4
Fourier Analysis in Hilbert Space

In the last section of Chapter 3 we introduced the Lebesgue L^p-spaces for general measures and discussed their most basic properties. The most important L^p-space, by far, is L^2. Its importance is its role in applications, especially in Fourier analysis. The material of this chapter lies at the foundation of the branch of mathematics called harmonic analysis.

In this chapter we will see that L^2 is a Hilbert space (we already really have all the bits of information we need to see this) and that in some sense the L^2-spaces (with different μ's) are the *only* Hilbert spaces. We will come to see how the problem that Fourier examined, about decomposing functions as infinite sums of other — somehow more basic — functions, is a problem best phrased and understood in the language of abstract Hilbert spaces. One of the triumphs of functional analysis is to take a very concrete problem — in this case Fourier decomposition — view it in an abstract setting, and use theoretical tools to obtain powerful results that can be translated back to the concrete setting. Fourier's work certainly holds an important spot at the roots of functional analysis, and it motivated much early work in the development of the field.

Further Hilbert space theory appears in Section 5.4.

4.1 Orthonormal Sequences

During the second half of the eighteenth century and first decade of the nineteenth century, infinite sums of sines and cosines appeared as solutions to physical problems then being studied. Daniel Bernoulli (1700–1782; Netherlands)[1] suggested that these sums were solutions to the problem of modeling the vibrating string, and Joseph Fourier (1768–1830; France) proposed them as solutions to the problem of modeling heat flow. It is not really until the response to Fourier's work that we see other mathematicians coming to grips with the challenge that these infinite sums truly posed: to understand the fundamental notions of *convergence* and *continuity*. Over the decades following the appearance of Fourier's works on heat, the field of "real analysis" would be born in large part out of efforts to respond to the challenges that Fourier's work raised in pure mathematics. Many of the great mathematicians of the period — perhaps most notably Cauchy, Riemann, and Weierstrass — did their most important work in the development of this field. For an excellent historical account of these mathematical developments, see [25]. Fourier begins with an arbitrary function f on the interval from $-\pi$ to π and states that *if* we can write

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx),$$

then it must be the case that the coefficients a_k and b_k are given by the formulas

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx, \quad k = 0, 1, 2, \ldots,$$

and

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx, \quad k = 1, 2, \ldots .$$

The big question is this: *When* is this decomposition actually possible? Even if the integrals involved make sense, does the series converge? If it does converge, what *type* of convergence (pointwise, uniform, etc.) do we get? Even if the series converges in some sense, does it converge to f?

The immediate goal is to show you how these questions about Fourier series can be treated in the abstract setting of an inner product space.

Let us now take stock of what we already know by gathering our information about L^2. First, recall that $L^2 = L^2(\mu)$, for any abstract measure space $(X, \mathcal{R}, \mu)$, denotes the collection of all measurable functions $f : X \to \mathbb{C}$ such that the integral

$$\int_X |f|^2 d\mu$$

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[1] Daniel Bernoulli is the nephew of James Bernoulli, who was mentioned at the beginning of Section 3.1. The Bernoulli family produced several distinguished mathematicians and physicists; at least twelve members of the family achieved distinction in at least one of these fields.

is finite. These functions are often called the "square integrable" functions on X. With norm

$$\|f\|_2 = \sqrt{\int_X |f|^2 d\mu},$$

this collection of functions becomes a Banach space. We can define an inner product on L^2 via

$$\langle f, g\rangle = \int_X f\overline{g}d\mu.$$

It is easily seen that this is an inner product, and that the norm does indeed come from this inner product. That is,

$$\|f\|_2 = \sqrt{\langle f, f\rangle} = \sqrt{\int_X |f|^2 d\mu}.$$

Theorem 3.21 shows that L^2 is a Hilbert space.

In the following definitions, the terminology should seem familiar from your experiences with $\mathbb{R}^n$.

Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space. We say that v and w in V are *orthogonal* if $\langle v, w\rangle = 0$. We say that v is *normalized* if $\|v\| = \sqrt{\langle v, v\rangle} = 1$. A sequence $\{v_k\}_{k=1}^{\infty}$ in V is an *orthonormal sequence* if $\langle v_k, v_j\rangle = \delta_{kj}$, $1 \le k, j < \infty$. The function δ_{kj} is defined to be 1 if $k = j$ and 0 if $k \ne j$.

In Exercise 4.1.1 you are asked to show that the *trigonometric system* (Figure 4.1)

$$\frac{1}{\sqrt{2\pi}}, \quad \frac{\cos(nx)}{\sqrt{\pi}}, \quad \frac{\sin(mx)}{\sqrt{\pi}}, \qquad n, m = 1, 2, \ldots,$$

is an orthonormal sequence in the inner product space $L^2([-\pi, \pi], m)$. From this, you should find it plausible that the goal of Fourier analysis in its general setting is this: Given an orthonormal sequence $\{f_k\}_{k=1}^{\infty}$ in an inner product space V and an $f \in V$, find complex numbers c_k such that

$$f = \sum_{k=1}^{\infty} c_k f_k.$$

The convergence of this infinite sum is in the norm induced by the inner product. Further, it would be desirable to be able to do this *for all* $f \in V$. In general, this cannot be done. Notice that Fourier was asserting that when $\{f_k\}_{k=1}^{\infty}$ is the trigonometric system, the coefficients are of form $\langle f, f_k\rangle$ (an appropriate indexing of the trigonometric system has not yet been established) whenever his decomposition works.

Let $\{f_k\}_{k=1}^{\infty}$ be an orthonormal sequence in V. If it is the case that for each $f \in V$ we can find constants c_k (depending on f) such that

$$f = \sum_{k=1}^{\infty} c_k f_k,$$

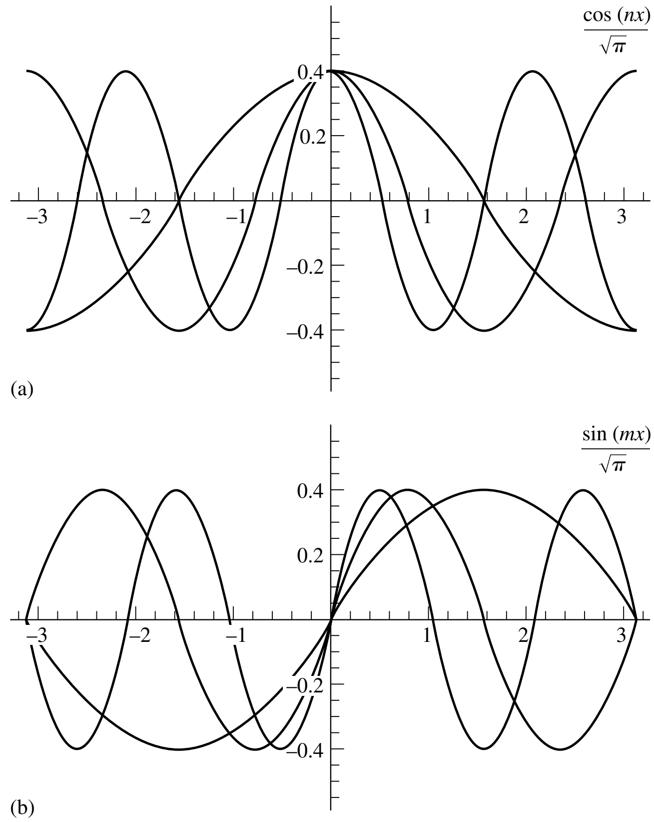

FIGURE 4.1. (a) the functions $\frac{\cos(nx)}{\sqrt{\pi}}$ for $n = 1, 2, 3$. (b) the functions $\frac{\sin(mx)}{\sqrt{\pi}}$ for $m = 1, 2, 3$.

then we say that the sequence $\{f_k\}_{k=1}^{\infty}$ is a *complete orthonormal sequence* in V.[2] A complete orthonormal sequence is sometimes called an *orthonormal basis* for V. The latter terminology can cause confusion since a complete orthonormal system is not a basis in the finite-dimensional sense discussed in Section 1.3.

The questions posed by Fourier's work are, to some degree, answered by the fact that the trigonometric system does indeed form a complete orthonormal sequence in L^2. This important result appears as Theorem 4.6.

The trigonometric system is certainly an important complete orthonormal sequence (for the Hilbert space $L^2([-\pi, \pi])$). But there are others, and we end this section with a brief description of a few of them ([43] is a good general reference for this topic). We can use the Gram–Schmidt process to construct an orthonormal sequence in any inner product space.

For our first example, the Hilbert space is $L^2([-1, 1])$. If one applies the Gram–Schmidt process to the functions $1, x, x^2, x^3, \ldots,$ one obtains the complete

[2] Note that this is a new usage of the word "complete"; we now have at least two ways we will use this adjective: a *complete* metric space, a *complete* orthonormal system.

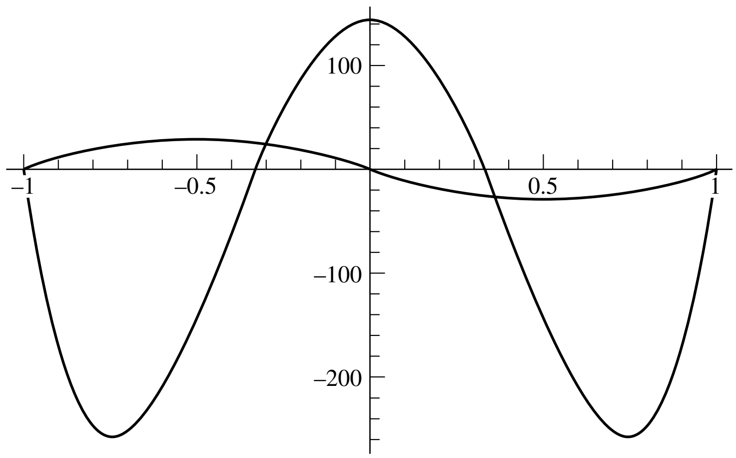

FIGURE 4.2. The Legendre polynomials, $n = 3, 4$.

orthonormal sequence of *Legendre polynomials* (Figure 4.2),

$$\sqrt{\frac{2n+1}{2}}\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \qquad n = 1, 2, \ldots .$$

These polynomials are named for Adrien-Marie Legendre (1752–1833; France).

Next, consider the Hilbert space $L^2((0,\infty))$. If one applies the Gram–Schmidt process to the the functions $x^n e^{-x}$, $n = 0, 1, \ldots$, one obtains the complete orthonormal sequence of *Laguerre functions*. These appear in quantum mechanics in the analysis of the hydrogen atom. This family is named for Edmond Laguerre (1834–1886; France).

For our third example, the Hilbert space is $L^2(\mathbb{R})$. If one applies the Gram–Schmidt process to the the functions $x^n e^{\frac{-x^2}{2}}$, $n = 0, 1, \cdots$, one obtains the complete orthonormal sequence of *Hermite functions*. These also appear in quantum mechanics. This family is named for Charles Hermite (1822–1901; France).

The Legendre, Laguerre, and Hermite functions all show up as eigenfunctions of certain linear operators (linear operators are the subject of the next chapter) related to the Sturm–Liouville problem in differential equations.

The final family we discuss is the complete orthonormal sequence of *Haar functions*. The Hilbert space is $L^2([0,1])$. This example is fundamentally different from the previous examples in that the functions in this family are not continuous, and they are not connected with differential equations. Haar functions appear in the study of "wavelets." Wavelet theory and its applications experienced explosive development in the 1980s. There are several good books, at varying levels, on the subject. A "brief" investigation of wavelets, their properties and uses, makes a good student project ([28] gives an excellent overview and introduction to wavelets). Wavelet series are used in signal and imaging processing and, in some contexts,

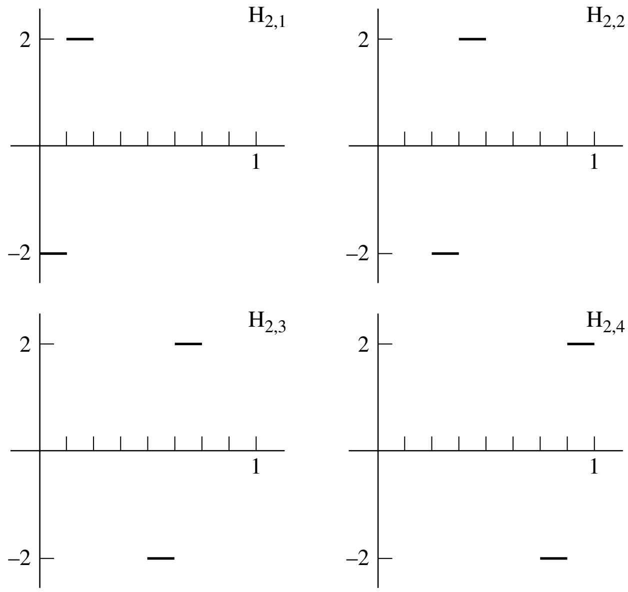

FIGURE 4.3. The Haar functions, $H_{2,k}(x)$.

are replacing the classical Fourier series. We define

$$H_{0,0}(x) = 1, \qquad H_{n,k}(x) = \begin{cases} -2^{\frac{n}{2}} & \text{if } \frac{k-1}{2^n} \le x < \frac{k-\frac{1}{2}}{2^n}, \\ 2^{\frac{n}{2}} & \text{if } \frac{k-\frac{1}{2}}{2^n} \le x < \frac{k}{2^n}, \\ 0 & \text{otherwise}, \end{cases}$$

for $n \geq 1$, $1 \leq k \leq 2^n$ (Figure 4.3). This family is named for Alfréd Haar (1885–1933; Hungary).

Jean Baptiste Joseph Fourier was born March 21, 1768, in Auxerre, France (Figure 4.4). His father had been a tailor, but both of his parents were dead by the time Fourier was ten. There seems to be some disagreement among authors as to exactly how many siblings Fourier had, but by all accounts he had many. According to [51], he was the nineteenth (and not the last) child in the family.

Fourier was distinguished in two fields: mathematics and Egyptology. He began both careers when he attended a military school run by the Benedictines. He showed great talent in many areas by the age of fourteen. He wanted to join the military, for some reason was rejected, and instead entered a Benedictine abbey to train for the priesthood. While there, he was able to work on mathematics and submitted his first paper in 1789. He never took his vows and returned to his school, teaching math, history, philosophy, and rhetoric. This was the time of the French Revolution, and Fourier became quite involved in revolutionary politics. In 1794 he was imprisoned and sentenced to be guillotined.

FIGURE 4.4. Joseph Fourier.

In 1795 the École Normale in Paris opened to train teachers in an effort to rehabilitate the system of higher education in France. The students were chosen and financed by the republic. Fourier was chosen to attend, and while there, he came into contact with very good professors: Lagrange, Laplace, and Gaspard Monge (1746–18181; France). Unfortunately, the school closed after a few months. At this time, Fourier went to teach at the École Polytechnique, which was designed as a military academy to train the military elite. During this period he was, for a second time, arrested and subsequently freed.

Over the next few years, Fourier taught (mathematics with military applications) and worked on mathematical research (mostly having to do with polynomials: extending Descartes's rule of signs, approximating values of real roots, detecting existence of complex roots).

In 1798 Fourier was recommended, by Monge and the chemist Claude Louis Berthollet (1748–1822; France), to be Napoleon Bonaparte's scientific advisor on his expedition to Egypt. Very soon after their arrival in Egypt, the Institut d'Égypte opened in Cairo, and Fourier was appointed *secrétaire perpetuel*. He had many duties in this post, including investigating ancient monuments and irrigation projects, but he managed to find time to continue mathematical research.

Napoleon left for France in 1799. Fourier followed in 1801 and was appointed by Napoleon to a government position in Grenoble. He held this post from 1802 until 1814. During this period, he devoted much time to the writing of a massive work entitled ***Description de l'Égypte***. This work was written by the team that Napoleon brought with him to Egypt and is very important in the birth of the modern field of Egyptology; it gave the most comprehensive account, to date, of ancient and contemporary Egypt. To put this accomplishment in perspective, the Rosetta Stone was discovered by this team, and it was in 1822 that hieroglyphics were fully deciphered.

It was also during his time in Grenoble that Fourier did his work on heat diffusion. This work, done primarily during the period 1804–1807, culminated in a monograph that was submitted to the Institut de France in Paris at the end of 1807. This paper caused a great deal of controversy. One complaint was from Lagrange and had to do with the convergence of his "Fourier series." Lagrange's skepticism was on target and, indeed, led to the rise of a new field of mathematics: "real analysis" (see [25]). The controversy caused Fourier to revise the paper and resubmit it in 1811. Eventually, his ***Théorie analytique de la chaleur*** was published in 1822. This work is Fourier's greatest contribution and certainly remains one of the masterpieces of mathematical physics. It is important not only for the physical explanations that it gives, but also for the mathematical techniques developed in the course of his attempting to explain the physics of heat flow. For example, he developed techniques to find solutions for many differential equations that, up until that point, had not been worked out.

In the last fifteen years of his life, Fourier continued to work on mathematics and on topics related to his work in Egypt. However, his most substantial contributions had already been made, and much of his mathematical work during his later years focused on consequences of his earlier work. One of his other important mathematical projects during this time was on problems that can now be viewed as precursors to the field of linear programming. He also did editorial work and wrote several biographies of mathematicians during this period.

Fourier died on May 16, 1830, after being in a state of deteriorating health for several years.

4.2 Bessel's Inequality, Parseval's Theorem, and the Riesz–Fischer Theorem

Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space, and $\{f_k\}_{k=1}^{\infty}$ a specified orthonormal sequence in V. Suppose we have an $f \in V$ that we can decompose as

$$f = \sum_{k=1}^{\infty} c_k f_k.$$

What, then, are the c_k's to be? We turn to the very simple case of $\mathbb{R}^3$, with its usual inner product, for inspiration. We take as our orthonormal family the three Euclidean basis vectors $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$. Then every vector in $\mathbb{R}^3$ can be written in form

$$\sum_{k=1}^{3} c_k e_k.$$

In this case we know that $c_1 = \langle v, e_1\rangle$, $c_2 = \langle v, e_2\rangle$, and $c_3 = \langle v, e_3\rangle$. This example illustrates the next theorem.

Theorem 4.1. *Suppose that $f = \sum_{k=1}^{\infty} c_k f_k$ for an orthonormal sequence $\{f_k\}_{k=1}^{\infty}$ in an inner product space V. Then $c_k = \langle f, f_k\rangle$ for each k.*

Before we prove this result, notice that this is, in fact, consistent with Fourier's assertion about the trigonometric system.

Proof. Let $s_n = \sum_{k=1}^{n} c_k f_k$. Our hypothesis is thus that

$$\lim_{n\to\infty} \|s_n - f\| = 0.$$

Fix an m and let $n \geq m$. Then

$$\lim_{n\to\infty} \langle s_n, f_m\rangle = \langle f, f_m\rangle. \tag{4.1}$$

This is because

$$\langle s_n, f_m\rangle - \langle f, f_m\rangle = \langle s_n - f, f_m\rangle,$$

and

$$|\langle s_n - f, f_m\rangle| \leq \|s_n - f\| \cdot \|f_m\| = \|s_n - f\|.$$

Therefore,

$$\langle s_n, f_m\rangle = \sum_{k=1}^{\infty}\langle c_k f_k, f_m\rangle = \sum_{k=1}^{\infty} c_k\langle f_k, f_m\rangle = \sum_{k=1}^{\infty} c_k\delta_{km} = c_m. \tag{4.2}$$

Combining (4.1) and (4.2) gives the desired result. □

Let $\{f_k\}_{k=1}^{\infty}$ be an orthonormal sequence in V, and let $f \in V$. We call $\sum_{k=1}^{\infty}\langle f, f_k\rangle f_k$ the *Fourier series* of f with respect to $\{f_k\}_{k=1}^{\infty}$, and $\langle f, f_k\rangle$ the *Fourier coefficients* of f with respect to $\{f_k\}_{k=1}^{\infty}$. These objects are defined without any assumptions or knowledge about convergence of the series.

The next theorem tells us something about the size of these coefficients.

Theorem 4.2 (Bessel's Inequality[3]). *Suppose that $\{f_k\}_{k=1}^{\infty}$ is an orthonormal sequence in an inner product space V. For every $f \in V$, the series (of nonnegative real numbers) $\sum_{k=1}^{\infty}|\langle f, f_k\rangle|^2$ converges and*

$$\sum_{k=1}^{\infty}|\langle f, f_k\rangle|^2 \leq \|f\|^2.$$

Proof. Consider the partial sum s_n of the Fourier series for f. Then

$$\begin{aligned}
\langle f - s_n, f_k\rangle &= \langle f, f_k\rangle - \langle s_n, f_k\rangle \\
&= \langle f, f_k\rangle - \Big\langle \sum_{j=1}^{n}\langle f, f_j\rangle f_j, f_k\Big\rangle \\
&= \langle f, f_k\rangle - \sum_{j=1}^{n}\Big\langle\langle f, f_j\rangle f_j, f_k\Big\rangle \\
&= \langle f, f_k\rangle - \sum_{j=1}^{n}\langle f, f_j\rangle(f_j, f_k\rangle \\
&= \langle f, f_k\rangle - \sum_{j=1}^{n}\langle f, f_j\rangle\delta_{jk} \\
&= \langle f, f_k\rangle - \langle f, f_k\rangle = 0.
\end{aligned}$$

This shows that $f - s_n$ is orthogonal to each f_k. Further,

$$\langle f - s_n, s_n\rangle = \Big\langle f - s_n, \sum_{k=1}^{n}\langle f, f_k\rangle f_k\Big\rangle$$

[3] Due to Friedrich Bessel (1784–1846; Westphalia, now Germany).

$$= \sum_{k=1}^{n} \Big\langle f - s_n, \langle f, f_k \rangle f_k \Big\rangle$$
$$= \sum_{k=1}^{n} \overline{\langle f, f_k \rangle} \langle f - s_n, f_k \rangle,$$

which equals zero by the previous argument. This shows that $f - s_n$ is orthogonal to s_n. Then, by Exercise 4.2.1,

$$\|f - s_n\|^2 + \|s_n\|^2 = \|f\|^2.$$

This shows that

$$\|s_n\|^2 \leq \|f\|^2.$$

Since

$$\|s_n\|^2 = \Big\| \sum_{k=1}^{n} \langle f, f_k \rangle f_k \Big\|^2,$$

which, by the same exercise and induction, is equal to

$$\sum_{k=1}^{n} \|\langle f, f_k \rangle f_k\|^2,$$

we have that

$$\|s_n\|^2 = \sum_{k=0}^{n} |\langle f, f_k \rangle|^2 \cdot \|f_k\|^2 = \sum_{k=1}^{n} |\langle f, f_k \rangle|^2.$$

Combining these last two sentences yields

$$\sum_{k=1}^{n} |\langle f, f_k \rangle|^2 \leq \|f\|^2.$$

Since this holds for each n,

$$\sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq \|f\|^2,$$

as desired. □

It is natural to want to determine conditions on $\{f_k\}_{k=1}^{\infty}$ under which equality in Bessel's inequality holds.

Theorem 4.3 (Parseval's Theorem[4]). *As in the preceding theorem, suppose that $\{f_k\}_{k=1}^{\infty}$ is an orthonormal sequence in an inner product space V. Then $\{f_k\}_{k=1}^{\infty}$ is a complete orthonormal sequence if and only if for every $f \in V$,*

$$\sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 = \|f\|^2.$$

[4] Due to Marc-Antoine Parseval des Chênes (1755–1836; France).

PROOF. This is left as Exercise 4.2.1(b). □

We end this section with a sort of converse to Bessel's inequality. Theorem 4.2 implies, as a special case, that if $f \in L^2$, then the sum of squares of the Fourier coefficients of f, with respect to the trigonometric system $\{f_k\}_{k=1}^{\infty}$, is always finite. The combination of Theorems 4.2, 4.3, 4.4, and 4.6 sets up a "linear isometry" between $L^2([-\pi, \pi], m)$ and ℓ^2. Specifically, for $f \in L^2$, define $Tf = \{\langle f, f_k\rangle\}_{k=1}^{\infty}$, where $\{f_k\}_{k=1}^{\infty}$ denotes the trigonometric system. Then $Tf \in \ell^2$ (Theorem 4.2), T is linear, one-to-one (Theorem 4.6, together with Theorem 4.5 (c)), onto (Theorems 4.4 and 4.6), and $\|f\|_{L^2} = \|Tf\|_{\ell^2}$ (Theorem 4.3), for all $f \in L^2$. This result, that L^2 and ℓ^2 are isometrically isomorphic, is referred to as the Riesz–Fischer Theorem (Theorem 4.6 sometimes goes by the same name).

Theorem 4.4. *Assume that*

(a) $\{d_k\}_{k=1}^{\infty}$ *is a sequence of real numbers such that* $\sum_{k=1}^{\infty} d_k^2$ *converges, and*
(b) V *is a Hilbert space with complete orthonormal sequence* $\{f_k\}_{k=1}^{\infty}$.

Then there is an element $f \in V$ *whose Fourier coefficients with respect to* $\{f_k\}_{k=1}^{\infty}$ *are the numbers* d_k, *and*

$$\|f\|^2 = \sum_{k=1}^{\infty} d_k^2.$$

PROOF. Define

$$s_n = \sum_{k=1}^{n} d_k f_k.$$

For $m > n$,

$$\|s_n - s_m\|^2 = \sum_{j=n+1}^{m} \sum_{k=n+1}^{m} d_j d_k \langle f_j, f_k\rangle = \sum_{k=n+1}^{m} d_k^2.$$

Therefore, $\{s_n\}_{n=1}^{\infty}$ is Cauchy. Because V is a Hilbert space, there is an $f \in V$ such that

$$\lim_{n\to\infty} \|s_n - f\| = 0.$$

This is what we mean when we write

$$f = \sum_{k=1}^{\infty} d_k f_k,$$

and Theorem 4.1 now says that $d_k = \langle f, f_k\rangle$.

The remaining identity now follows from Parseval's theorem. □

In order for the Riesz–Fischer theorem to be true, we would need to know that the trigonometric system is, in fact, a *complete* orthonormal sequence in $L^2([-\pi, \pi], m)$. This is the goal of the next section.

4.3 A Return to Classical Fourier Analysis

We now return to the classical setting, and the orthonormal family

$$\frac{1}{\sqrt{2\pi}}, \quad \frac{\cos(nx)}{\sqrt{\pi}}, \quad \frac{\sin(mx)}{\sqrt{\pi}}, \qquad n, m = 1, 2, \dots,$$

in $L^2([-\pi, \pi], m)$.

Our first theorem of this section is proved mainly for its use in the proof of Theorem 4.6 (that is why it appears here and not in the preceding section), but it is interesting in its own right. Parseval's theorem gives an alternative way to think about "completeness" of an orthonormal family; this theorem gives a few more ways. We state it only for orthonormal families in the specific Hilbert space $L^2([-\pi, \pi], m)$; the result can be generalized to arbitrary Hilbert spaces.

Theorem 4.5. *For an orthonormal sequence $\{f_k\}_{k=1}^{\infty}$ in $L^2([-\pi, \pi], m)$, the following are equivalent:*

(a) *$\{f_k\}_{k=1}^{\infty}$ is a complete orthonormal sequence.*
(b) *For every $f \in L^2$ and $\epsilon > 0$ there is a finite linear combination*

$$g = \sum_{k=1}^{n} d_k f_k$$

such that $\|f - g\|_2 \le \epsilon$.
(c) *If the Fourier coefficients with respect to $\{f_k\}_{k=1}^{\infty}$ of a function in L^2 are all 0, then the function is equal to 0 almost everywhere.*

Proof. It should be clear from the definition that (a) implies (b).

To prove that (b) implies (c), let f be a square integrable function such that $\langle f, f_k\rangle = 0$ for all k. Let $\epsilon > 0$ be given and choose g as in (b). Then

$$\begin{aligned}\|f\|_2^2 &= \left| \|f\|_2^2 - \left\langle f, \sum_{k=1}^{n} d_k f_k \right\rangle \right| = |\langle f, f - g\rangle| \\ &\le \|f\|_2 \cdot \|f - g\|_2 \le \epsilon \|f\|_2.\end{aligned}$$

This implies that $\|f\|_2 \le \epsilon$. Since ϵ was arbitrary, f must be 0 almost everywhere.

To prove that (c) implies (a), let $f \in L^2$ and put

$$s_n = \sum_{k=1}^{n} \langle f, f_k\rangle f_k.$$

As in the proof of Theorem 4.4, we see that $\{s_n\}_{n=1}^{\infty}$ is Cauchy in L^2. And Theorem 3.21 then tells us that there is a function $g \in L^2$ such that

$$\lim_{n\to\infty} \|s_n - g\|_2 = 0.$$

That is,

$$g = \sum_{k=1}^{\infty} \langle f, f_k\rangle f_k.$$

Theorem 4.1 then tells us that the Fourier coefficients of g are the same as the Fourier coefficients of f with respect to $\{f_k\}_{k=1}^{\infty}$, i.e., $\langle g, f_k\rangle = \langle f, f_k\rangle$. By (b), $f - g$ must equal 0 almost everywhere. In other words,

$$f = \sum_{k=1}^{\infty} \langle f, f_k\rangle f_k.$$

Since f was arbitrary, $\{f_k\}_{k=1}^{\infty}$ is a complete orthonormal sequence. □

Theorem 4.6. *The trigonometric system*

$$\frac{1}{\sqrt{2\pi}}, \quad \frac{\cos(nx)}{\sqrt{\pi}}, \quad \frac{\sin(mx)}{\sqrt{\pi}}, \qquad n, m = 1, 2, \ldots,$$

forms a complete orthonormal sequence in $L^2([-\pi, \pi], m)$. That is, if f is such that $|f|^2$ is Lebesgue integrable, then its (classical) Fourier series converges to f. The convergence is convergence in the norm $\|\cdot\|_2$, i.e.,

$$\lim_{n\to\infty} \left[\int_{-\pi}^{\pi} \left[f(x) - \left(\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kx) + \sum_{k=1}^{n} b_k \sin(kx) \right) \right]^2 dx \right] = 0.$$

This type of convergence is often called "in mean" convergence.

Proof. In Exercise 4.1.1 you are asked to prove that the trigonometric system is orthonormal. We complete the proof of the theorem by verifying that condition (c) of Theorem 4.5 holds. First consider the case that f is continuous and real-valued, and that $\langle f, f_k\rangle = 0$ for each f_k. If $f \neq 0$, we then know that there exists an x_0 at which $|f|$ achieves a maximum, and we may assume that $f(x_0) > 0$. Let δ be small enough to ensure that $f(x) > \frac{f(x_0)}{2}$ for all x in the interval $(x_0 - \delta, x_0 + \delta)$. Consider the function

$$t(x) = 1 + \cos(x_0 - x) - \cos(\delta).$$

This function is a finite linear combination of functions in the trigonometric system; such functions are called "trigonometric polynomials." It is straightforward to verify

(i) $1 < t(x)$, for all x in $(x_0 - \delta, x_0 + \delta)$, and
(ii) $|t(x)| \le 1$ for all x outside of $(x_0 - \delta, x_0 + \delta)$.

Since f is orthogonal to every member of the trigonometric system, f is orthogonal to every trigonometric polynomial and, in particular, is orthogonal to t^n for every positive integer n. This will lead us to a contradiction. Notice that

$$\begin{aligned} 0 = \langle f, t^n\rangle &= \int_{-\pi}^{\pi} f(x)t^n(x)dx \\ &= \int_{-\pi}^{x_0-\delta} f(x)t^n(x)dx + \int_{x_0-\delta}^{x_0+\delta} f(x)t^n(x)dx + \int_{x_0+\delta}^{\pi} f(x)t^n(x)dx. \end{aligned}$$

By (ii) above, the first and third integrals are bounded in absolute value for each n by $2\pi f(x_0)$. The middle integral, however, is greater than or equal to $\int_a^b f(x)t^n(x)dx$, where $[a, b]$ is any closed interval in $(x_0 - \delta, x_0 + \delta)$. Since t is continuous on $[a, b]$, we know that t achieves a minimum value, m, there. By (i) above, $m > 1$. Then

$$\int_a^b f(x)t^n(x)dx \geq \frac{f(x_0)}{2} \cdot m^n \cdot (b - a),$$

which grows without bound as $n \to \infty$. This contradicts the assumption that $0 = \langle f, t^n \rangle$ for all n. Thus, any continuous real-valued function that is orthogonal to every trigonometric polynomial must be identically 0.

If f is continuous but not real-valued, our hypothesis implies that

$$\int_{-\pi}^{\pi} f(x)e^{-ikx}dx = 0, \quad k = 0, \pm 1, \pm 2, \ldots,$$

and thus also that

$$\int_{-\pi}^{\pi} \overline{f(x)}e^{-ikx}dx = 0, \quad k = 0, \pm 1, \pm 2, \ldots.$$

If we add and subtract these two equations, we see that the real and imaginary parts of f are orthogonal to each of the members of the trigonometric system. By the first part of the proof, the real and imaginary parts of f are identically 0; hence f is identically 0.

Finally, we no longer assume that f is continuous. Define the continuous function

$$F(x) = \int_{-\pi}^{x} f(t)dt.$$

For now let us assume that $f_k(x) = \frac{\cos(kx)}{\sqrt{\pi}}$. Our hypothesis implies

$$0 = \int_{-\pi}^{\pi} f(x)\cos(kx)dx.$$

Integration by parts yields

$$\int_{-\pi}^{\pi} F(x)\sin(kx)dx = \frac{1}{k}\int_{-\pi}^{\pi} f(x)\cos(kx)dx = 0.$$

Similarly, we can show that

$$\int_{-\pi}^{\pi} F(x)\cos(kx)dx = 0.$$

We now have shown that F, and hence $F - C$ for every constant C, is orthogonal to each of the nonconstant members of the trigonometric system. We now take care of the member $\frac{1}{\sqrt{2\pi}}$. Let

$$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)dx.$$

Then $F - C_0$ is easily seen to be orthogonal to *every* member of the trigonometric system. Since F is continuous, $F - C_0$ is also continuous, and the first part of the proof shows that $F - C_0$ is identically 0. From this it follows that $f = F'$ is 0 almost everywhere. □

Is this theorem "good"? Note that $\|\cdot\|_2$-convergence does not necessarily imply either uniform or pointwise convergence (Exercise 4.1.4). With uniform convergence, for example, we know that we cannot get the same result because the partial sums of the Fourier series of f are always continuous functions, and if the convergence of the series were uniform, then f would have to be continuous, too. Since $L^2([-\pi, \pi], m)$ contains discontinuous functions, we see that uniform convergence cannot always be achieved.

Theorem 4.6 has an important corollary, which we state as our next theorem. See Exercise 3.6.8 for an alternative proof of this same result.

Theorem 4.7. *$C([-\pi, \pi])$ is dense in L^2.*

PROOF. This is immediate, since the trigonometric polynomials are each continuous, and Theorem 4.6 shows that the smaller set is dense (see Theorem 4.5(b)). □

This theorem gives us an alternative way to define L^2. First, it is not hard to see that the interval $[-\pi, \pi]$ can be replaced by any other closed and bounded interval $[a, b]$. One can define $L^2([a, b], m)$ as the *completion* of $C([a, b])$ with respect to the norm $\|\cdot\|_2$. The advantage of this definition is that it gives a way of discussing the very important Hilbert space L^2 without ever mentioning general measure and integration theory. Specifically, we define L^2 to be the collection of functions f defined on the interval $[a, b]$ such that

$$\lim_{n\to\infty} \|f_n - f\|_2 = 0$$

for some sequence $\{f_n\}_{n=1}^{\infty} \in C([a, b])$. Actually, L^2 must be considered to be the equivalence classes of such functions, where two functions are equivalent if and only if they are equal almost everywhere (see the discussion preceding Theorem 3.17). Therefore, it is not entirely true that this definition avoids discussing measure. However, we can give this definition with only an understanding of "measure zero," and not general measure. (And measure zero can be defined in a straightforward manner and is much simpler to understand than general measure.)

Exercises for Chapter 4

Section 4.1

4.1.1 Show that the trigonometric system

$$\frac{1}{\sqrt{2\pi}}, \quad \frac{\cos(nx)}{\sqrt{\pi}}, \quad \frac{\sin(mx)}{\sqrt{\pi}}, \qquad n, m = 1, 2, \ldots,$$

is an orthonormal sequence in $L^2([-\pi, \pi], m)$.

4.1.2 In this exercise you will actually compute a classical (i.e., with respect to the orthonormal sequence of Exercise 1) Fourier series, and investigate its convergence properties. The function given is a basic one; in the next exercise you are asked to do the same procedure with another very basic function. You are being asked to do these by hand, and you can no doubt appreciate that the computations get quite laborious once we depart from even the most basic functions. There are tricks for doing these computations; the interested reader can learn more about such techniques in a text devoted to classical Fourier series.

(a) Let

$$f(x) = \begin{cases} 1 & \text{if } -\pi \le x < 0, \\ 0 & \text{if } 0 \le x < \pi. \end{cases}$$

Show that its Fourier series is

$$\frac{1}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{((-1)^n - 1)}{n} \sin(nx).$$

(b) Explain why this series converges in mean to f.
(c) What can you say about the *pointwise* and *uniform* convergence of this series?
(d) Why are the coefficients of the cosine terms all zero?

4.1.3 In this exercise you will compute another Fourier series and investigate its convergence properties.

(a) Let $f(x) = x^2$. Show that its classical Fourier series is

$$\frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx).$$

(b) Explain why this series converges in mean to f.
(c) What can you say about the *pointwise* and *uniform* convergence of this series?
(d) Why are the coefficients of the sine terms all zero?

4.1.4 For a sequence $\{f_n\}_{n=1}^{\infty}$ in $L^2([-\pi, \pi], m)$, we have seen three ways for $\{f_n\}_{n=1}^{\infty}$ to converge:

(i) "pointwise,"
(ii) "uniformly,"
(iii) "in mean."

The point of this exercise is to understand the relation between these three types of convergence. For the counterexamples asked for below, use whatever finite interval $[a, b]$ you find convenient. Please make an effort to supply "easy" examples.

(a) Prove that uniform convergence implies pointwise convergence.

(b) Give an example to show that pointwise convergence does not imply uniform convergence.
(c) Prove that uniform convergence implies convergence in mean.
(d) Give an example to show that pointwise convergence does not imply convergence in mean.
(e) Give an example to show that convergence in mean does not imply pointwise convergence. (Note that the same example shows that convergence in mean does not imply uniform convergence.)

4.1.5 Apply the Gram–Schmidt process to the functions $1, x, x^2, x^3, \ldots$ to obtain formulas for the first three Legendre polynomials. Then verify that they are indeed given by the formula

$$\sqrt{\frac{2n+1}{2}}\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \qquad n = 1, 2, 3.$$

4.1.6 Prove that the Haar family is an orthonormal family in the Hilbert space $L^2([0, 1])$.

4.1.7 **(a)** Show that the sequence

$$\frac{e^{inx}}{\sqrt{2\pi}}, \quad n = 0, \pm 1 \pm 2, \ldots,$$

is a complete orthonormal sequence in $L^2([-\pi, \pi])$.

(b) Show that the sequence

$$\sqrt{\frac{2}{\pi}}\cos(nx), \quad n = 1, 2, 3, \ldots,$$

is a complete orthonormal sequence in $L^2([0, \pi])$. (Observe that $\sqrt{\frac{2}{\pi}}\cos(nx)$ can be replaced by $\sqrt{\frac{2}{\pi}}\sin(nx)$.)

Section 4.2

4.2.1 **(a)** Prove that in any inner product space $(V, \langle\cdot,\cdot\rangle)$, f and g orthogonal implies

$$\|f\|^2 + \|g\|^2 = \|f + g\|^2.$$

Here, as usual, $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$.

(b) Prove Parseval's theorem.

4.2.2 Assume that $\{f_n\}_{n=1}^{\infty}$ is a sequence in L^2 and that $f_n \to f$ in mean. Prove that $\{\|f_n\|_2\}_{n=1}^{\infty}$ is a bounded sequence of real numbers.

4.2.3 Assume that $f_1, f_2, \ldots, f_n$ is an orthonormal family in an inner product space. Prove that $f_1, f_2, \ldots, f_n$ are linearly independent.

4.2.4 For f and g in an inner product space, $g \neq 0$, the *projection of f on g* is the vector

$$\frac{\langle f, g\rangle}{\|g\|^2}g.$$

Show that the two vectors

$$\frac{\langle f, g\rangle}{\|g\|^2} g \quad \text{and} \quad f - \frac{\langle f, g\rangle}{\|g\|^2} g$$

are orthogonal.

4.2.5 **(a)** Show that the classical Fourier series of $f(x) = x$ is

$$2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx).$$

(b) Use your work in (a), together with Parseval's identity, to obtain Euler's remarkable identity

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(Note: The same procedure can be applied to the Fourier series of x^2 to obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90},$$

and so on!)

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