vertex x (land mass), you use two edges incident with x . In your walk around Königsberg, then, using all the bridges, you have shown that every vertex except at most two has to have even degree. But none of the vertices of Figure 8 has even degree. There was therefore no way for the Königsbergians to walk over each bridge exactly once.

• • • **BREAK 8**

- (1) What is the least number of edges that you need to add to the graph of Figure 8 so that there is a route around the graph that starts and ends at *different* vertices but uses every edge?
- (2) If you want to start and finish at the *same* vertex, what is the minimum number of edges that must be added.

Being the good mathematician that he was, Euler generalized his Königsberg bridge problem to *any* multigraph. Let's say that a graph has an **Euler tour** if there is a way of going over every edge once and only once and returning to the starting point. Then Euler claimed the following result.

Theorem (Euler, 1736) *A connected multigraph has an Euler tour if and only if every vertex has even degree.*

Euler proved part of this but didn't prove it all. Using the argument we used above, he showed that if a multigraph *has* an Euler tour, then every vertex *has* to have even degree. But he didn't quite manage the converse. This is not too difficult. It can be done several ways, but we outline a proof by induction on the number of edges in a multigraph.

To get the induction step to work we need to show that a multigraph G in which every vertex has even (non-zero) degree has a cycle. That argument goes like this. First, if G has a multiple edge, it has a cycle C_2 , so we exclude multiple edges from here on. So let us start with any vertex v_1 . Then v_1 has even degree, so v_1 is adjacent to some vertex v_2 . Similarly, v_2 has even degree, so there is a vertex $v_3 \neq v_1$ that is adjacent to v_2 . Then v_3 is adjacent to $v_4 \neq v_2$. If $v_4 = v_1$, we have a cycle. If not, continue. So v_4 is adjacent to $v_5 \neq v_3$. If $v_5 = v_1$ or v_2 we have a cycle. If not, continue. Since we are dealing with a finite number of vertices, we will eventually have to use one of the vertices we have already used. Hence we have a cycle.

Suppose then that we have a cycle C in G . Every vertex in C has degree 2. Remove C from G . By induction, there is an Euler tour in each of the components of the multigraph we have left. We can then patch together these Euler tours and C to give a tour of G .

• • • BREAK 9

Suppose we want to start and end at different vertices of a multigraph and pass over every edge once and only once. Find necessary and sufficient conditions for this to be possible. Prove this.

5.5 WHY GRAPHS?

We started out worrying about ways to color the counties of England, and then we sidetracked you into thinking about graphs. Presumably, there's some relation here. What is it?

To see the advantage of graphs, we need to turn our maps "inside out." To see how to do this, look at Figure 9(a). Here we have a fairly simple map. It only has six regions (or faces), counties, or countries. They are labeled A, B, C, D, E, F .

Suppose the regions are countries. Then mark the capital city of country X with a dot labeled x and join up two capital cities if and only if their countries have a common border. This leads to the picture in Figure 9(b). Note that we always think of the "outside" region A as being a country, too.

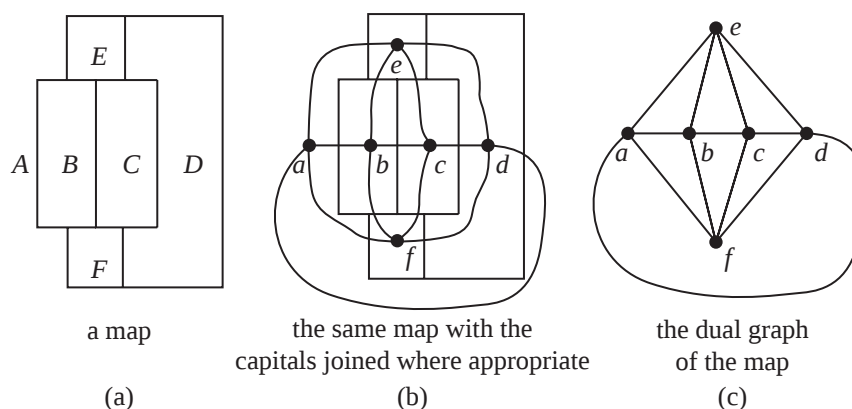


FIGURE 9

Now extract from Figure 9(b) the diagram consisting of capital cities and joins. This gives us Figure 9(c), which is a graph. Actually we call it the **dual graph** of the map.

But what has the dual graph got to do with coloring maps? Suppose our map in Figure 9(a) can be 4-colored. What does this tell us about the dual graph in Figure 9(c)? Since the vertices in some sense stand for the countries or regions, why not color the vertices of the dual graph? And we need to color them so that two adjacent vertices have different colors, because that corresponds to two countries with a common border having different colors.

Hence if we can color the *regions* of the map with four or fewer colors, we can color the *vertices* of the dual graph with four or fewer colors. Is the converse true? That is, is it true that if we can color the vertices of the dual graph in four or fewer colors, then we can color the map in the same way?

Yes, of course, this is quite straightforward. Whatever color we have on vertex a , we use over the whole of region A . Whatever color we have on vertex b , we use over the whole of region B , and so on. Now if vertex u is adjacent to vertex v they will have different colors. In that case, adjacent regions U and V will have different colors.

We've just discovered that the regions of the map can be colored in k colors so that no two neighboring regions have the same color if and only if the vertices of the dual graph can be colored in k colors so that no two adjacent vertices have the same color. Hence we've transformed the map-coloring problem into a graph-coloring problem. But of what value

is that? Have we really made any progress? Is graph-coloring somehow easier to understand than map-coloring?

But maybe we're running a little ahead of ourselves. It's just possible that the map of Figure 9(a) isn't 4-colorable. Let's check it and see.

Without loss of generality let region A be colored red. Then B , D , E , and F can't be red, and since B and E are neighbors they have to have different colors. So let's assume that B is colored blue and E green.

It ought to be clear that we can save colors by letting F be colored green, too. After all, F and E are not neighbors. Now C can't be blue or green, but since it's not adjacent to A , it can be red. Then we can finish off the map by coloring D the same color as B , namely blue. It looks as if we can color this map in only three colors. And we can shift these colors over to the dual graph of Figure 9(c) by coloring a and c red, e and f green, and b and d blue.

• • • BREAK 10

- (1) All the vertices in the dual graph of Figure 9(c) have degree 4. Do all dual graphs have this property?
- (2) Can the graph of Figure 9 be colored in fewer than 3 colors? How low can you go?

So how low can graph-colorings go? Can we color a graph in one color or two colors? We know that we can do some in three (see Figure 9(c)).

Just so that we know precisely what we are talking about, we'll say that a graph is **k -colorable** if its vertices can be colored in k colors so that no two adjacent vertices have the same color.

Going back to Figure 9(c), that graph can be colored in 6 colors. We can always give each vertex a different color so that it is 6-colorable. But we know that it can also be colored in 3 colors, so it is also 3-colorable. On the other hand, it can't be colored in 2 colors, because a , b , and e are mutually adjacent. So the graph of Figure 9(c) is *not* 2-colorable.

Fine, but are there 1-colorable or 2-colorable graphs? What would a 1-colorable graph look like? Surely, as soon as a graph has an edge it has to have at least 2 colors. So a 1-colorable graph is just a collection of isolated vertices.

So let's try for 2-colorable graphs. Suppose the vertices are colored red and blue. If we started off from a blue vertex, we'd have to move to a red vertex. In fact, all the vertices adjacent to a blue vertex would have to be red vertices. And in the same way, all the red vertices would have to be

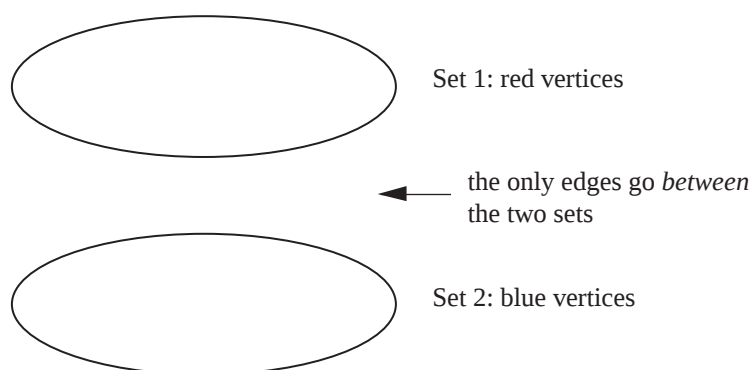


FIGURE 10 Schema for a 2-colorable graph

adjacent to blue vertices. So a 2-coloring divides the vertices into two sets, the red vertices and the blue vertices. What's more, no two red vertices are adjacent, and no two blue vertices are adjacent. The schematic picture of such a graph is shown in Figure 10. Such graphs are called **bipartite** graphs.

The first two graphs in Figure 11 are familiar. Figure 11(a) is C_8 , while Figure 11(b) is the graph N from Figure 5. As far as Figure 11(c) is concerned, it may not be obvious at first that it is bipartite. You may want to check that out. Remember, all you need to do is to show that it's 2-colorable. But one thing also worth noting is that the graph of Figure 11(c) is a tree. This is because it is connected and has no cycles.

If you go back to Figure 10 for a minute, suppose we have a bipartite graph with r red vertices and b blue ones. If it has as many edges as possible, it has b edges coming out of each red vertex and r edges coming out of each blue vertex. We call such a graph a **complete bipartite** graph

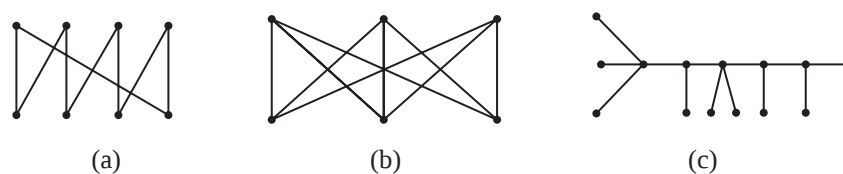


FIGURE 11 Bipartite graphs

with parts of size r and b . We denote that graph by $K_{r,b}$. So the graph of Figure 11(b) is actually $K_{3,3}$, as are the graphs M and N of Figure 5.

• • • **BREAK 11**

- (1) What does a map look like that has a bipartite graph as its dual graph?
- (2) Show that all trees are bipartite graphs.

5.6 ANOTHER CONCEPT

So if we start off with a map, we can produce a graph (the dual graph) from it. And coloring the *faces* of the map gives a coloring of the *vertices* of the graph. Now if all maps are 4-colorable, are all graphs 4-colorable too? Surely this can't be true. If a vertex has 4 neighbors, then won't the graph need 5 colors, one color for the vertex and one for each neighboring vertex? No. We've seen that this isn't the case in Figure 9. The graph there is shown again in Figure 12. Here v has degree 4, but the graph can be colored using only three colors!

Hang on! If all the vertices other than v were joined to each other, then we *would* need more than four colors. Look at the graph in Figure 13.

If v is colored 1, then a could be colored 2. But no other vertex can be colored 1 or 2 because every other vertex is joined both to v and to a . This means that b has to have a new color, 3 say. Arguing in this way we see that each new vertex we look at in Figure 13 has to have a new color. So we need five colors altogether.

Isn't this a difficulty for us? Surely the map we get from the graph in Figure 13 has to have five colors, too. What is that map? How do we get

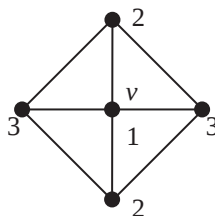


FIGURE 12

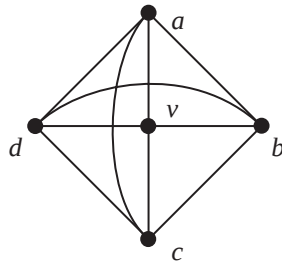


FIGURE 13 A graph that needs 5 colors

it from the graph? Let's try to get the map from the graph and see what it looks like.

The diagram in Figure 14 tells the story. You might as well put V as the middle region, which has to be surrounded by A , B , C , and D . Since A is already touching B and D , it has to be made to touch C . This accounts for the "handle" that goes across between A and C . But then look at B . It's already next to A and C , so how can we make sure that B and D have a common boundary? We obviously need a "handle" between them.

At this stage, there's clearly a problem. The two "handles" have to overlap. So how can we produce a map corresponding to the graph of

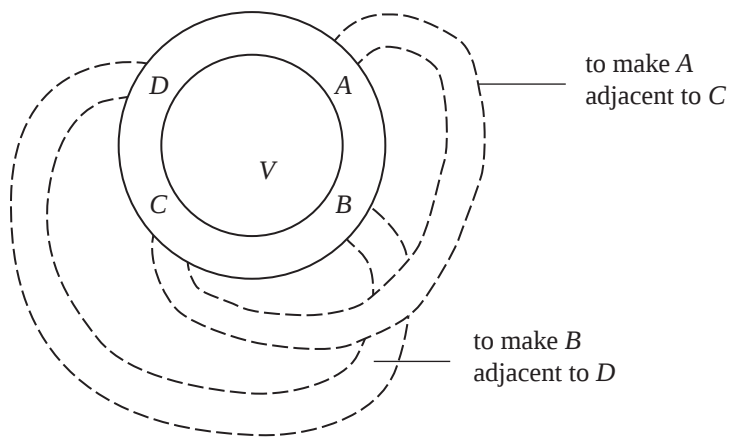


FIGURE 14 To find the map in the graph of Figure 13

Figure 13? It doesn't look as if we can. Maps can't have regions that cross each other.

This obviously raises the question of which graphs are dual graphs of a map? When they are, we say that the graph has a dual map.

• • • BREAK 12

- (1) Are there some graphs that are 6-colorable? Can you find a graph that is n -colorable for any n ?
- (2) What graphs have dual maps and why?

5.7 PLANARITY

What essential properties do dual graphs have? We know that the degrees of the vertices are not restricted because we can surround a country by as many countries as we like. But we also know that not all graphs can be dual graphs. Figure 13 is an example of a graph that turns out not to have a corresponding map. So how do we know when the graph we have is a dual graph? One very simple property of a dual graph is that, when we draw the graph on top of its corresponding map so that vertex x lies in X , then none of the edges cross. This is because we only join vertices in regions that have a common boundary. We will say that graphs that can be drawn so that no two edges cross are **planar**. In Figure 15 we give some examples of planar graphs. Of course, all dual graphs are planar.

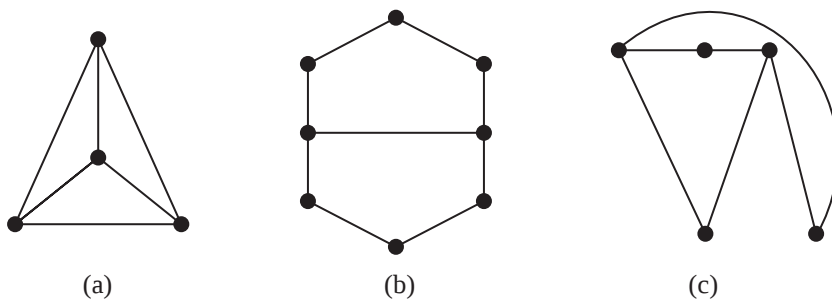


FIGURE 15 Examples of planar graphs

• • • **BREAK 13**

- (1) If you have a graph drawn so that its edges *do* cross, does that necessarily mean that it is not planar?
- (2) The Hiltons, the Holtons, and the Pedersens have bought three neighboring houses. They all want to have gas, electricity, and water connected from the three main points for these utilities. Can the pipes be laid in such a way that no two cross over each other at any point?

Now it's easy to see that dual graphs are planar, but it's another matter to decide, in general, whether or not a given graph is planar. After all, for a graph with a large number of vertices, you might take ages trying to draw it so that no two edges cross. Even if you couldn't find a planar drawing, how would you know that no such drawing existed? You might have missed it somehow.

Let's go back for a moment to the graph in Figure 13. We've drawn it again in Figure 16(a), along with another graph (Figure 16(b)). Now, of course, from what we've said before, Figure 16(a) is the complete graph K_5 , and Figure 16(b) is the complete bipartite graph $K_{3,3}$.

Now we have essentially shown that K_5 is non-planar in the discussion relating to Figure 13. Meanwhile, you have considered the planarity of $K_{3,3}$ under another guise in BREAK 13. But we'll now outline an argument that shows that $K_{3,3}$ is not planar.

Let's assume that $K_{3,3}$ is planar. In the final "planar" drawing of $K_{3,3}$ we will have to have the cycle (a, d, b, e, c, f) as shown in Figure 16(c). The question then is, where are the three remaining edges?

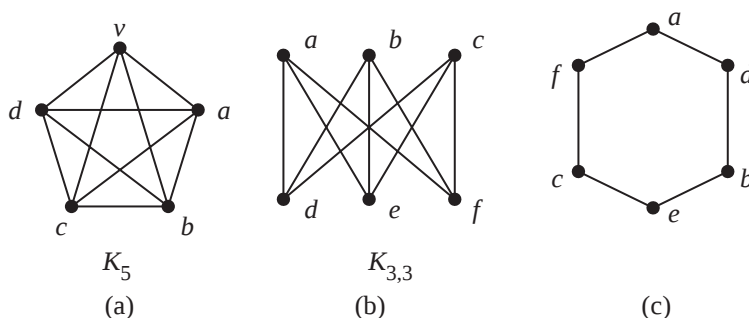


FIGURE 16

Now, at the moment, there is a symmetry between the edges ae , bf , and cd and so we can first think about the edge ae without any loss of generality. This edge is either drawn *inside* the cycle or *outside*. But if the edge is actually *outside*, we can interchange the inside and outside of the cycle, so we can assume that ae is inside. This forces cd to be outside the cycle; otherwise, it would have to cross ae . This gives us the situation of Figure 17.

The problem is where to put the edge bf ? If it is *inside* the cycle, it has to cross the edge ae . If it is *outside* the cycle, it has to cross the edge cd . These both contradict the assumption that $K_{3,3}$ is planar. Hence that assumption is false. So $K_{3,3}$ is not planar!

Discovering that K_5 and $K_{3,3}$ are not planar is no big deal. However, the Polish topologist Kazimierz Kuratowski was able to show that, in a sense, these are the *only* non-planar graphs. Kuratowski's Theorem says that a graph G is non-planar if and only if, in a special sense to be explained, it contains K_5 or $K_{3,3}$ as a subgraph.

To explain to you this special sense, let's see why the graph of Figure 18, the so-called Petersen graph, is non-planar. The highlighted edges in Figure 18 form the subgraph $K_{3,3}$, "in a special sense." Hence, by Kuratowski's Theorem, this graph is non-planar.

Let's think about this special sense. If we take the highlighted subgraph of Figure 18, we get the graph of Figure 19(a). This can be rearranged (see Figure 19(b)) so that it looks more like $K_{3,3}$.

The graph in Figure 19(b) clearly shows $K_{3,3}$ with a small case of measles. To four edges of $K_{3,3}$ have been added vertices of degree 2. Intuitively, it's clear that adding vertices of degree 2 to $K_{3,3}$ cannot suddenly

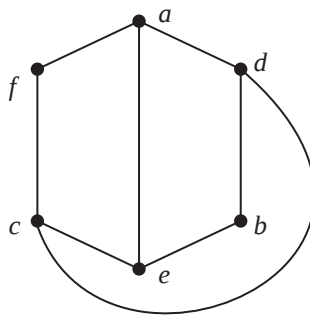


FIGURE 17 $K_{3,3}$ without the edge bf

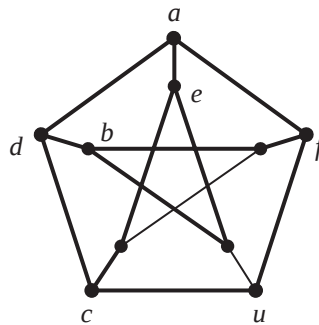


FIGURE 18

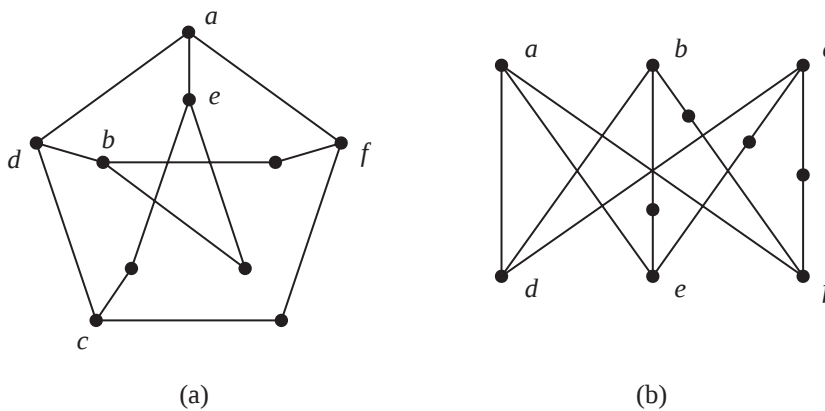


FIGURE 19

make it planar. There is no way that adding vertices of degree 2 can *undo* a crossing of edges.

Now the same thing can be said of K_5 , too. If K_5 sprouts measles in the form of an arbitrary number of vertices of degree 2, then the resulting graph is still non-planar. Thus Kuratowski's Theorem says that G is non-planar if and only if it contains a subgraph that is either K_5 with measles or $K_{3,3}$ with measles. (Of course, as in much of mathematical language, "with measles" includes the possibility of no measles.)

• • • **BREAK 14**

- (1) Show that K_5 is non-planar, using an argument similar to the one we used on $K_{3,3}$.
- (2) Show, in three ways, that the graph in Figure 20 is non-planar.

5.8 THE END

We started out with a map. We turned that into a dual graph. We noticed that dual graphs are planar. If we could prove that all planar graphs could be colored with four (or fewer) colors, then, of course, so could all dual graphs. But then all maps would be 4-colorable because dual graphs are. How might we prove that all planar graphs are 4-colorable?

Let's not worry about that for a moment. In 1879, the Englishman Kempe proved that the Four Color Conjecture is true! As a reward for this and other work, Kempe was made a Fellow of the Royal Society. The Royal Society is a body that was set up for the promotion of science in Britain under the Royal Charter of King Charles II in 1662. Only a limited number of Fellows exist at any one time. Existing Fellows elect new Fellows on the basis of their contribution to science. To become an F.R.S. is a singular honor.

Kempe's proof therefore was generally applauded and certainly was widely accepted. However, the story wasn't over. In 1890 another English mathematician, Percy John Heawood, found an error in Kempe's proof! Heawood was, however, able to salvage an important idea, that of Kempe chains (see [6] or [8]), from Kempe's proof. Using Kempe's work, Heawood was able to prove that every map is 5-colorable. (See [2], [3], [4], [6], [8], or [9].)

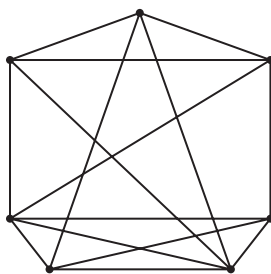


FIGURE 20

For nearly 90 more years mathematicians, both amateur and professional, tried to solve the Four Color Conjecture. Some progress was made. For instance, a group of people concentrated on the number of regions in the map. One mathematician provided his new wife with maps for her to color on their honeymoon! By the mid-1970's it was known that every map with at most 99 regions is 4-colorable.

Then in 1976, Appel and Haken [1] completed the proof using a method strongly based on Kempe's ideas. The major tool that Appel and Haken used that wasn't available to Kempe was a computer. Before they used it, though, they had to exploit certain carefully chosen mathematical reductions to make the problem finite; otherwise, the computer would have been of no use. But, even so, they had to run their machine for 3 months in order to cover all the cases required by their approach.

The nature of this proof raised certain mathematicians' hackles. Up to this time, it had been possible for mathematicians to check every line of a proof and, consequently, to accept the proof or reject it. Here was a proof by computer that, if written out in full, could not have been checked even if all of the world's mathematicians and their apprentices had given the task all the years they had at their disposal. In many quarters this "new" proof took some time to be accepted.

In the meantime, various mathematicians completed computer proofs similar to that of Appel and Haken. To date the nicest attempt at tidying up a computer proof has come from Robertson, Sanders, Seymour, and Thomas (see [8]). But there are still mathematicians who would like an elegant proof without recourse to a computer. A "traditional" proof would make many people happy.

• • • **BREAK 15**

Find, and check, a proof of the Five Color Theorem.

5.9 COLORING EDGES

The Scottish mathematician P.G. Tait had some crazy ideas, but in 1880 he had an idea that might just work. There is still a long way to go, but here is the idea, and here is how things stand up to the publication of this book.

Instead of looking at the dual graph of a map, we'll look at its **underlying** graph. In other words, we'll look at the graph whose edges are the boundaries of the regions and whose vertices are the points where three or more boundaries meet.

The first thing to notice is that we may assume that precisely three boundaries meet at every vertex of the underlying graph. We show how to do this in Figure 21. We put a new “polygonal” region at every vertex to give an underlying graph all of whose vertices are of degree 3. We call the corresponding map a *trivalent map*.

Suppose the trivalent map is 4-colored. Then regions A, B, C, D, E are colored such that A and B have different colors, B and C have different colors, C and D have different colors, D and E have different colors, and E and A have different colors. Hence the 4-coloring of the trivalent map leads to a 4-coloring of the map without region X . This means that we only have to worry about coloring trivalent maps.

So now let's concentrate on trivalent maps. And let's turn to looking at *edge*-coloring their underlying graphs, each vertex of which has degree 3. (By an edge-coloring we mean a coloring of the edges so that edges that meet at a common vertex have different colors.) What we want to show is that 4-coloring the *regions* of the map is equivalent to 3-coloring the *edges* of the underlying graph. This was Tait's idea, and there's a rather nice proof.

Before we can examine the proof, though, we need to restrict the kind of map we are dealing with. In Figure 22 we have a map where the same region lies on two sides of the edge e .

Remove e and the regions on the right. Suppose the left regions plus the outside face are 4-colorable. Remove e and the regions on the left. Suppose the right regions plus the outside face are 4-colorable. Let the outside face in each case have the same color. Then putting the two colorings back together gives a 4-coloring of the original map. The Four Color Conjecture now depends on coloring the left and right submaps. So we'll assume from

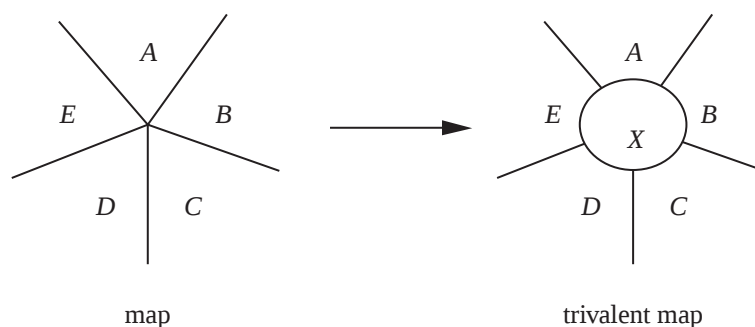


FIGURE 21

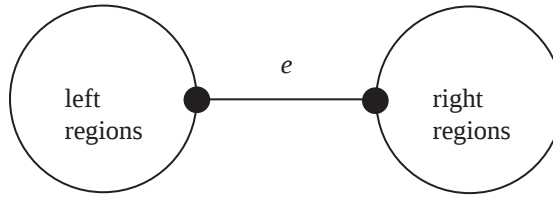


FIGURE 22

now on that we are dealing with the smallest map that is not 4-colorable. Such a map will not have a boundary like e in Figure 22.

To show that a 4-face coloring supplies a 3-edge coloring, we need to take another small detour. We ask for the reader's patience.

Consider the set $W = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. We'll define the addition of elements of W according to the table below; and we'll write $\oplus$ for this addition.³

$\oplus$	$(0, 0)$	$(0, 1)$	$(1, 0)$	$(1, 1)$
$(0, 0)$	$(0, 0)$	$(0, 1)$	$(1, 0)$	$(1, 1)$
$(0, 1)$	$(0, 1)$	$(0, 0)$	$(1, 1)$	$(1, 0)$
$(1, 0)$	$(1, 0)$	$(1, 1)$	$(0, 0)$	$(0, 1)$
$(1, 1)$	$(1, 1)$	$(1, 0)$	$(0, 1)$	$(0, 0)$

Table 1. Addition in W

There are three very useful properties of the set W under the operation $\oplus$. First of all, $c_1 \oplus c_2 = c_2 \oplus c_1$ for all $c_1, c_2 \in W$. You should be able to see, too, that if $c_1 \oplus c_2 = (0, 0)$, then $c_1 = c_2$, and vice versa. Finally, you can check that if $c_1 \oplus c_2 = c_1 \oplus c_3$, then $c_2 = c_3$. We'll find these properties useful in a moment.

Now suppose that we have colored all the faces of a map using four colors represented by the elements of W . We then color the edges of the underlying graph by the sum of the colors on the faces on either side of the edges (see Figure 23).

The thing to note here is that the edge receiving the color $c_1 \oplus c_2$ is not an edge like the edge e in Figure 22. Hence the faces on either side of the edge are different. This means that $c_1 \neq c_2$. From what we said above, we

³ W is a well-known object in linear algebra, a vector space of dimension 2 over the field with 2 elements, that is to say, the pairs are added componentwise mod 2.

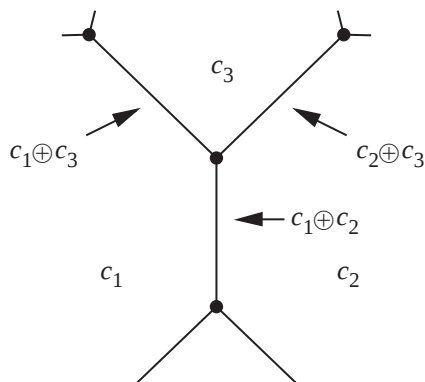


FIGURE 23 Coloring edges using face coloring

now know that $c_1 \oplus c_2 \neq (0, 0)$. This argument is true for *all* edges, and so the colors on the edges of the underlying map must therefore come from the set $\{(0, 1), (1, 0), (1, 1)\}$.

Now, can two adjacent edges have the same color? Suppose $c_1 \oplus c_2 = c_1 \oplus c_3$. From what we know of W , $c_2 = c_3$. Hence we have an edge that has the same face on either side of it. We know that this is not possible. So adjacent edges have different colors. This strange method of coloring edges, given the colors on the faces, has led to a 3-coloring of the edges. The edge colors come from $\{(0, 1), (1, 0), (1, 1)\}$. So we see that a 4-face coloring implies a 3-edge coloring.

How can we go the other way? We now want to 3-edge color the underlying graph of a trivalent map and produce a 4-coloring of its faces. So we can assume that we have the 3-coloring of the edges. Suppose the colors are r , s , and t . The trick here is to pick two colors r and s , and to look first at the edges colored r or s . By themselves, what sort of a graph do they form? Because every vertex of the underlying graph is of degree three, and because we have three colors, all three colors *have* to be present on the three edges around each vertex. So ignoring edges with color t is like throwing those edges away. What that does is to leave us with a graph, all of whose vertices have degree 2. Such a graph has to be a cycle or a collection of cycles. A map consisting of cycles can be face colored with just two colors. Let these two colors be a and b .

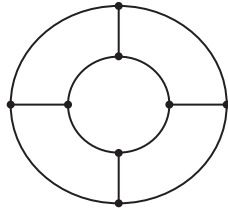


FIGURE 24

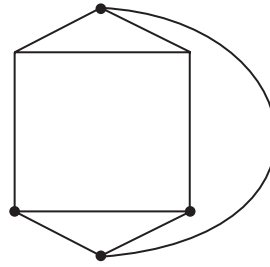


FIGURE 25

Now repeat the process by looking just at the edge colors r and t . Again these produce a graph that is a collection of cycles. These can be face colored using colors c and d .

The final trick is to notice that any region in the original map is the intersection of faces from the two lots of cycle maps that we have just described. So color a face ac if it is the intersection of an a -colored face and a c -colored face, or color it ad , bc , bd for the various other intersections. Hence we've used at most four colors. By the method of coloring the cycle graphs, no two faces colored ac , ad , bc , or bd can be adjacent. So a 3-edge coloring gives us a 4-face coloring.

This shows the equivalence of the 4-face coloring problem and the 3-edge coloring problem.

• • • BREAK 16

- (1) Show that $c_1 \oplus c_2 = c_2 \oplus c_1$.
 Show that $c_1 \oplus c_2 = (0, 0)$ implies that $c_1 = c_2$.
 Show that $c_1 \oplus c_2 = c_1 \oplus c_3$ implies that $c_2 = c_3$.
- (2) Show that the edges of the map-graph in Figure 24 are 3-edge colorable by first finding a 4-face coloring.
- (3) In the graph of Figure 25 produce a 4-face coloring from a 3-edge coloring.

5.10 A BEGINNING?

Suppose we have an arbitrary planar graph all of whose vertices have degree 3. How can we show that it is 3-edge colorable? If we can find a way to do this, we will have found a way of proving the Four Color

Conjecture. If we can find a method that does not require the use of a computer, that would be a bonus.

Of course, we don't know how to do this . . . yet. But there might be a possible approach hidden in what we do next. Maybe you'll see how to do it. Let us know if and when you do.

There are two key steps in the plan. The first comes from a theorem due to the Russian mathematician V.G. Vizing. He proved that the graphs we are looking at need only three or four colors in order to color their edges! That cuts down the whole business considerably. We don't have to worry about these graphs having to be 5- or 500-edge colored.

Suppose we take one of our planar graphs of degree 3. If it's 3-edge colored, we're finished by what we did in the last section. If it's 4-edge colored . . . ? Ay, there's the rub! So let us assume that it's 4-edge colored. What do 4-edge colored graphs of degree 3 look like? If we knew the answer to that, we might be finished.

Actually, by various tricks we can reduce the kind of 4-edge colored graphs of degree 3 we need to look for. Surprisingly, they turn out to be hard to find. At one stage in the 1970s only five of them were known! Martin Gardner, who wrote a column in *Scientific American* for many years, suggested that searching for these graphs was like hunting for snarks, referring to a poem by Lewis Carroll. Hence the restricted graphs we are looking for came to be known as **snarks**.

The smallest snark is the graph of Figure 18, which we showed to be non-planar. This graph, as you know, is called the Petersen graph; it is named after the Scandinavian mathematician Julius Peter Christian Petersen, who did some early work with it. We show the Petersen graph again in Figure 26.

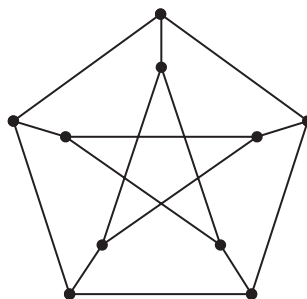


FIGURE 26 The Petersen graph

• • • **BREAK 17**

Show that the Petersen graph is 4-edge colorable but not 3-edge colorable.

Perhaps by giving snarks some publicity in *Scientific American*, Gardner prompted the mathematical community into action. Relatively soon after his column appeared, more snarks were discovered. In fact, it was soon established that there is an *infinite* number of them. This was done by showing how to construct new snarks from old ones. We give such a construction below.

This construction is called the **dot product** of two graphs and was produced by R. Isaacs [5] in 1975. He first took two snarks, any two snarks, L and R . Then he

1. removed a pair of adjacent vertices x and y from L ;
2. removed a pair of independent edges ab and cd from R ;
3. joined vertices between L and R by means of new edges ra, sb, tc, ud , as shown in Figure 27, to produce a graph, $L.R$, which is regular of degree 3.

It turns out that $L.R$ is a snark.

In Figure 27 we have chosen to join r to a , s to b , t to c , and u to d , but we would have obtained a snark, which we would still refer to as $L.R$, if instead of joining the vertices of $\{r, s\}$ to the vertices of $\{a, b\}$ and the vertices of $\{t, u\}$ to the vertices of $\{c, d\}$ we had joined the vertices of $\{r, s\}$ to the vertices of $\{c, d\}$, and the vertices of $\{t, u\}$ to the vertices of $\{a, b\}$.

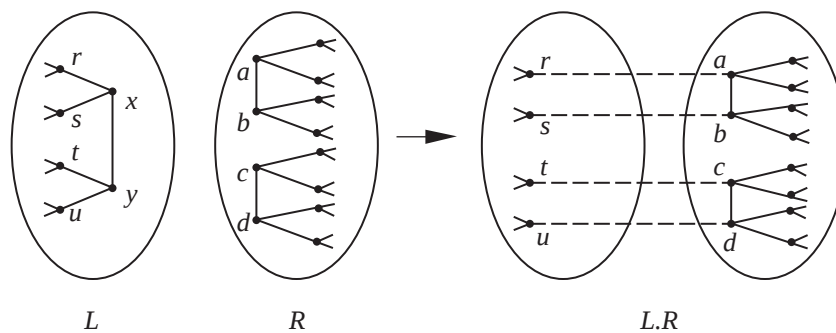


FIGURE 27

To see how this works out in practice, let's construct an example of $P.P$, where P is the Petersen graph. The graph shown in Figure 28 is one of the Blanuša snarks, named after the Croatian mathematician Danila Blanuša. This graph is one of the five first-known snarks we mentioned above.

Using P alone as the starting snark we can produce an infinite collection of snarks via $P.P$, $P.(P.P)$, $P.(P.(P.P))$, $\dots$. But these are not the only snarks known. For many more examples see [7].

• • • BREAK 18

Find another snark of the form $P.P$ that is not isomorphic to the graph in Figure 28.

The surprising thing about all the snarks that we know today is that, in some sense, they *all* contain the Petersen graph! Here we have to be vague about what “in some sense” means. Roughly, it means that the Petersen graph is a subgraph of them all.

This led the British mathematician W.T. (Bill) Tutte, who spent much of his working life at Waterloo University in Canada, to make the following conjecture.

Tutte's Conjecture *Every snark contains a copy of the Petersen graph.*

This is our second key step. If this is true, then our imagined 4-edge colorable, planar graph of degree 3 has the Petersen graph inside it somewhere. If it does, then, by Figure 18, the graph contains a copy of $K_{3,3}$. Hence the graph is non-planar. But this contradicts the assumption that

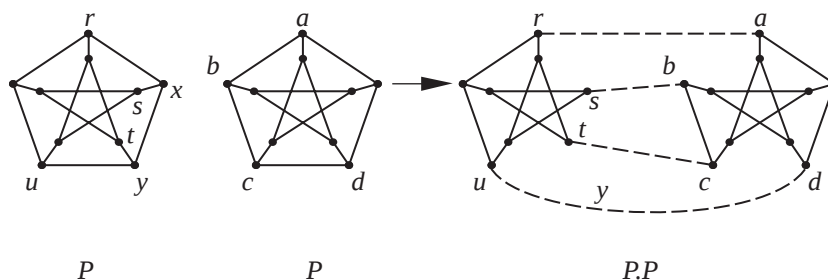


FIGURE 28

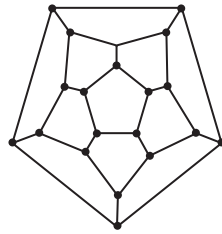


FIGURE 29

it was *planar*! Hence it can't be 4-edge colorable. And we've proved the Four Color Theorem by another route!

• • • BREAK 19

- (1) Find all graphs on n vertices that are regular of degree 1.
- (2) Find all graphs on n vertices that are regular of degree 2.
- (3) Find all graphs on n vertices that are regular of degree 3. (If this is too hard, try it for $n = 4$ and $n = 6$.)
- (4) Show that the “dodecahedron”⁴ in Figure 29 is **Hamiltonian**. That is, show that there is a cycle in the graph that passes through all the vertices of the graph.
- (5) Use Kuratowski's theorem to show that K_n is non-planar for all $n \geq 5$. Which complete bipartite graphs are planar?
- (6) Find all connected graphs on 5 vertices that have no cycles.
- (7) What is the smallest number of colors that can be used to edge-color $K_{3,3}$?
- (8) Show that if every planar graph that is regular of degree 3 were Hamiltonian, then the Four Color Theorem would follow. (See question (4) for a definition of Hamiltonian.)

REFERENCES

1. Appel, K. and W. Haken, The solution of the four-color-map problem. *Scientific American*, 237(4), 1977, 108–121.
2. Beineke, L.W. and R.J. Wilson (eds), *Selected Topics in Graph Theory* 1, 2, Academic Press, London, 1978, 1983.

⁴The figure shows the edges of a regular dodecahedron as viewed up close through one face. This is called the **Schlegel diagram** of the dodecahedron.

3. Biggs, N.L., E.K. Lloyd, and R.J. Wilson, *Graph Theory 1736–1936*, Clarendon Press, Oxford, 1986.
4. Clark, J. and D.A. Holton, *A First Look at Graph Theory*, World Scientific, Singapore, 1996.
5. Isaacs, R., Infinite families of non-trivial trivalent graphs which are not Tait colorable, *Amer. Math. Monthly*, **82**, 1975, 221–239.
6. Fritsch, R and Fritsch, G., *The Four-Color Theorem*, translated from the German by J. Peschke, Springer Verlag, New York, 1998.
7. Holton, D.A. and J. Sheehan, *The Petersen Graph*, Cambridge University Press, Cambridge, 1987.
8. Robertson, N., D.P. Sanders, P.D. Seymour, and R. Thomas, The Four-Color Theorem, *J. Comb. Th. (B)*, **70**, 1997, 2–44.
9. Saaty, T.L. and P.C. Kainen, *The Four-Color Problem*, McGraw-Hill, New York, 1977.
10. Wilson, R.J., *Introduction to Graph Theory*, Third Edition, Longman, Harlow, England, 1983.

Information on the web regarding The Four Color Theorem can be found at <http://www.astro.virginia.edu/~eww6n/math/Four-ColorTheorem.html> and at <http://www.math.gatech.edu/~thomas/FC/fourcolor.html>



<http://www.springer.com/978-0-387-95064-8>

Mathematical Vistas

From a Room with Many Windows

Hilton, P.; Holton, D.; Pedersen, J.

2002, XIV, 337 p., Hardcover

ISBN: 978-0-387-95064-8