

8

C H A P T E R

Symmetry

8.1 INTRODUCTION: A REALLY BIG IDEA

The concept of symmetry plays a strong role today in many of the exact sciences. Thus, for example, theoretical physicists, in searching for a unified field theory, have been led to the notion of *supersymmetry*, applied to the (super)strings, which, as some believe, are the fundamental building blocks of the universe. Perhaps the foremost exponent of this position is the American physicist Edward Witten, of the Princeton Institute for Advanced Study, who, a few years ago, won a Fields Medal — the most prestigious award that can be given to a mathematician¹ — for his fundamental theoretical contributions to superstring theory. Even more recently (August 1998, at the International Congress of Mathematicians held in Berlin), the Cambridge mathematician Richard Borcherds was awarded a Fields Medal for his contribution to the development of symmetry theory, especially with respect to Witten theory and its relation to the advanced mathematical theory of sporadic finite groups.

What, then, *is* symmetry? In this chapter we attempt to make the idea precise, keeping our applications of the concept at a level where, as we hope, they will be appreciated by our readers. Nevertheless, there is much

¹It is a curious fact that the only mathematician to receive a Nobel Prize for mathematical work was John Nash, whose work on equilibrium had highly significant consequences in the study of economics. So John Nash was awarded a Nobel Prize as an economist; there is no Nobel Prize for mathematics.

intrinsic difficulty in the mathematics of this chapter, so do not expect to be able to understand it without careful attention to the arguments and the examples.²

We will confine ourselves to the use of the symmetry concept within mathematics; and we must first of all emphasize that notions of symmetry, while fundamental to geometry, are certainly, and importantly, to be found in mathematics outside geometry. Thus, for example, any function of two variables $f(x, y)$ may be described as *symmetric* if $f(x, y) = f(y, x)$; for example, $f(x, y) = x^2 + y^2$ or $f(x, y) = 5xy$. Similarly, any function of three variables $f(x, y, z)$ may be described as *symmetric* if its value remains unchanged under any permutation (there are 6 if one includes the identity permutation) of the variables x, y, z ; for example, $f(x, y, z) = x^4 + y^4 + z^4 - 3xyz$. As you will have seen in Chapter 6, the binomial coefficient $\binom{n}{r}$ becomes a symmetric function if one regards it as a function of r and s , where $r + s = n$. Thus the relation $\binom{n}{r} = \binom{n}{n-r}$ takes the *symmetric* form

$$\binom{n}{r \ s} = \binom{n}{s \ r}$$

if one writes $\binom{n}{r \ s}$, with $r + s = n$, instead of the (apparently) simpler $\binom{n}{r}$. Similarly, the trinomial coefficient $\binom{n}{r \ s \ t}$, $r + s + t = n$, is a symmetric function of the variables³ r, s, t .

As another example, symmetric functions of the roots of polynomial equations

$$x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$$

play an essential role in the algebra of polynomials. We know that a polynomial equation of degree n over the complex numbers has n roots. A crucial theorem states that if $\alpha_1, \alpha_2, \dots, \alpha_n$ are the roots, then any symmetric polynomial in $\alpha_1, \alpha_2, \dots, \alpha_n$, with integer coefficients may be expressed, uniquely, as a polynomial in $a_1, a_2, \dots, a_n$ with integer coefficients. Thus, for example, with $n = 3$, we have

$$\alpha_1 + \alpha_2 + \alpha_3 = -a_1$$

²We advise you to read Chapter 4 before studying this chapter systematically. We have found that many students — but not all — understand the arguments about symmetry much better when they have a concrete geometrical model actually in their hands.

³In order to stress the symmetry, the eminent French mathematician Henri Cartan employs the simple, but revealing, notation $(r \ s)$, $(r \ s \ t)$ for binomial and trinomial coefficients, respectively. Here the n is suppressed. Why is this legitimate?

$$\begin{aligned}\alpha_1^2 + \alpha_2^2 + \alpha_3^2 &= a_1^2 - 2a_2 \\ \alpha_1^3 + \alpha_2^3 + \alpha_3^3 &= -a_1^3 + 3a_1a_2 - 3a_3\end{aligned}$$

• • • **BREAK 1**

- (1) Check the above formulae, using the identity

$$x^3 + a_1x^2 + a_2x + a_3 = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$$

(You may like to check them first in a simple particular case like $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3$.)

- (2) Continue the sequence above by expressing $\alpha_1^4 + \alpha_2^4 + \alpha_3^4$ in terms of a_1, a_2, a_3 ; remember that $n = 3$.
- (3) Use the crucial theorem quoted to prove that, if F_n is the n th Fibonacci number, then $F_m | F_n$ if $m | n$; and that, if L_n is the n th Lucas number, then $L_m | L_n$ if $m | n$ with odd quotient (we say that $m | n$ *oddly*). [Hint: Recall the Binet formulae from Chapter 3,

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n$$

where α, β are the roots of $x^2 - x - 1 = 0$.]

However, while symmetry is valuable outside geometry, our main purpose in this chapter will, in fact, be to explain the nature and role of symmetry within geometry. It is our belief that its role in other parts of mathematics will then become easier to understand. Interestingly, it turns out that, in order to describe the nature of symmetry in geometry, it is necessary to introduce some basic concepts in a very important part of algebra known as *group theory*. Thus, in Section 2, we describe these concepts and explain how they enable us to define the key notion of the *symmetry group* of a geometric configuration A . Let us add that it is not at all surprising that we need basic concepts from group theory in geometry. For the distinguished German mathematician Felix Klein (1849–1925), in his famous manifesto, called (in German) the Klein Erlanger Programm, *defined* geometry in terms of the group of allowed transformations of a given set of points. We will be adopting Klein's point of view. For example, from this point of view, the *Euclidean Geometry* of the plane $\mathbb{R}^2$ is to be understood as the set of properties of configurations in $\mathbb{R}^2$ invariant under the group of transformations of $\mathbb{R}^2$ generated by *translations*, *rotations*, and *reflections*.

In Section 3, we use the group theory presented in Section 2 to introduce the concept of *homologue*, due to the great mathematician and expositor

George Pólya (1887–1985). Pólya never wrote this idea down, but asked two of us (PH and JP), near the end of his life, to do so on his behalf, so we take this chance to do so. In fact, our interpretation of Pólya’s idea makes that idea applicable in a more general situation, namely, whenever one considers groups acting on sets — though the most vivid examples are going to come from geometry.

The precise details of our presentation of homologues are to be found at the end of Section 4, where we describe a very important theorem of combinatorics, the **Pólya Enumeration Theorem**. This theorem is described, with justification, in the *Handbook of Applicable Mathematics* [3] as the *Redfield–Pólya Theorem*.⁴ It is very often applied in a geometric context (our own applications will be geometric); however, once again, it is not in essence a theorem of geometry at all, but rests on the foundations of the idea of a (finite) group acting on a finite set — a basic idea of combinatorics.

Section 5 is a treatment of the idea of *odd* and *even* permutations. Those who would have preferred to get their group theory in one piece rather than two may, of course, read this section immediately after they have studied

⁴To see why, consult the article by E. Keith Lloyd, J. Howard Redfield 1879–1944, *J. Graph Theory*, **8** (1984), 195–203. There is a poem “Enumerational” in C. Berge’s *Principles of Combinatorics* (Academic Press, 1971) that, somewhat amusingly, refers to the history of this part of mathematics.

*Pólya had a theorem
(Which Redfield proved of old).
What Secrets sought by graphmen
Whereby the theorem told!*

*So Pólya counted finite trees
(As Redfield did before).
Their number is exactly such,
And not a seedling more.*

*Harary counted finite graphs
(Like Redfield, long ago).
And pointed out how very much
To Pólya’s work we owe.*

*And Read piled graph on graph on graph
(Which is what Redfield did).
So numbering the graphic world
That nothing could be hid.*

*Then hail, Harary, Pólya, Read,
Who taught us graphic lore,
And spare a thought for Redfield too,
Who went too long before.*

Blanche Descartes

Section 2; but we thought that, though the basic idea of the section plays an important role in the study of polyhedral symmetry, it might be rather indigestible if offered to the reader before much of the geometry had been treated.

8.2 SYMMETRY IN GEOMETRY

Although, as we have said, the concept of symmetry may be found in all parts of mathematics, and in all those areas of science to which mathematics makes an essential contribution, it remains true that its best known applications are in geometry. One finds the idea of symmetry frequently referred to in good elementary treatments of geometry (e.g., [8]), and also in good treatments of geometry suitable for those making a serious study of mathematics at the university level (e.g., [7]). But many elementary treatments of the symmetry of geometric figures are confusing and misleading, largely because those treatments never make it clear what geometry is (nor what symmetry is).

So we want to give a precise definition of *geometry*. Of course, we are not advocating that this be done the way we are doing it here if you are teaching elementary students, but it does seem to us appropriate to introduce these fundamental mathematical ideas to the readers of our book. We maintain, with Klein, that no treatment of *geometry* is complete without the idea of *symmetry*; and that no clear idea of symmetry is possible without the basic notion of a *group*.

Thus we start this section with the definition of a group.

Definition 1 Let G be a set and let $*$ be a binary operation on G ; that is, given two elements g, h of G , then $g * h$ is itself an element of G . We then say that $(G, *)$ is a *group* (often abbreviated to “ G is a group” if $*$ may be understood) if it satisfies the following 3 axioms:

- I (associative law) for all g, h, k in G , $(g * h) * k = g * (h * k)$;
- II (existence of two-sided identity) there exists an element e in G such that

$$g * e = e * g = g, \quad \text{for all } g \text{ in } G;$$

- III (existence of two-sided inverses) there exists, to each g in G , an element $\bar{g}$ in G such that

$$g * \bar{g} = \bar{g} * g = e$$

Let us immediately give two examples.

Example 2 Let G be the set of integers (positive, negative, and zero) and let $g * h$ mean $g + h$, the ordinary addition of integers. Then G is a group, with $e = 0$ and $\bar{g} = -g$. Notice that the set of integers does not form a group under multiplication, since there are, in general, no multiplicative inverses, hence the introduction of rational numbers, represented by fractions.

Example 3 Consider the set S of permutations of the set $\{1, 2, 3\}$. There are 6 such permutations; we specify each permutation s by saying where s sends 1, 2, 3. Thus

$$\begin{aligned} s_1 : 1 \rightarrow 1, 2 \rightarrow 2, 3 \rightarrow 3; & \quad s_2 : 1 \rightarrow 1, 2 \rightarrow 3, 3 \rightarrow 2; \\ s_3 : 1 \rightarrow 2, 2 \rightarrow 1, 3 \rightarrow 3; & \quad s_4 : 1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow 1; \\ s_5 : 1 \rightarrow 3, 2 \rightarrow 1, 3 \rightarrow 2; & \quad s_6 : 1 \rightarrow 3, 2 \rightarrow 2, 3 \rightarrow 1. \end{aligned}$$

The binary operation $s_i * s_j$ ($1 \leq i, j \leq 6$) simply produces the permutation s_i *followed by* the permutation s_j . It is obvious that the associative law is satisfied.⁵ The identity permutation is s_1 . Finally, the inverses are as follows:

The inverse of s_1 is s_1
 The inverse of s_2 is s_2
 The inverse of s_3 is s_3
 The inverse of s_4 is s_5
 The inverse of s_5 is s_4
 The inverse of s_6 is s_6

It is customary to write, e.g., $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ for the permutations s_3, s_5 . There is also a convenient shorthand notation for a *cycle* like $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, in which $1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow 1$. (We feel sure you will understand why it's called a cycle.) This shorthand notation is simply $(1 \ 2 \ 3)$; of course, this is the same permutation as $(2 \ 3 \ 1)$ or $(3 \ 1 \ 2)$.

If it is clear from the context that our permutation is operating on the set $\{1, 2, 3\}$, it is convenient to write $(1 \ 2)$ for the permutation s_3 , which is a cycle of *length* 2. Of course, very strictly speaking — and sometimes we have to speak strictly — s_3 is the composition of a cycle of length 2 and a cycle of length 1.

⁵When the binary operation $g * h$ says, “Do g , then do h ,” it is always associative. (Do you understand?)

Notice what these two examples have in common, and where they differ. Example 2 is an *infinite* group, and the group operation is *commutative*, that is,

$$g * h = h * g, \quad \text{for all } g, h \text{ in } G.$$

Example 3 is a *finite* group, and the group operation is not commutative, thus

$$s_2 * s_3 = s_4, \quad s_3 * s_2 = s_5$$

In both cases, the group has only one identity element, and each element has only one inverse; moreover, if $\bar{g}$ is the inverse of g , then g is the inverse of $\bar{g}$.

• • • **BREAK 2**

- (1) Prove that in both of the above examples
 - (a) There is only one identity element.
 - (b) Each element has only one inverse.
 - (c) If $\bar{g}$ is the inverse of g , then g is the inverse of $\bar{g}$.
- (2) Try to prove (a), (b), (c) for *any* group G .

Where the group is finite, we can display the entire group law by a square *group table*. Thus, for our second example, the group table is

*	s_1	s_2	s_3	s_4	s_5	s_6
s_1	s_1	s_2	s_3	s_4	s_5	s_6
s_2	s_2	s_1	s_4	s_3	s_6	s_5
s_3	s_3	s_5	s_1	s_6	s_2	s_4
s_4	s_4	s_6	s_2	s_5	s_1	s_3
s_5	s_5	s_3	s_6	s_1	s_4	s_2
s_6	s_6	s_4	s_5	s_2	s_3	s_1

(We read off $s_i * s_j$ by looking in row i and column j .) Notice that each row and each column is a permutation of the list of group elements. This is so for any finite group (sometimes this information can be used to complete a table without working out all the individual cases). The *order* of a group is the number of elements in the group, so the order is infinite (Example 2) or some finite number (in the case of Example 3 the order is 6).

Just so that you will feel reassured that we have not forgotten our purpose in introducing you to the idea of a group, let us tell you that the group of Example 3 is called the **symmetric group** on 3 symbols, and written S_3 . It will certainly reappear—for example, you will later recognize it as the group of symmetries of an equilateral triangle.

But before we turn to geometry, we require one more result from group theory. This result, due to the great French mathematician Joseph-Louis Lagrange (1736–1813), relates the order of a finite group G and the order of a *subgroup* H of G .

If $(G, *)$ is a group and H is a subset of G , we say that H is a **subgroup** of G if $g_1 * g_2$ belongs to H whenever g_1, g_2 belong to H , and if the induced binary operation on H is a group operation. For this last requirement to be satisfied, it is necessary and sufficient for the identity element to belong to H , and for the inverse of any element in H to belong to H .

Example 2 (revisited) We have the additive group $\mathbb{Z}$ of integers. The subset consisting of even integers is a subgroup, usually written $2\mathbb{Z}$. The subset consisting of non-negative integers is closed under addition, but is *not* a subgroup because the additive inverse of a positive integer is negative.

Before stating Lagrange's Theorem, we introduce some important standard notation. It is customary to write a group G *multiplicatively*, especially if it is not commutative. That is, we write the group operation as multiplication and even (as in ordinary algebra) suppress the product symbol, so that $g_1 * g_2$ is simply written as $g_1 g_2$. Sometimes we may even go further and write the identity element as 1 instead of e ; but we will not adopt this notation unless we judge there is no risk at all of confusion. We will, however, always write g^{-1} for the inverse of g . Notice that, since the associative law holds, the positive powers of g , that is, $g, g^2, g^3, \dots$, have an unambiguous meaning, and so, indeed, do the negative powers if we define

$$g^{-n} = (g^n)^{-1}, \quad \text{for } n \text{ positive} \quad (1)$$

Of course, (1) then also holds for n negative (just as in ordinary algebra).

Notice, too, that the inverse satisfies the basic law

$$(gh)^{-1} = h^{-1}g^{-1}, \quad g, h \text{ in } G \quad (2)$$

Here in (2) we have been careful to state the law so that we do *not* require commutativity.

Now let G be a finite group of order n and let H be a subgroup of G . Of course, H is also a finite group; let its order be m . Then Lagrange's Theorem is this:

Theorem 4 *The order of H divides the order of G , that is, $m|n$.*

Proof⁶ Let g be an element of G (we write $g \in G$), and let gH be the set of all elements gh , as h ranges over the elements of H . We call the set gH a (left) *coset* of H in G , and g a **representative** of the coset gH .

We now prove that, if g_1H, g_2H are two (left) cosets, then either they are disjoint or they coincide. For if $g \in g_1H$, then $g = g_1h_0$ for some $h_0 \in H$, so, for all $h \in H$,

$$gh = g_1h_0h, \quad \text{with } h_0h \in H$$

so $gh \in g_1H$, and thus $gH \subseteq g_1H$. But $g_1 = gh_0^{-1}$, with $h_0^{-1} \in H$, so we may repeat the argument with the roles of g and g_1 exchanged (and h_0^{-1} replacing h_0), concluding that $g_1H \subseteq gH$, so that, finally,⁷ $g_1H = gH$. Thus if $g \in g_1H \cap g_2H$, then $g_1H = gH = g_2H$, establishing that g_1H and g_2H coincide if they are not disjoint.

We have next to prove that every coset gH has exactly the same number of elements as H itself. For consider the function φ from H to gH that sends h to gh . This function certainly maps H onto gH ; for the elements of gH are exactly the elements gh , as h ranges over H . Moreover, φ is one-to-one; for if $gh_1 = gh_2$, then $h_1 = g^{-1}(gh_1) = g^{-1}(gh_2) = h_2$. Thus φ sets up a one-to-one correspondence⁸ between H and gH .

Now suppose that H has m elements. Then every (left) coset has m elements, and G , as a set, is the disjoint union of a certain number of cosets. If, then, G is the disjoint union of k cosets, we have proved that

$$n = km \tag{3}$$

establishing Lagrange's Theorem. $\square$

⁶We advise you to read this argument carefully. It is really subtle.

⁷We have given you here an argument that does *not* depend on G or H being finite.

⁸See previous footnote.

Remarks

- (i) The number k that appears in (3) is called the **index** of H in G .
- (ii) Notice that we could have argued with right cosets Hg . Obviously, we would have arrived at the same relationship (3), so the index is also the number of disjoint right cosets. This is significant because it is by no means true in general that a left coset is a right coset.

• • • **BREAK 3**

- (1) Let G be the additive group of remainders modulo 12 (think of a 12-hour clock). What is the order of G ? Find all the subgroups of G and list their orders.
- (2) Set up a one-to-one correspondence between the set of left cosets of H in G and the set of right cosets of H in G . (Be careful — there is no well-defined function sending the coset gH to the coset Hg .)

This is as far as we need to take the group theory in order to be precise as to what we mean by a *geometry* on a configuration A , and the symmetry of the resulting geometric figure. However, some further (and obviously relevant) group theory will be found in Section 5.

We have been guided in our definitions by the approach of the German mathematician Felix Klein (1849–1925) to explaining the nature of geometry. Consider, for example, the usual plane Euclidean geometry, in which we study the properties of planar figures that are invariant under certain *Euclidean motions*. These motions certainly include *translation* and *rotation* in the plane of the figure, but it is a matter of choice whether they include *reflection*. For example, the elaborate 7-gon⁹ of Figure 1(a) is invariant under rotations through $\frac{2\pi}{7}$ about its center, but, unlike the ordinary regular 7-gon, not under reflection about any axis through its center. Thus, to define our geometry, we must decide whether we allow Euclidean motions that reverse orientation. Of course, if we allow certain Euclidean motions, we must also allow compositions and inverses of such motions, so we postulate a certain *group* G of allowed motions of the *ambient* plane, the plane containing our planar figure. If A is such a planar figure, then,

⁹Those of you who are interested in knowing how to construct this figure (by simply folding paper) may wish to consult Chapter 4.

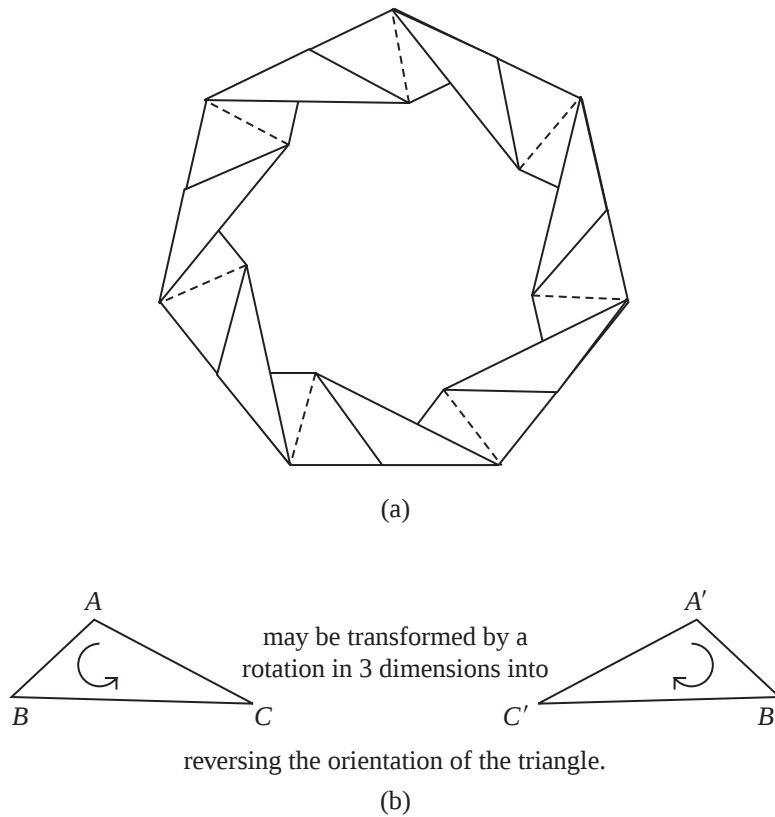


FIGURE 1

for any $g \in G$, Ag is again a planar figure.¹⁰ In the G -geometry of A , we study the properties of the figure A that it shares with all the figures Ag as g varies over G . Such properties are called the **G -invariants** of A , abbreviated to **invariants** if, but only if, the group G may be understood. It is thus the group G that, according to Klein, determines the geometrical nature of A .

Example 5 Let G be the group of motions of the plane generated by translations, rotations, and reflections (in a line); we call this the **Euclidean**

¹⁰We prefer to write Ag for the image of A under the motion g , rather than gA . This is because, in the group G , gh means “first g , then h .”

group in 2 dimensions and write it E_2 . Then the Euclidean geometry of the plane is the study of the properties of subsets of the plane that are invariant under the motions in E_2 . For example, the property of being a polygon is a Euclidean property; the number of vertices and sides of a polygon is a Euclidean invariant. On the other hand, as we have hinted, orientation is not invariant with respect to this group, though it would be if we disallowed reflections. Thus, by means of a motion in E_2 the triangle ABC may be turned over (flipped) to form the triangle $A'B'C'$ as shown in Figure 1(b). But the orientation $\overrightarrow{ABC}$ is counterclockwise, while the orientation $\overrightarrow{A'B'C'}$ is clockwise.

Example 6 We may step up a dimension, passing to the group E_3 of Euclidean motions in 3-dimensional space. Notice that it is natural to think of reflections in a line (of a planar figure) as a motion in space, since it can be achieved by a rotation in some suitable ambient 3-dimensional space containing the plane figure — you will surely see this by looking again at Figure 1(b). However, it requires a greater intellectual effort to think of reflection in a plane (of a spatial figure) as a motion in some ambient 4-dimensional space! Who would think of turning the golden dodecahedron¹¹ (see Figure 2(c)) inside out? Thus it is common not to include such reflections in defining 3-dimensional geometry. This preference is, of course, a consequence of our experience of living in a 3-dimensional world and has no mathematical basis. However, whenever we are highlighting the construction of actual physical models of geometrical configurations, it is entirely reasonable to omit motions to which the models themselves cannot be subjected.

We now move to a precise definition of symmetry. Let a geometry be defined on the ambient space of a configuration A by means of the group of motions G . Then the **symmetry group** of A , relative to the geometry defined by G , is the subgroup G_A of G consisting of those motions $g \in G$ such that $Ag = A$, that is, those motions that map A onto itself, or, as we say, under which A is *invariant*. Thus, for example, if our geometry is defined by rotations and translations in the plane, and if A is an equilateral triangle, then its symmetry group G_A consists of rotations about its center through 0° , 120° , and 240° ; if, in our geometry, we also allow reflections, then the symmetry group has 6 elements instead of 3, and is, in fact, the

¹¹We have included Figures 2(a), 2(b) as well for the benefit of those readers who might wish to construct the golden dodecahedron. See Figure 33 of Chapter 4.

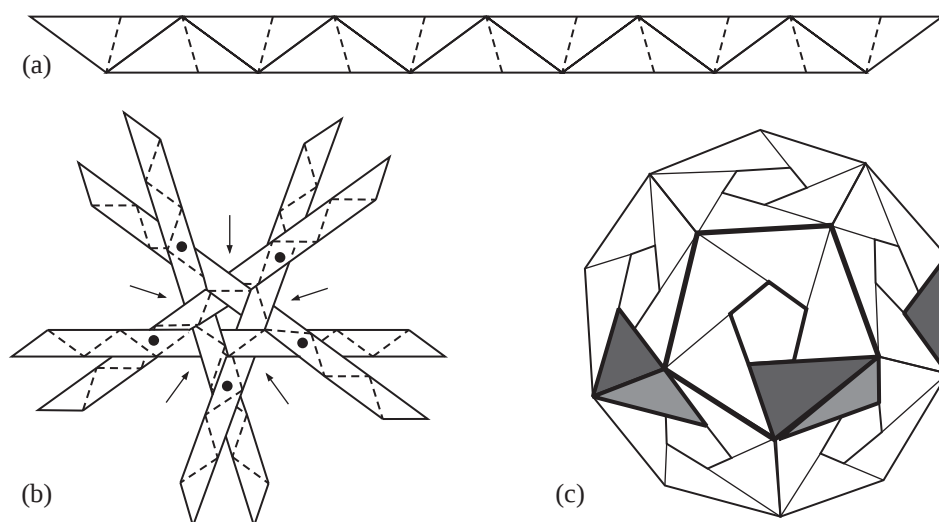


FIGURE 2 The golden dodecahedron. This extraordinarily beautiful model may be made by braiding together six colored strips like the one shown in Figure 2(a). Begin by laying out five of the strips as shown in Figure 2(b) and then attach paper clips at the places marked by the arrows. Next continue to braid these strips together, remembering what the finished model will look like (attaching paper clips, in the middle of the faces—not at the vertices—wherever needed to hold everything in position). At a certain stage you will need to insert the sixth strip about the “equator.” When you are finished, every strip should go over and under the strips it meets all the way around the model, and there should be no loose ends sticking out. For more detailed information about how to prepare the strips, consult Chapter 4.

very well-known group S_3 (recall Example 3), called the **symmetric group on 3 symbols** — the symbols may be thought of in this case as the vertices of the triangle. We repeat, for emphasis, that the symmetry group G_A of the configuration A is a *relative* notion, depending on the choice of geometry G .

It is plain that no compact (bounded) configuration can possibly be invariant under a non-zero translation. Thus when we are considering the symmetry group of such a figure we may suppose G to be generated by rotations and, perhaps, reflections. Moreover, any such motion in the plane is determined by its effect on 3 independent points, and any such motion in 3-dimensional space is determined by its effect on 4 independent points. Since a (plane) polygon has at least 3 vertices and a polyhedron has at least 4 vertices, and since any element of the symmetry group of a polygon or

a polyhedron must map vertices to vertices, it follows that the symmetry group of a polygon or a polyhedron is *finite* (compare the symmetry groups of a circle or a sphere).

The symmetry group of any polygon with n sides is, by the argument above, a subgroup of S_n , the group of permutations of n symbols, also called the **symmetric group** on n symbols. If G is generated by rotations alone, and the polygon is regular, its symmetry group is the cyclic group¹² of order n , often written C_n , generated by a rotation through an angle of $\frac{2\pi}{n}$ radians about the center of the polygonal region. If G also includes reflections, then the symmetry group has $2n$ elements and includes n reflections; this group is called the **dihedral group** of order $2n$ and is usually written D_n .

In discussing the symmetry groups of *polyhedra*, however, we will, as indicated earlier, always assume that the geometry is given by the group G generated by rotations in 3-dimensional space. Then the symmetry group of the regular tetrahedron is the so-called **alternating group** A_4 . In general, A_n is the subgroup of S_n consisting of the *even* permutations¹³ of n symbols; it is of index 2 in S_n , whose order is $n!$, so that its order is $\frac{1}{2}(n!)$. Thus the order of A_4 is 12.

The cube and the regular octahedron have the same symmetry group, namely S_4 . It is easy to see why the symmetry groups are the same; for the centers of the faces of a cube are the vertices of a regular inscribed octahedron, and the centers of the faces of a regular octahedron are the vertices of an inscribed cube. Likewise, and for the same reason, the regular dodecahedron and the regular icosahedron have the same symmetry group, which is A_5 . It is a matter of interest and relevance here that the elements of the symmetry group of the diagonal cube (Figure 3, which is Figure 27 of Chapter 4) permute the four braided strips from which the model is made, and thus the symmetry group is S_4 . (We'll have you study this closer in Break 6.)

We are now in a position to give at least one possible precise meaning to the statement "Figure A is more symmetric than Figure B ." We assume that A and B are defined as geometric figures by the same group of motions G . If it happens that the symmetry group G_A of A strictly contains the symmetry group G_B of B , then we are surely entitled to say that A is "more symmetric" than B . Notice that the situation described may, in fact, occur because B is obtained from A by adding features that destroy some

¹²A cyclic group of order n is a group of order n , all of whose elements are powers of a given element g , called a **generator** of the group.

¹³See Section 5 of this chapter for a careful treatment of even and odd permutations.

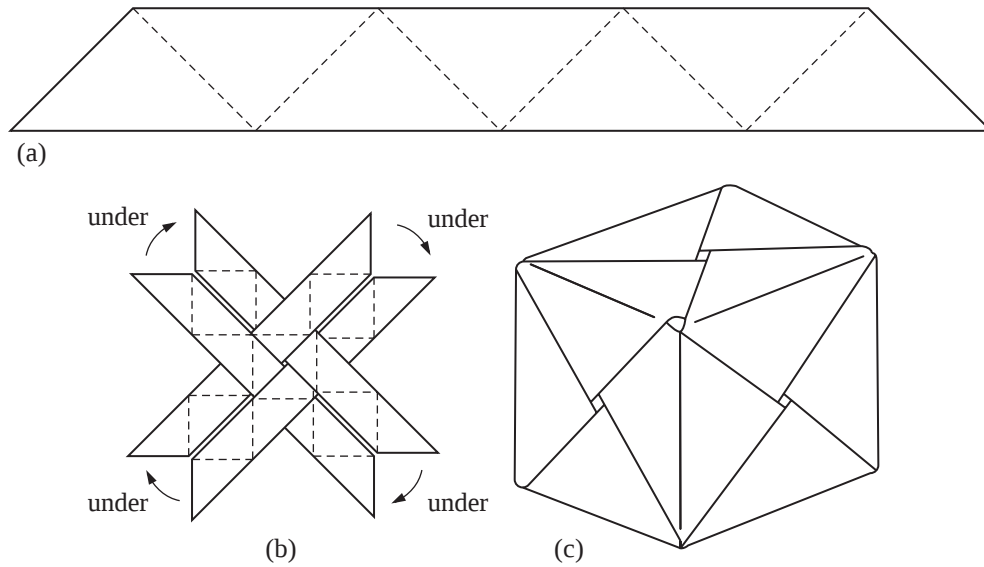


FIGURE 3 To construct this model, begin with four strips, each one consisting of 7 isosceles right triangles as shown in Figure 3(a). Each strip should be of a different color. Then fold the strips so that there are “mountain” folds on the colored side of the paper. With the mountain folds down, lay the strips out as shown in Figure 3(b). Secure the center of the configuration with a small piece of tape and then continue to “braid” the strips together to form the diagonal cube shown in Figure 3(c). Every face should have four colors visible, and every strip should go over and under the strips it meets as it goes around the cube. Finally, all the ends should be tucked in.

of the symmetry of A . For example, Figure 4(a) shows a typical strip that may be used to braid each of the Platonic solids. (Precise details about the construction of these models appear in Chapter 4.) However, the resulting braided Platonic solids of Figure 4(b) will all have their symmetry reduced if we braid them from strips of different colors (or different patterns). It is natural to ask whether it is possible to braid the tetrahedron, octahedron, and icosahedron in such a way as to retain all the symmetry of the original polyhedron.¹⁴ We will address this question in the next break.

Meanwhile, returning to the discussion of symmetry, we note that the notion “more symmetric” above is really too restrictive. For we would like to be able to say that the regular n -gon becomes more symmetric as n increases. We are thus led to a weaker notion that will be useful if we

¹⁴Notice that we already know how to do this for the cube (Figure 3) and the dodecahedron (Figure 2).

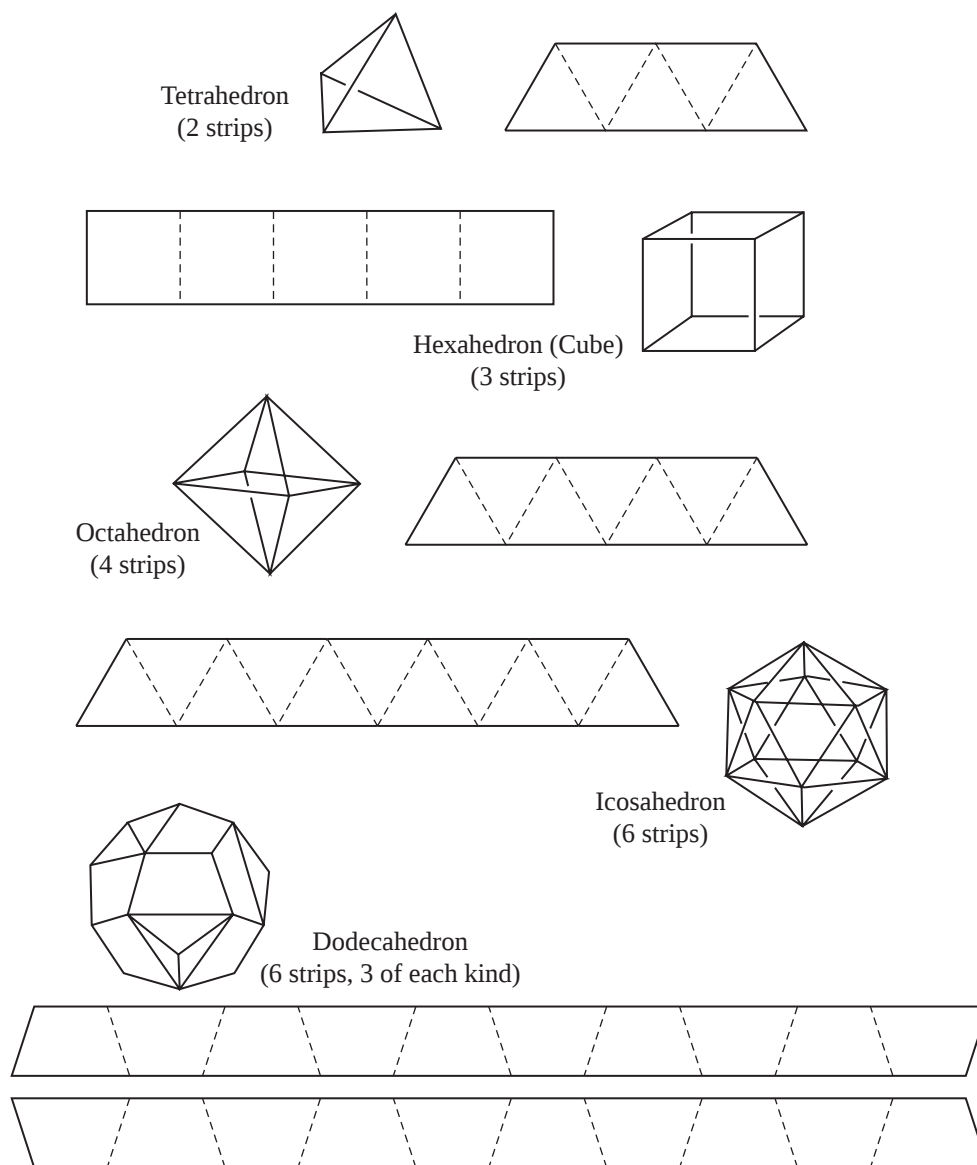


FIGURE 4 (a) Pattern pieces for constructing the Platonic solids.

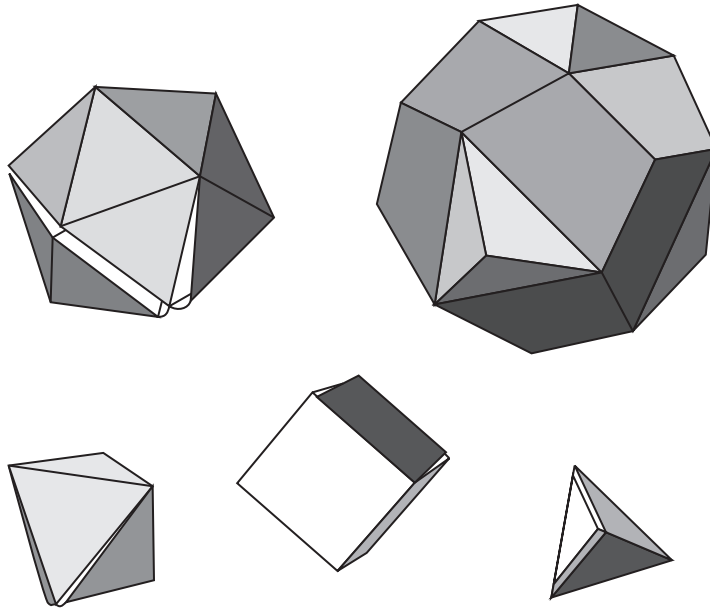


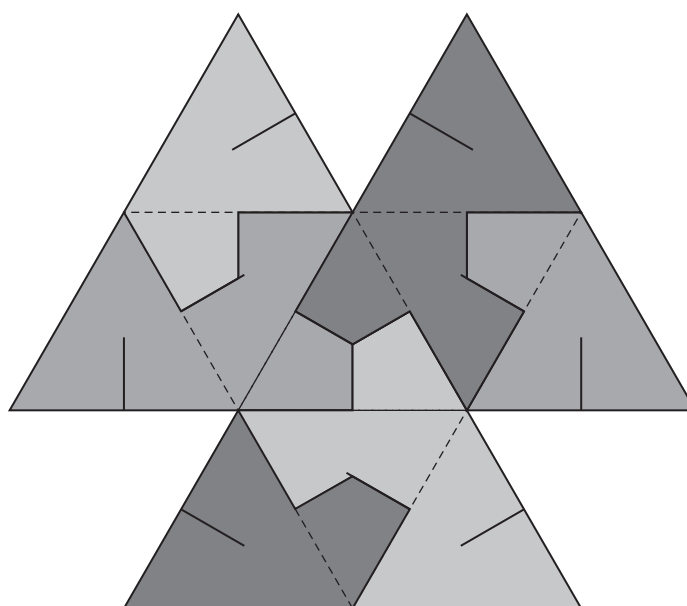
FIGURE 4 (b) The completed Platonic solids.

are dealing with figures with finite symmetry groups (e.g., polygons and polyhedra). We could then say — and do say — that A is more symmetric than B if G_A has more elements than G_B . Thus we have, in fact, two notions whereby we may compare symmetry — and they have the merit of being consistent. Indeed, if A is more symmetric than B in the first sense, it is more symmetric than B in the second sense — but not conversely.

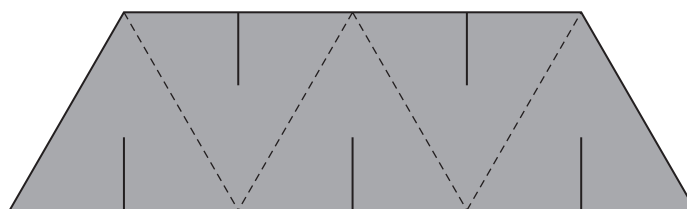
Notice that we deliberately avoid the statement — often to be found in popular writing — “ A is a symmetric figure.” We regard this statement as having no precise meaning!

• • • BREAK 4

- (1) Why should we wish to say that the regular n -gon becomes more symmetric as n increases? What happens to the regular n -gon as n increases indefinitely?
- (2) Figure 5(b) shows a typical straight strip of 5 equilateral triangles with a slit in each triangle from the top (or bottom) edge to (just



(a)



(b)

FIGURE 5

past) the center.¹⁵ A tetrahedron with symmetry group A_4 may be constructed out of three of these strips. Figure 5(a) shows how the three strips are interlaced initially. We leave the completion of the model as a challenge to you.

¹⁵Theoretically, the slit could go just to the center, but the model is then impossible to assemble. You need to have some leeway for the pieces to be free to move during the process of construction — although they will finally land in a symmetric position, so that it *looks* as though the slit need not have gone past the center.

- (3) **Harder.** Figure 6(a) shows the layout of three strips for the beginning of the construction of an octahedron with symmetry group S_4 . Can you build it? We'll give you one more hint. When you use the layout of Figure 6(a) remember that the strip shown below it in Figure 6(b) has to be braided into the figure above it.
- (4) **Much harder.** An icosahedron with symmetry group A_5 may be constructed from 6 strips of the type shown in Figures 5 and 6, where each strip has 11 equilateral triangles. Over to you! But

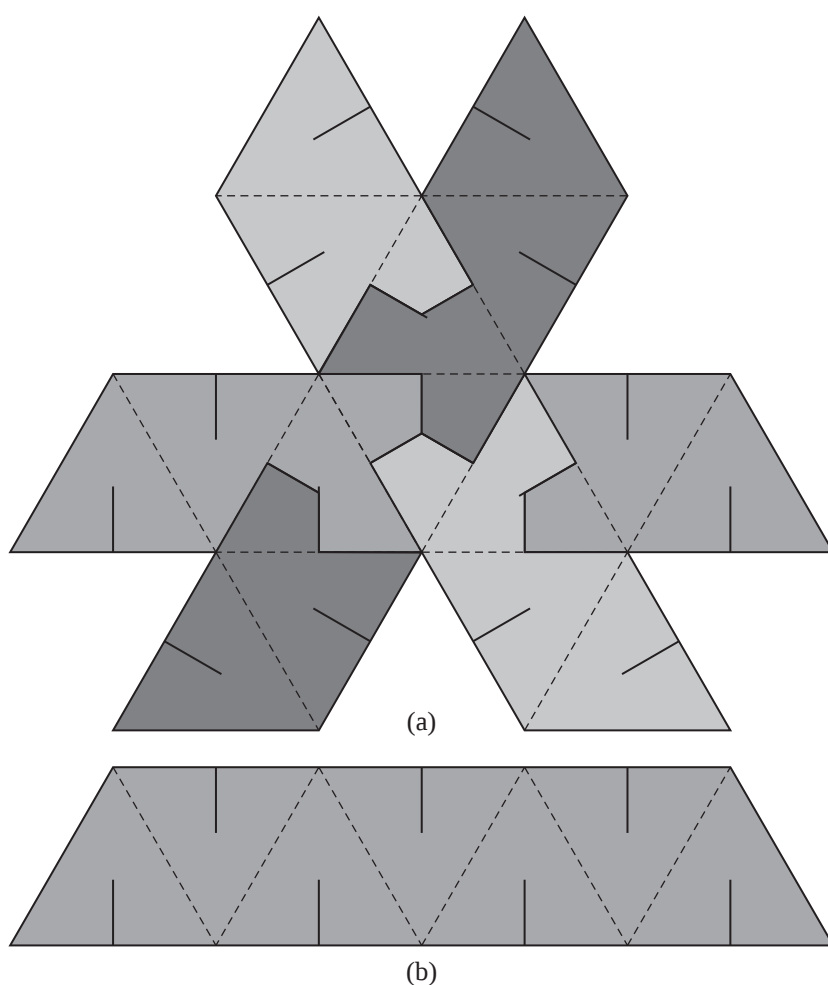


FIGURE 6

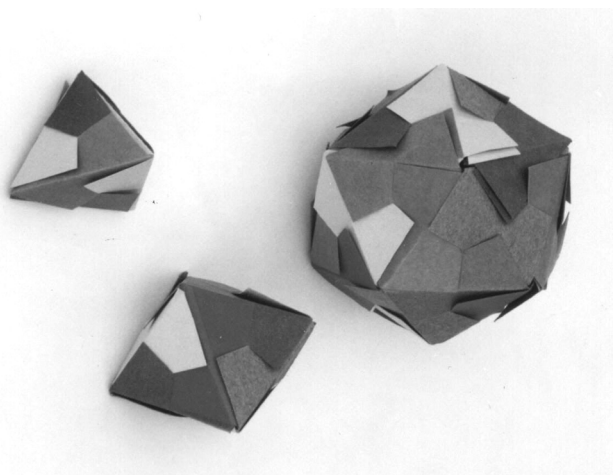


FIGURE 6 (c)

take heart — these models take several hours to construct. Just to prove that they really do exist we show a photo of them in Figure 6(c).

8.3 HOMOLOGUES

George Pólya, who made great contributions not only to mathematics itself, but also to the understanding of how and why we do mathematics — or perhaps one should say, “how and why we should do mathematics” — was particularly fascinated by the Platonic solids and first introduced his notion of *homologues* (orally) in connection with the study of their symmetry. We know that the idea of homologues played a key role in his thinking about one of his greatest contributions to the branch of mathematics known as *combinatorics*, namely, the **Pólya Enumeration Theorem** (see [5] for an intuitive account and [3] for a detailed account). Let us describe this notion of homologues in terms of symmetry groups. We believe that we are thereby increasing the scope of the notion while entirely maintaining the spirit of Pólya’s original idea.¹⁶

Let A be a geometrical configuration in a geometry given by a group Γ , and let the symmetry group of A with respect to this geometry be Γ_A ; and

¹⁶We will ourselves be discussing the Pólya Enumeration Theorem in the next section.

let B be a subset of A . Thus, for example, A may be a polyhedron and B a face of that polyhedron. We consider the subgroup Γ_{AB} of Γ_A consisting of those motions in the symmetry group Γ_A of A that map B to itself. We consider a right coset of Γ_{AB} in Γ_A , that is, a set $\Gamma_{AB}g$, $g \in \Gamma_A$. (Be sure you understand what Γ_{AB} and $\Gamma_{AB}g$ mean before you read on.) Every element in $\Gamma_{AB}g$ sends B to the same subset Bg of A . The collection of these subsets is what Pólya called the collection of **homologues** of B in A . We see that the set of homologues of B in A is in one-to-one correspondence with the set of right cosets of Γ_{AB} in Γ_A .

Example 7 Consider the pentagonal dipyrmaid A of Figure 7(b). We may specify any motion in the symmetry group of A by the resulting permutation of its vertices 1, 2, 3, 4, 5, 6, 7; note that the polar vertices are 6 and 7. In fact, Γ_A is the dihedral group D_5 , with 10 elements, given by the following permutations:

$$(1\ 2\ 3\ 4\ 5\ 6\ 7) \left\{ \begin{array}{ll} \rightarrow (1\ 2\ 3\ 4\ 5\ 6\ 7) & \text{(identity)} \\ \rightarrow (2\ 3\ 4\ 5\ 1\ 6\ 7) & \text{(rotation through } \frac{2\pi}{5} \text{ about axis } 67) \\ \rightarrow (3\ 4\ 5\ 1\ 2\ 6\ 7) & \text{(rotation through } \frac{4\pi}{5} \text{ about axis } 67) \\ \rightarrow (4\ 5\ 1\ 2\ 3\ 6\ 7) & \bullet \\ \rightarrow (5\ 1\ 2\ 3\ 4\ 6\ 7) & \bullet \\ \rightarrow (5\ 4\ 3\ 2\ 1\ 7\ 6) & \text{(interchanging the poles)} \\ \rightarrow (4\ 3\ 2\ 1\ 5\ 7\ 6) & \text{(interchange plus rotation} \\ & \text{through } \frac{2\pi}{5}) \\ \rightarrow (3\ 2\ 1\ 5\ 4\ 7\ 6) & \text{(interchange plus rotation} \\ & \text{through } \frac{4\pi}{5}) \\ \rightarrow (2\ 1\ 5\ 4\ 3\ 7\ 6) & \bullet \\ \rightarrow (1\ 5\ 4\ 3\ 2\ 7\ 6) & \bullet \end{array} \right.$$

First, let B be the edge 16. Then $\Gamma_{AB} = \{\text{Id}\}$, since only the identity sends the subset $\{1, 6\}$ to itself. Thus the index of Γ_{AB} in Γ_A is ten, and there are ten homologues of the edge 16; these are the ten “spines” of the dipyrmaid (i.e., we exclude the edges around the equator). Second, let B be the edge 12. Then Γ_{AB} has two elements, since there are two elements of Γ_A , namely the identity and $(1\ 2\ 3\ 4\ 5\ 6\ 7) \rightarrow (2\ 1\ 5\ 4\ 3\ 7\ 6)$, which send the subset $\{1, 2\}$ to itself. Thus the index of Γ_{AB} in Γ_A is five, and there are five homologues of the edge 12; these are the five edges around the equator.

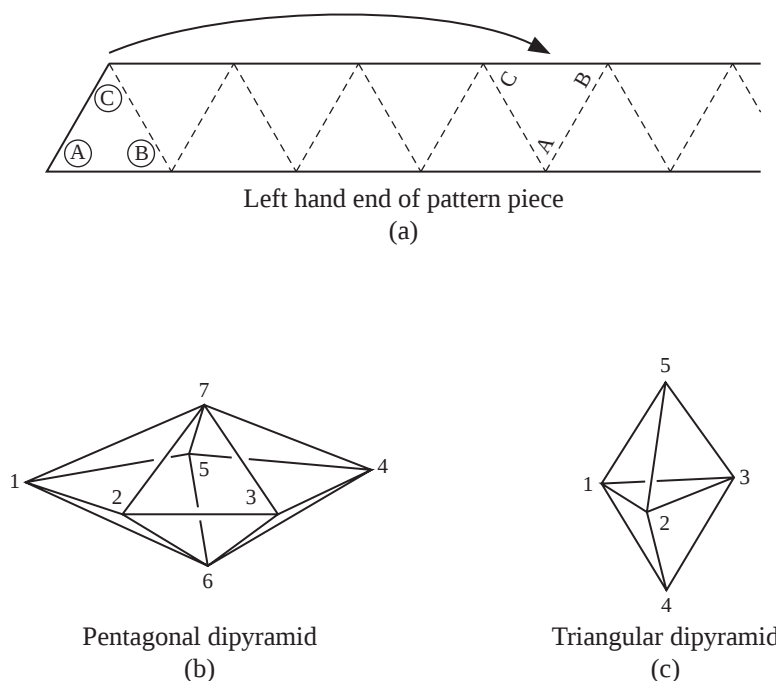


FIGURE 7 The pentagonal dipyramid may be made from a strip of 31 equilateral triangles. You begin the construction by placing the first triangle over the eighth triangle as shown in Figure 7(a). At that point just let the rest of the strip fall around the model, remembering what it should look like when you are finished. The last triangle will tuck into a slot produced by an edge of the tape from a previous crossing of the last face. A similar construction of the triangular dipyramid exists, beginning with a strip of 19 equilateral triangles. More detailed instructions for preparing the strips may be found in Chapter 4.

Third, let B be the face 126. Then $\Gamma_{AB} = \{\text{Id}\}$, so that, as in the first case, there are 10 homologues of the face 126; in other words, all the (triangular) faces are homologues of each other.

• • • BREAK 5

- (1) Consider the triangular dipyramid P of Figure 7(b). Specify the motions of the symmetry group by writing down the resulting permutations of its vertices 1, 2, 3, 4, 5; note that the polar vertices are 4 and 5. In fact, Γ_P is the dihedral group D_3 , with 6 elements.

- (2) How many homologues does the edge 12 have? How many homologues does the vertex 4 have? How many homologues does the face 124 have?

Let us now explain the Pólya Enumeration Theorem — actually, there are two theorems — and see how the notion of homologue fits into the story.

8.4 THE PÓLYA ENUMERATION THEOREM

Let S be a finite set; the reader might like to keep in mind the set of vertices (or edges, or faces) of a polygon or polyhedron. Let G be a finite (symmetry) group acting on S . Suppose S has n elements, and that G has m elements; we write $|S| = n$, $|G| = m$. We may represent the elements of the set S by the integers $1, 2, \dots, n$. If $g \in G$, then g acts as a *permutation* of $\{1, 2, \dots, n\}$. Now every permutation is uniquely expressible as a composition of cyclic permutations on mutually exclusive subsets of the elements of S . For example, the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 2 & 4 & 7 & 1 & 3 & 11 & 5 & 6 & 8 & 10 & 9 \end{pmatrix} \quad (4)$$

of the set S of integers $\{1, 2, \dots, 11\}$ is the composite

$$(1 \ 2 \ 4)(3 \ 7 \ 5)(6 \ 11 \ 9 \ 8)(10)$$

where, e.g., $(1 \ 2 \ 4)$ denotes the cyclic permutation

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \end{pmatrix}$$

Thus the permutation (4) is the composite of one cyclic permutation of length 1, two cyclic permutations of length 3, and one cyclic permutation of length 4, the cyclic permutations acting on disjoint subsets of the set S . In general, a permutation of S has the *type* $\{a_1, a_2, \dots, a_n\}$ if it consists of a_1 permutations of length 1, a_2 permutations of length 2, $\dots$, a_n permutations of length n , the permutations having disjoint domains of action; notice that $\sum_{i=1}^n i a_i = n$. For example, the permutation (4) has type $\{1, 0, 2, 1, 0, 0, 0, 0, 0, 0, 0\}$. If g has type $\{a_1, a_2, \dots, a_n\}$, we define the **cycle index** of g to be the monomial

$$Z(g) = Z(g; x_1, x_2, \dots, x_n) = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$$

The **cycle index of G** is $Z(G) = Z(G; x_1, x_2, \dots, x_n) = \frac{1}{m} \sum_{g \in G} Z(g)$.

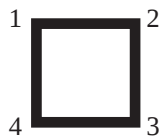


FIGURE 8 We denote this labeling of the vertices of the square by $\begin{smallmatrix} 1 & 2 \\ 4 & 3 \end{smallmatrix}$.

We give an example that we will revisit periodically throughout this section.

Example 8 We consider the symmetries¹⁷ of the square that is shown in Figure 8.

The group G of symmetries is the group D_4 of order 8, which we here describe as a group of permutations of the set S of vertices $\{1, 2, 3, 4\}$. Thus $n = 4$, $m = 8$, and the elements of G are

g_1 (identity)	$(1)(2)(3)(4)$	cycle index	x_1^4
g_2	$(1\ 2\ 3\ 4)$	cycle index	x_4
g_3	$(1\ 3)(2\ 4)$	cycle index	x_2^2
g_4	$(1\ 4\ 3\ 2)$	cycle index	x_4
g_5	$(1\ 3)(2)(4)$	cycle index	$x_1^2 x_2$
g_6	$(2\ 4)(1)(3)$	cycle index	$x_1^2 x_2$
g_7	$(1\ 2)(3\ 4)$	cycle index	x_2^2
g_8	$(1\ 4)(2\ 3)$	cycle index	x_2^2

Thus the cycle index of G , obtained from the calculations above, is

$$\frac{1}{8} (x_1^4 + 2x_1^2 x_2 + 3x_2^2 + 2x_4)$$

We leave the example temporarily and return to the general case. Suppose we want to *color* the elements of S . That is, we have a finite set Y of colors, $|Y| = r$, and then a *coloring* of S is a function¹⁸ $f : S \rightarrow Y$. For any $g \in G$, we regard the colorings f and fg as indistinguishable or

¹⁷Recall that, in considering the symmetries of a polygon, we permit reflections in three-dimensional space about a line.

¹⁸We speak of a *coloring* of S ; this may be literally true, or it may merely be a metaphor for a rule for dividing the elements of S into disjoint classes.

equivalent; and a *pattern* is an equivalence class of colorings. Then Pólya's first theorem is as follows.

Theorem 9 *For a given group of symmetries G and a given set of colors Y with $|Y| = r$, the number of patterns is $Z(G; r, r, \dots, r)$.*

Example 8 (continued) Suppose the vertices of the square are to be colored using r colors. Then the number of patterns is $\frac{1}{8}(r^4 + 2r^3 + 3r^2 + 2r)$. If, in particular, the vertices are to be colored red or blue, then $r = 2$, and the number of patterns is $\frac{1}{8}(16 + 16 + 12 + 4) = 6$. In fact, the patterns are represented by the 6 colorings

$$\begin{array}{cc} R & R \\ R & \blacksquare R \end{array} \quad \begin{array}{cc} B & R \\ R & \blacksquare R \end{array} \quad \begin{array}{cc} B & B \\ R & \blacksquare R \end{array} \quad \begin{array}{cc} B & R \\ R & \blacksquare B \end{array} \quad \begin{array}{cc} R & B \\ B & \blacksquare B \end{array} \quad \begin{array}{cc} B & B \\ B & \blacksquare B \end{array}$$

Notice that we regard the colorings $\begin{smallmatrix} B & R \\ R & \blacksquare R \end{smallmatrix}$ and $\begin{smallmatrix} R & B \\ R & \blacksquare R \end{smallmatrix}$, for example, as indistinguishable or equivalent; we are sure you see why.

We now describe Pólya's second theorem. This is really the big theorem, and the first theorem is, in fact, deducible from it. Let us enumerate the elements of Y (the colors) as $y_1, y_2, \dots, y_r$.

Theorem 10 (The Pólya Enumeration Theorem) *Evaluate the cycle index*

$$Z(G; x_1, x_2, \dots, x_n) \quad \text{at} \quad x_i = \sum_{j=1}^r y_j^i, \quad i = 1, 2, \dots, n.$$

Then the coefficient of $y_1^{n_1} y_2^{n_2} \dots y_r^{n_r}$ is the number of patterns assigning the color y_j to n_j elements¹⁹ of S .

Example 8 (continued) For the symmetries of the square we know that

$$Z(G) = \frac{1}{8}(x_1^4 + 2x_1^2 x_2 + 3x_2^2 + 2x_4)$$

Let $Y = \{R, B\}$; then the evaluation of $Z(G)$ at $x_i = R^i + B^i, i = 1, 2, 3, 4$, yields

$$\begin{aligned} & \frac{1}{8}((R+B)^4 + 2(R+B)^2(R^2+B^2) + 3(R^2+B^2)^2 + 2(R^4+B^4)) \\ &= R^4 + R^3B + 2R^2B^2 + RB^3 + B^4 \end{aligned}$$

¹⁹Of course, $\sum_{j=1}^r n_j = n$.

(It is, of course, no coincidence that this polynomial is homogeneous, of degree $|S|$, and symmetric; and has integer coefficients.)

Thus the Pólya Enumeration Theorem tells us that there is one pattern with 4 red vertices (obvious); one pattern with 3 red vertices and 1 blue vertex (represented by the coloring $\begin{smallmatrix} B \\ R \end{smallmatrix} \blacksquare \begin{smallmatrix} R \\ R \end{smallmatrix}$); 2 patterns with 2 red vertices and 2 blue vertices (represented by the colorings $\begin{smallmatrix} B \\ R \end{smallmatrix} \blacksquare \begin{smallmatrix} B \\ R \end{smallmatrix}$ and $\begin{smallmatrix} B \\ R \end{smallmatrix} \blacksquare \begin{smallmatrix} R \\ B \end{smallmatrix}$); and the remaining possibilities are most easily analyzed by considering the symmetric roles of R and B .

Consider the various coloring functions $S \rightarrow Y$ representing a given pattern. These functions all have the form $fg : S \rightarrow Y$, where f is a fixed coloring and g ranges over the elements of G . It will turn out that the colorings fg are essentially the *homologues* of f . Let us first revert to our example.

Example 8 (continued) As we have seen, there is one coloring in which all vertices of the square are colored red. There is only one homologue, namely, $\begin{smallmatrix} R \\ R \end{smallmatrix} \blacksquare \begin{smallmatrix} R \\ R \end{smallmatrix}$.

There is one coloring in which 3 vertices are colored red and one blue. There are 4 homologues, namely,

$$\begin{smallmatrix} B \\ R \end{smallmatrix} \blacksquare \begin{smallmatrix} R \\ R \end{smallmatrix} \quad \begin{smallmatrix} R \\ R \end{smallmatrix} \blacksquare \begin{smallmatrix} B \\ R \end{smallmatrix} \quad \begin{smallmatrix} R \\ R \end{smallmatrix} \blacksquare \begin{smallmatrix} R \\ B \end{smallmatrix} \quad \begin{smallmatrix} R \\ B \end{smallmatrix} \blacksquare \begin{smallmatrix} R \\ R \end{smallmatrix}$$

There are two colorings in which 2 vertices are colored red and 2 blue. In the first there are 4 homologues, namely,

$$\begin{smallmatrix} B \\ R \end{smallmatrix} \blacksquare \begin{smallmatrix} B \\ R \end{smallmatrix} \quad \begin{smallmatrix} R \\ R \end{smallmatrix} \blacksquare \begin{smallmatrix} B \\ B \end{smallmatrix} \quad \begin{smallmatrix} R \\ B \end{smallmatrix} \blacksquare \begin{smallmatrix} R \\ B \end{smallmatrix} \quad \begin{smallmatrix} B \\ B \end{smallmatrix} \blacksquare \begin{smallmatrix} R \\ R \end{smallmatrix}$$

In the second there are 2 homologues, namely,

$$\begin{smallmatrix} B \\ R \end{smallmatrix} \blacksquare \begin{smallmatrix} R \\ B \end{smallmatrix} \quad \begin{smallmatrix} R \\ B \end{smallmatrix} \blacksquare \begin{smallmatrix} B \\ R \end{smallmatrix}$$

You may complete the analysis by considerations of symmetry.

Let us now show how this concept of homologues agrees with our earlier definition. We are given the group G of permutations²⁰ of S . Given a coloring $f : S \rightarrow Y$, we consider the subset G_0 of G consisting of those g such that $fg = f$, that is, those movements of S that preserve the coloring. It is easy to see (just as easy as in our earlier, simpler situation) that G_0 is a *subgroup* of G . Corresponding to each coset G_0g of G_0 in G we have a coloring fg of S , and these colorings run through the pattern determined by f . We describe the set of colorings $\{fg\}$ in this pattern as the set of *homologues* of the coloring f ; just as in our geometric definition

²⁰Recall, from Section 3, the group Γ_A of symmetries of the configuration A .

in Section 3, they are in one-to-one correspondence with the set of right cosets of G_0 in G .

• • • **BREAK 6**

- (1) Construct the diagonal cube²¹ shown in Figure 3.
- (2) Use this cube to fill in the following table:

Description of rotation about the axis through the centers of	The amount of rotation	The number of rotations of this type
Opposite faces	$\pm \frac{1}{4}$ turn	6
Opposite faces	$\frac{1}{2}$ turn	
Opposite vertices	$\pm \frac{1}{3}$ turn	
Opposite edges	$\frac{1}{2}$ turn	
Identity	0	
		Total number of rotations =

- (3) Fill in the following table about the permutations of four objects, and make the obvious comparison of the two tables.

Description of the type of permutation	The number of permutations of this type
(_ _ _ _) (one cycle of length 4)	6
(_ _)(_ _) (two cycles, each of length 2)	
(_ _ _)(_) (one cycle of length 3 and one cycle of length 1)	
(_ _)(_)(_) (one cycle of length 2 and two cycles of length 1)	
(_)(_)(_)(_) (four cycles of length 1, i.e., the identity)	
	Total number of permutations =

²¹Notice that the diagonal cube is so-called because of the appearance of the diagonals on each of its faces. We are not referring to the *interior* diagonals of the cube.

- (4) Notice anything? (Hint: Think of the strips of your cube as being numbered 1, 2, 3, 4 and observe what happens to the strips when you perform the rotations in part (2). This is one very vivid way to see why the symmetry group of the cube is S_4 . Of course, there are other ways, too. For example, the interior diagonals of the cube might be labeled 1, 2, 3, 4, and you would see that the rotations of the cube simply permute these diagonals. It is just a little harder to “see” the interior diagonals than it is to see the strips of the diagonal cube.
- (5) Write the cycle index for the 4 strips of the diagonal cube.
- (6) Use the Pólya Enumeration Theorem to determine how many different diagonal cubes can be made with four strips if you may choose any one of four colors for each strip.
- (7) Write the cycle index for the 6 strips of the Golden Dodecahedron.
- (8) Use the Pólya Enumeration Theorem to determine how many different Golden Dodecahedra can be made with six strips, each a different color.

★ We are now prepared to offer our general definition of a homologue. To help you to follow this, we first repeat briefly the two contexts in which this notion has already arisen in this chapter.

First, consider a geometric configuration Z with respect to a geometry given by the group Γ . Let G be the subgroup of Γ consisting of those $g \in \Gamma$ such that $Zg = Z$. So G is the symmetry group of Z in the geometry given by Γ ; G acts on Z .

Set $X = 2^Z = \text{set of subsets of } Z$.

Now G acts on X . Then $x \in X$ is a subset of Z , and G_x is the subgroup of G consisting of those g that map x to itself.²² Hence we obtain the set of homologues of x ; and the homologues are in one-to-one correspondence with the cosets of G_x in G .

Second, recall that, in this section, we have discussed the set of functions $f : S \rightarrow Y$, or colorings of S , and a group G acting on S .

Let $X = Y^S = \text{set of colorings of } S, \text{ i.e., functions from } S \text{ to } Y$

Then G acts on X by the rule

$$fg(s) = f(sg), \quad g \in G, \quad f : S \rightarrow Y, \quad s \in S$$

²²In Section 3 we had $Z = A$, $G = \Gamma_A$, $x = B$, $G_x = \Gamma_{AB}$.

Now, for any f , $G_f = \{g \mid fg = f\}$, and we get a one-to-one correspondence between the homologues of f , as described in this section, and the cosets of G_f in G . The homologues of f are the colorings of S determining the same pattern as f .

Thus the common description, covering both applications, is this: A group G acts on a set S . Let Y be a fixed set and consider the induced action of G on Y^S . If $f \in Y^S$, that is, if f is a function from S to Y , let G_f be the subgroup of G consisting of those $g \in G$ that fix f , that is, such that $fg = f$. Then the **homologues** of f (with respect to G) are the functions fg , as g ranges over G ; and they are in one-to-one correspondence with the cosets of G_f in G , under the rule $G_f g \longleftrightarrow fg$. In our first example above, we had $S = Z$, $Y = \{0, 1\}$ (so that Y^S is the set of subsets of Z), and $f = x$.

This fulfills our promise to George Pólya to write up his idea of **homologues**.

We will give you a further example of homologues in Break 10; but before we can make that example clear we need to describe *even* and *odd* ★ permutations and the *alternating group*.

8.5 EVEN AND ODD PERMUTATIONS

We described the symmetry group of a regular tetrahedron as the alternating group A_4 consisting of *even* permutations of the set $(1, 2, 3, 4)$. We owe you at the very least a definition of this concept! The reason we postponed giving you this is that the *precise* definition is difficult to understand. See if you agree with us.

Let A be the *alternating form in n variables*, that is, the expression

$$A = \prod_{1 \leq i < j \leq n} (x_i - x_j) \quad (5)$$

Then any permutation ρ in S_n acts on A by permuting the suffixes $i, j, \dots$ appearing in (5) by means of ρ ; thus

$$A\rho = \prod_{1 \leq i < j \leq n} (x_{i\rho} - x_{j\rho}) \quad (6)$$

We claim that $A\rho$ is either A or $-A$. For, given any factor $x_i - x_j$ of $A\rho$, then $i = h\rho$, $j = k\rho$, for some (unique) h, k between 1 and n , and either $x_h - x_k$ or $x_k - x_h$ occurred in A . Thus we are led to the following crucial definition:

Definition 11 The permutation ρ is *even* if $A\rho = A$; it is *odd* if $A\rho = -A$.

We call the evenness or oddness of a permutation its **parity**.

Example 12 Consider the *cyclic* permutation $(1\ 2\ 3)$, that is, $\rho(1) = 2$, $\rho(2) = 3$, $\rho(3) = 1$. Then if $A = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$,

$$A\rho = (x_2 - x_3)(x_2 - x_1)(x_3 - x_1) = A$$

Thus ρ is even. On the other hand, if σ is the permutation $(1\ 3)$, that is, $\sigma(1) = 3$, $\sigma(2) = 2$, $\sigma(3) = 1$, then

$$A\sigma = (x_3 - x_2)(x_3 - x_1)(x_2 - x_1) = -A,$$

so σ is odd.

Now in the next break you will see that every permutation may be expressed as a composition of *transpositions*, that is, of cycles that (like $(1\ 3)$ above) merely interchange two numbers. Granted this, one may show the following:

Theorem 13 *A permutation is even if and only if it may be expressed as a composition of an even number of transpositions.*

You may ask—why don't we simply *define* an even permutation as one that may be expressed as a composition of an even number of transpositions? For that is surely the reason for the use of the name “even.” The answer is that it is by no means obvious that, if a permutation can be expressed as a composition of an *even* number of transpositions, it cannot also be expressed as a composition of an *odd* number of transpositions. It is, indeed, Theorem 13 that makes that fact clear.

However, before we prove Theorem 13, we'll take the promised break and put you to work seeing why every permutation may be expressed as a composition of transpositions.²³

• • • BREAK 7

- (1) Satisfy yourselves, by looking at a particular but not special case, that every permutation of a set of numbers may be expressed as a composition of cycles acting on disjoint sets of numbers. (See our example (4).)

²³We adopt the usual convention that the identity permutation is the *empty* composition of transpositions.

- (2) We now need to show that every cycle $(\iota_1 \iota_2 \cdots \iota_r)$ is a composition of transpositions. Of course, this holds if $r = 2$. Now, if $r \geq 3$, show that

$$(\iota_1 \iota_2 \cdots \iota_r) = (\iota_1 \iota_2 \cdots \iota_{r-1})(\iota_1 \iota_r)$$

(recall that fg means “do f , then do g ”) and hence deduce, by induction on r , that the cycle $(\iota_1 \iota_2 \cdots \iota_r)$ is a composition of transpositions.

- (3) Find an explicit expression for $(\iota_1 \iota_2 \cdots \iota_r)$ as a composition of transpositions.

Now we return to the proof of Theorem 13. It is obvious that even and odd permutations under composition $*$ act like ordinary integers under addition, that is,

$$\text{even} * \text{even} = \text{even},$$

$$\text{odd} * \text{odd} = \text{even},$$

$$\text{even} * \text{odd} = \text{odd},$$

$$\text{odd} * \text{even} = \text{odd}.$$

It therefore follows immediately that, if we can show that every transposition is odd, then it must be true that any permutation expressible as a composition of an *even* number of transpositions is *even*, and any permutation expressible as a composition of an *odd* number of transpositions is *odd* — and this will prove Theorem 13. Thus we are left to prove that *every transposition is odd*.

First, we claim that the transposition $(1\ 2)$ is odd. For if we apply $(1\ 2)$ to the form $A = \prod_{1 \leq i < j \leq n} (x_i - x_j)$, then the factor $(x_1 - x_2)$ is transformed into its negative, while all the other factors are merely permuted among themselves with no change of sign. Thus $(1\ 2)$ is an odd permutation. Second, consider the transposition $(i\ j)$, $i < j$. We claim that

$$(2\ j)(1\ 2)(2\ j) = (1\ j), \quad j \geq 3,$$

$$(1\ j)(1\ 2)(1\ j) = (2\ j), \quad j \geq 3,$$

$$(1\ i)(2\ j)(1\ 2)(1\ i)(2\ j) = (i\ j), \quad i \geq 3, \quad j \geq 3.$$

Thus, since $(1\ 2)$ is odd, so is every $(i\ j)$, and Theorem 13 is completely proved.²⁴

²⁴Notice that the parity of a permutation is unaffected by the choice of n in Definition 11.

• • • **BREAK 8**

Let σ be an arbitrary element of S_n and let ρ be the cycle $(\iota_1 \iota_2 \cdots \iota_r)$ in S_n . Show that $\sigma^{-1}\rho\sigma$ is the cycle $(\iota_1\sigma \ \iota_2\sigma \ \cdots \ \iota_r\sigma)$, where $m\sigma$ is the effect of the permutation σ on m .

Now it is plain that the set of even permutations of n objects is a *subgroup* of S_n ; it is called the **alternating group on n objects** and written A_n . Moreover, if $n \geq 2$ and ρ is any *odd* permutation, then $A_n\rho$ is the set of *all* odd permutations. For certainly every permutation in $A_n\rho$ is odd; and, conversely, if σ is odd, then $\sigma\rho^{-1}$ is even, $\sigma\rho^{-1} \in A_n$, and $\sigma = (\sigma\rho^{-1})\rho$. Thus S_n is the disjoint union of the two cosets A_n and $A_n\rho$, consisting of the even and odd permutations, respectively, and hence A_n is a subgroup of S_n of index 2, as claimed in Section 2.

Notice that we could have argued above using ρA_n instead of $A_n\rho$; indeed, and remarkably, $\rho A_n = A_n\rho$. It is a special, and very important, property of some subgroups H of a group G that $gH = Hg$ for all $g \in G$. We call such subgroups **normal**. Our argument above can easily be generalized to show that every subgroup of index 2 of an arbitrary group G is normal.

• • • **BREAK 9**

- (1) Let P be a subgroup of S_n and let Q be the subset of P consisting of the even permutations in P . Show that Q is a subgroup of P of index 1 or 2. Give an example of each possibility.
- (2) Show that, if H is a subgroup of G of index 2, then, for any $g \notin H$, we have $gH = Hg$.

Finally, let us remark here that the reason that even permutations figure so prominently in the discussion of symmetry in geometry is this: any symmetry of a polyhedron, and any orientation-preserving symmetry of a polygon, will induce an *even* permutation of the vertices. Try it and see!

• • • **BREAK 10**

- (1) It is a fact that the regular tetrahedron (T) may be inscribed in the regular hexahedron or cube (H), as shown in Figure 9. Look at the rotations of the cube (from Break 6) that leave T occupying

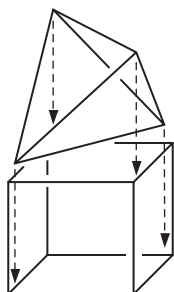


FIGURE 9

		<i>T</i>	<i>H</i>	<i>O</i>	<i>D</i>	<i>I</i>
<i>F</i>		4	6	8	12	20
<i>V</i>		4	8	6	20	12
<i>E</i>		6	12	12	30	30
Axes of sym- metry	$\frac{2\pi}{2}$	?	6			?
	$\frac{2\pi}{3}$	?	4			?
	$\frac{2\pi}{4}$		3			
	$\frac{2\pi}{5}$					?
Planes of symmetry			6 3			?
Face angles		?	24			?
Diagonals (interior)			4	3	60 30 10	? ?

FIGURE 10 Here *T* is “tetrahedron,” *H* is “hexahedron” (or “cube”), etc.

its original position within the cube. You should see that these rotations are simply the *even* permutations of the four strips of the diagonal cube (and that all the odd permutations move the tetrahedron from its original position to the same new position). Thus you have a dramatic confirmation that the symmetry group of *T* is A_4 , the alternating subgroup of S_4 , the symmetry group of *H*.

- (2) Figures 10 and 11, which are about the Platonic solids (see Figure 4), were copied by JP from Pólya’s personal notebook. Figure 11 is even in Pólya’s own handwriting. See if you can fill in the blanks in Figure 10; and try to work out what Pólya means in Figure 11.

Page from Polya's notebook

FIGURE 11 As before, T is “tetrahedron,” H is “hexahedron” (or “cube”), etc.

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