

# 7

## Velocity-Pressure Equations

The solution of the Navier-Stokes equations in velocity-pressure variables is the subject of this chapter. This set of equations is of broader application than the vorticity-streamfunction equations which are restricted to two-dimensional flows. First, the Fourier method for computing fully periodic flows is discussed. Then the major part of the chapter is devoted to the case of one or more nonperiodic directions. In such a situation, the classical difficulty is the determination of the pressure field ensuring that the velocity field is solenoidal. This problem is discussed according to the kind of method used for the advancement in time. Time-discretization methods leading, at each time-cycle, either to a Stokes problem or to a combination of Helmholtz and Darcy problems will be considered. Various solution methods will be discussed and compared in typical examples. Finally, an application to the calculation of a three-dimensional rotating flow will be presented.

### 7.1 Introduction

This chapter is devoted to the solution of the Navier-Stokes equations in the primitive variable formulation. These equations (see Chapter 5) are

$$\partial_t \mathbf{V} + \mathbf{A}(\mathbf{V}, \mathbf{V}) + \nabla p - \nu \nabla^2 \mathbf{V} = \mathbf{f} \quad (7.1)$$

$$\nabla \cdot \mathbf{V} = 0, \quad (7.2)$$

where  $\mathbf{V} = (u, v, w)$  is the velocity vector and  $p$  is the pressure. The initial condition associated with Eqs.(7.1) and (7.2) is

$$\mathbf{V} = \mathbf{V}_0 = (u_0, v_0, w_0) \quad (7.3)$$

and typical boundary conditions have been discussed in Chapter 5. Several configurations will be considered in the present chapter : (1) fully periodic three-dimensional flows, (2) flow in a two-dimensional channel with a periodic direction, (3) flow in a two-dimensional square domain.

## 7.2 Fourier-Fourier-Fourier method

In this section we present the Fourier method for the calculation of the three-dimensional spatially periodic solutions. This kind of solution is adapted to the study of homogeneous turbulence. The application of truncated Fourier series expansion to the direct numerical simulation (DNS) of turbulence has been initiated by the works by Orszag (1969, 1971, 1972) and Orszag and Patterson (1972). The success of these pioneering works was due to a clever combination of the efficiency of the FFT and of high-speed computing facilities.

Two points are important in the design of an efficient method : (1) the time-discretization scheme, and (2) the form and treatment of the nonlinear term. These questions are addressed now.

We assume that the forcing term  $\mathbf{f}$  in Eq.(7.1) and the initial velocity field  $\mathbf{V}_0$  are  $2\pi$ -periodic in the three spatial directions. The velocity vector  $\mathbf{V}(\mathbf{x}, t)$  and the pressure  $p(\mathbf{x}, t)$  with  $\mathbf{x} = (x_1, x_2, x_3)$  are approximated according to

$$\mathbf{V}(\mathbf{x}, t) \cong \mathbf{V}_K(\mathbf{x}, t) = \sum_{\mathbf{k}} \hat{\mathbf{V}}_{\mathbf{k}}(t) e^{i\mathbf{k} \cdot \mathbf{x}}, \quad (7.4)$$

$$p(\mathbf{x}, t) \cong p_K(\mathbf{x}, t) = \sum_{\mathbf{k}} \hat{p}_{\mathbf{k}}(t) e^{i\mathbf{k} \cdot \mathbf{x}}. \quad (7.5)$$

In these expressions,  $\mathbf{k} = (k_1, k_2, k_3)$  is the wavenumber vector, and the sums are taken over  $k_1, k_2$  and  $k_3$  such that  $-K_j \leq k_j \leq K_j$ ,  $j = 1, 2, 3$ . Therefore, the notation  $\hat{\phi}_{\mathbf{k}}$  means  $\hat{\phi}_{k_1, k_2, k_3}$ , and we set  $k^2 = |\mathbf{k}|^2 = k_1^2 + k_2^2 + k_3^2$ .

Now we apply the Fourier Galerkin method to the Navier-Stokes equations (7.1) and (7.2) with the approximation (7.4)-(7.5). We get a set of differential equations for determining the Fourier coefficients  $\hat{\mathbf{V}}_{\mathbf{k}}$  and  $\hat{p}_{\mathbf{k}}$  :

$$d_t \hat{\mathbf{V}}_{\mathbf{k}} + \hat{\mathbf{A}}_{\mathbf{k}} + i\mathbf{k} \hat{p}_{\mathbf{k}} + \nu k^2 \hat{\mathbf{V}}_{\mathbf{k}} = \hat{\mathbf{f}}_{\mathbf{k}} \quad (7.6)$$

$$i\mathbf{k} \cdot \hat{\mathbf{V}}_{\mathbf{k}} = 0, \quad (7.7)$$

to be solved for  $t > 0$ , with the initial condition

$$\hat{\mathbf{V}}_{\mathbf{k}}(0) = \hat{\mathbf{V}}_{0,\mathbf{k}}. \quad (7.8)$$

In Eqs.(7.6) and (7.8),  $\hat{\mathbf{A}}_{\mathbf{k}}$ ,  $\hat{\mathbf{f}}_{\mathbf{k}}$ , and  $\hat{\mathbf{V}}_{0,\mathbf{k}}$  are the Fourier coefficients of  $\mathbf{A}$ ,  $\mathbf{f}$ , and  $\mathbf{V}_0$ , respectively. The pressure is eliminated by applying the divergence operator in the Fourier space to Eq.(7.6), that is, by taking the scalar product of Eq.(7.6) with  $\underline{i}\mathbf{k}$ , and using Eq.(7.7). We obtain

$$-k^2 \hat{p}_{\mathbf{k}} = \underline{i}\mathbf{k} \cdot \hat{\mathbf{f}}_{\mathbf{k}} - \underline{i}\mathbf{k} \cdot \hat{\mathbf{A}}_{\mathbf{k}}$$

and Eq.(7.6) becomes

$$d_t \hat{\mathbf{V}}_{\mathbf{k}} + \nu k^2 \hat{\mathbf{V}}_{\mathbf{k}} = \hat{\mathbf{g}}_{\mathbf{k}} - \hat{\mathbf{C}}_{\mathbf{k}} \quad (7.9)$$

with

$$\hat{\mathbf{g}}_{\mathbf{k}} = \hat{\mathbf{f}}_{\mathbf{k}} - \frac{\mathbf{k} \cdot \hat{\mathbf{f}}_{\mathbf{k}}}{k^2} \mathbf{k}, \quad (7.10)$$

$$\hat{\mathbf{C}}_{\mathbf{k}} = \hat{\mathbf{A}}_{\mathbf{k}} - \frac{\mathbf{k} \cdot \hat{\mathbf{A}}_{\mathbf{k}}}{k^2} \mathbf{k}. \quad (7.11)$$

It is assumed that  $k^2 \neq 0$ . If  $\mathbf{k} = \mathbf{0}$ , Eq.(7.7) is automatically satisfied and Eq.(7.6) determines completely the velocity  $\hat{\mathbf{V}}_{\mathbf{0}}$ . The pressure  $\hat{p}_{\mathbf{0}}$  is arbitrary in agreement with the property of the Navier-Stokes equations. We may simply set  $\hat{p}_{\mathbf{0}} = 0$ , implying that the mean value of the pressure is zero.

### 7.2.1 Time-discretization schemes

In the commonly used time-discretization schemes the nonlinear term  $\hat{\mathbf{C}}_{\mathbf{k}}$  is treated in an explicit way. For example, Orszag and Patterson (1972) used the second-order Leap-Frog/Crank-Nicolson scheme (4.50). The advantage of this scheme is that it does not introduce numerical dissipation. On the other hand, the leap-frog time-stepping slightly uncouples even and odd time-cycles, thus producing oscillations. This is generally avoided by averaging the numerical solution periodically every few time-cycles by a procedure that conserves the second-order accuracy (see Peyret and Taylor, 1983). Moreover, the Crank-Nicolson scheme presents the drawback of not damping sufficiently the high-frequency modes. Higher-order methods can be used like the Runge-Kutta method (Rogallo, 1977). We describe now a “semi-exact” scheme proposed by Basdevant (1982) and used in some studies of homogeneous turbulence (Roy, 1982, 1986 ; Vincent and Meneguzzi, 1991).

The semi-exact scheme consists of integrating formally Eq.(7.9) between  $t_\star = (n+1-m)\Delta t$  and  $t_{n+1} = (n+1)\Delta t$ . Then the integrals involving the right-hand side of Eq.(7.9) are approximated by means of classical integration formulas. More precisely, the integration of Eq.(7.9) gives

$$\hat{\mathbf{V}}_{\mathbf{k}}^{n+1} = \hat{\mathbf{V}}_{\mathbf{k}}^{n+1-m} e^{-\nu k^2 m \Delta t} + e^{-\nu k^2 (n+1) \Delta t} [\mathbf{I}_1(t_{n+1}, t_\star) - \mathbf{I}_2(t_{n+1}, t_\star)] ,$$

where

$$\begin{aligned} \mathbf{I}_1(t_{n+1}, t_\star) &= \int_{t_\star}^{t_{n+1}} e^{\nu k^2 \tau} \hat{\mathbf{g}}_{\mathbf{k}}(\tau) d\tau , \\ \mathbf{I}_2(t_{n+1}, t_\star) &= \int_{t_\star}^{t_{n+1}} e^{\nu k^2 \tau} \hat{\mathbf{C}}_{\mathbf{k}}(\tau) d\tau . \end{aligned}$$

The choice of  $m$ , namely,  $t_\star$ , and that of the integration formula define the scheme. For  $m = 2$ ,  $\mathbf{I}_1$  and  $\mathbf{I}_2$ , evaluated with the midpoint rule, give the second-order scheme of leap-frog type :

$$\hat{\mathbf{V}}_{\mathbf{k}}^{n+1} = \hat{\mathbf{V}}_{\mathbf{k}}^{n-1} e^{-2\nu k^2 \Delta t} - 2\Delta t e^{-\nu k^2 \Delta t} (\hat{\mathbf{C}}_{\mathbf{k}}^n - \hat{\mathbf{g}}_{\mathbf{k}}^n) . \quad (7.12)$$

Another choice, considered by Vincent and Meneguzzi (1991), consists of setting  $m = 1$ , namely,  $t_\star = t_n$ . Then,  $\mathbf{I}_1(t_{n+1}, t_n)$  is evaluated with the trapezoidal rule. The integral  $\mathbf{I}_2(t_{n+1}, t_n)$  is written as

$$\mathbf{I}_2(t_{n+1}, t_n) = \mathbf{I}_2(t_{n+1}, t_{n-1}) - \mathbf{I}_2(t_n, t_{n-1}) ,$$

then  $\mathbf{I}_2(t_{n+1}, t_{n-1})$  is evaluated with the midpoint rule and  $\mathbf{I}_2(t_n, t_{n-1})$  with the trapezoidal rule. Finally we get the second-order scheme of Adams-Bashforth type

$$\begin{aligned} \hat{\mathbf{V}}_{\mathbf{k}}^{n+1} = \hat{\mathbf{V}}_{\mathbf{k}}^n e^{-\nu k^2 \Delta t} &- \frac{\Delta t}{2} \left( 3e^{-\nu k^2 \Delta t} \hat{\mathbf{C}}_{\mathbf{k}}^n - e^{-2\nu k^2 \Delta t} \hat{\mathbf{C}}_{\mathbf{k}}^{n-1} \right) \\ &+ \frac{\Delta t}{2} \left( \hat{\mathbf{g}}_{\mathbf{k}}^{n+1} + e^{-\nu k^2 \Delta t} \hat{\mathbf{g}}_{\mathbf{k}}^n \right) . \end{aligned} \quad (7.13)$$

### 7.2.2 The nonlinear term

The form and numerical treatment of the nonlinear term  $\hat{\mathbf{C}}_{\mathbf{k}}$  play a fundamental part in the stability of the scheme and in the quality of the numerical results. As mentioned in Chapter 5, the nonlinear term in the Navier-Stokes equations characterizing the convection of momentum can be considered in any equivalent form : convective, conservative, skew-symmetric, or rotational. This latter form is, however, somewhat different because the term  $|\mathbf{V}|^2/2$  is included in the pressure term. In the discrete problem, the equivalence may be lost so that different behaviours can be expected. This question has been discussed by Canuto *et al.* (1988), Zang (1991), Blaisdell *et*

*al.* (1996), and by Kravchenko and Moin (1997). We resume here the main results of these investigations.

In case of zero viscosity ( $\nu = 0$ ) and without the forcing term ( $\mathbf{f} = \mathbf{0}$ ), the resulting Euler equations conserve momentum  $Q$  and energy  $E$ , namely,

$$Q = \int_{\Omega} \mathbf{V} d\Omega, \quad E = \frac{1}{2} \int_{\Omega} |\mathbf{V}|^2 d\Omega, \quad \Omega = [0, 2\pi]^3. \quad (7.14)$$

It is highly desirable that these conservation properties be preserved in the discrete framework. The conservation of energy is especially important because it controls the stability of the numerical approximations (when  $\nu = 0$ ). These properties are denominated “semiconservation properties” when the Navier-Stokes equations are considered.

If the nonlinear term  $\hat{\mathbf{C}}_{\mathbf{k}}$  was approximated by the Galerkin method, the various form would be equivalent and the semiconservation properties would be satisfied. In practical calculations, however, the nonlinear term  $\hat{\mathbf{C}}_{\mathbf{k}}$  at time-levels  $n$  or  $n - 1$  is evaluated through the pseudospectral technique (Section 2.8). Due to aliasing errors, the resulting discrete term is different according to the form of the nonlinear term, thus implying different properties. More precisely, it is found that the momentum  $Q$  and energy  $E$  are semiconserved by the skew-symmetric and rotational forms. On the other hand, only  $Q$  is semiconserved by the conservative form and neither  $Q$  nor  $E$  is semiconserved by the convective form. Consequently, we may expect these last two forms to be unstable for low viscosity, while the first two forms are stable. This has effectively been observed in the numerical calculations reported in the above-referenced works.

It is interesting to compare the discrete version of the conservative and convective forms with a simple one-dimensional example (Blaisdell *et al.*, 1996 ; Kravchenko and Moin, 1997). Let us consider  $u(x)$  and  $v(x)$ , and the equivalent two quantities

$$C^{(1)} = (uv)', \quad C^{(2)} = uv' + vu'.$$

When determined with the pseudospectral technique, the Fourier coefficients  $\hat{C}_k^{(1)}$  and  $\hat{C}_k^{(2)}$  are, respectively, [see Eq.(2.56)] :

$$\begin{aligned} \hat{C}_k^{(1)} = \underline{i}k \sum_{\substack{p,q \\ p+q=k}} \hat{u}_p \hat{v}_q + \underline{i} \left[ \sum_{\substack{p,q \\ p+q=k+N}} (p+q) \hat{u}_p \hat{v}_q \right. \\ \left. + \sum_{\substack{p,q \\ p+q=k-N}} (p+q) \hat{u}_p \hat{v}_q \right] \end{aligned} \quad (7.15)$$

and

$$\begin{aligned} \hat{C}_k^{(2)} = i k \sum_{\substack{p,q \\ p+q=k}} \hat{u}_p \hat{v}_q + i \left[ \sum_{\substack{p,q \\ p+q=k+N}} (p+q-N) \hat{u}_p \hat{v}_q \right. \\ \left. + \sum_{\substack{p,q \\ p+q=k-N}} (p+q+N) \hat{u}_p \hat{v}_q \right], \end{aligned} \quad (7.16)$$

where the sums are taken over  $p = -K, \dots, K$ ,  $q = -K, \dots, K$ , and  $N = 2K + 1$ . In each of these expressions, the first sum is exactly the Fourier coefficient which would be obtained with the Galerkin method. The other sums in square brackets are aliasing terms. Moreover, it is easy to see that the aliasing terms in  $\hat{C}_k^{(1)}$  and  $\hat{C}_k^{(2)}$  have opposite signs. Indeed, the first sums in the square brackets are taken for  $0 \leq p+q \leq 2K$  so that  $-(2K+1) \leq p+q-N \leq -1$  and the second sums are taken for  $-2K \leq p+q \leq 0$  so that  $1 \leq p+q+N \leq 2K+1$ . Therefore, the aliasing error associated with the skew-symmetric form  $(\hat{C}_k^{(1)} + \hat{C}_k^{(2)})/2$  is much reduced, although not completely removed. This explains why the skew-symmetric form, even without aliasing removal, generally gives stable and accurate results and is recommended.

Concerning the rotational form, it has been observed (Zang, 1991 ; Kravchenko and Moin, 1997) that, even with aliasing errors, the numerical integration is stable but the results show an inability to maintain the turbulence at the proper level. Nevertheless, Vincent and Meneguzzi (1991), using a high spatial resolution, have obtained satisfactory results which did not significantly differ from those obtained after removing the aliasing errors.

When aliasing is removed using the 3/2 rule, for example (or other techniques as discussed by Canuto *et al.*, 1988), the discrete version of any form of the nonlinear term is equivalent to the Galerkin approximation and, consequently, is stable.

### 7.3 Fourier-Chebyshev method

The present section is devoted to the solution of the Navier-Stokes equations in a two-dimensional channel with one periodic direction. Besides its intrinsic physical interest, such a configuration allows us to discuss the fundamentals of the numerical methods in a simple case where the necessary algebra remains simple.

### 7.3.1 Flow in a plane channel

In this section, we consider the flow in an infinite channel in the  $x$ -direction with  $2\pi$ -periodicity in this direction. Thus, the domain  $\Omega$  in which Eqs. (7.1) and (7.2) are solved is defined by  $0 \leq x \leq 2\pi$ ,  $-1 \leq y \leq 1$ . The forcing term  $\mathbf{f} = (f_u, f_v)$  and the initial velocity field

$$\mathbf{V} = \mathbf{V}_0 \quad (7.17)$$

are assumed to be  $2\pi$ -periodic with respect to  $x$ . The boundary conditions are

$$\mathbf{V}(x, \pm 1, t) = \mathbf{V}_\Gamma^\pm(x, t) = (u_\Gamma^\pm, v_\Gamma^\pm), \quad (7.18)$$

where  $\mathbf{V}_\Gamma^\pm$  is also  $2\pi$ -periodic. The conservation of the total flow rate demands

$$\int_0^{2\pi} (v_\Gamma^+ - v_\Gamma^-) dy = 0, \quad (7.19)$$

that is, the mean values of  $v_\Gamma^+$  and  $v_\Gamma^-$  must be equal. Note that in the case of no-slip, impermeable walls, we have simply

$$\mathbf{V}(x, \pm 1, t) = \mathbf{0}. \quad (7.20)$$

Also, the case of a shear-stress-free wall may be considered, as defined by Eq.(6.14).

Now the solution of the problem is such that the velocity  $\mathbf{V}$  and the pressure gradient  $\nabla p$  are  $2\pi$ -periodic. Under this circumstance, the mean pressure gradient  $\gamma$  is assumed to be known (see Section 6.3.4).

### 7.3.2 Time-dependent Stokes equations

For a presentation of the basic properties of the solution methods, we begin with the time-dependent Stokes equations

$$\partial_t \mathbf{V} - \nu \nabla^2 \mathbf{V} + \nabla p = \mathbf{f}, \quad -1 < y < 1 \quad (7.21)$$

$$\nabla \cdot \mathbf{V} = 0, \quad -1 < y < 1 \quad (7.22)$$

to be solved with the initial condition (7.17) and the boundary conditions (7.18). The solution  $(\mathbf{V}, p)$  is sought in the form of truncated Fourier series with respect to  $x$ :

$$\mathbf{V}_K(x, y, t) = \sum_{k=-K}^K \hat{\mathbf{V}}_k(y, t) e^{ikx},$$

$$p_K(x, y, t) = \gamma x + \sum_{k=-K}^K \hat{p}_k(y, t) e^{ikx}.$$

The Fourier Galerkin equations (see Section 2.6.1), satisfied by  $(\hat{\mathbf{V}}_k, \hat{p}_k)$ ,  $k = -K, \dots, K$ , with  $\hat{\mathbf{V}}_k = (\hat{u}_k, \hat{v}_k)$ , are

$$\partial_t \hat{u}_k - \nu (\partial_{yy} - k^2) \hat{u}_k + \underline{i}k \hat{p}_k = \hat{f}_{u,k} \quad (7.23)$$

$$\partial_t \hat{v}_k - \nu (\partial_{yy} - k^2) \hat{v}_k + \partial_y \hat{p}_k = \hat{f}_{v,k} \quad (7.24)$$

$$\underline{i}k \hat{u}_k + \partial_y \hat{v}_k = 0, \quad (7.25)$$

where  $\gamma$  has been included in  $\hat{f}_{u,k}$ . These equations are solved in  $-1 < y < 1$  and  $t > 0$  with the boundary conditions

$$\hat{u}_k(\pm 1, t) = \hat{u}_{\Gamma,k}^{\pm}(t), \quad \hat{v}_k(\pm 1, t) = \hat{v}_{\Gamma,k}^{\pm}(t) \quad (7.26)$$

and the initial conditions

$$\hat{u}_k(y, 0) = \hat{u}_{0,k}(y), \quad \hat{v}_k(y, 0) = \hat{v}_{0,k}(y) \quad (7.27)$$

satisfying the incompressibility condition

$$\underline{i}k \hat{u}_{0,k} + \partial_y \hat{v}_{0,k} = 0. \quad (7.28)$$

The usual time-discretization schemes for the Stokes (or Navier-Stokes) equations can be classified into two categories according to the type of elliptic problem which has to be solved at each time-cycle. The first category leads to the solution of a Stokes problem and the second one to the solution of Helmholtz and Darcy problems. These two types of time-discretizations and the associated solution methods are now discussed. The discussion will concern, more precisely, the following points : (1) the formulation of the discrete problem, (2) the resolvability of the resulting algebraic system, (3) the stability of the time-scheme and (4) the algorithm for solving the algebraic system.

#### (a) Time-schemes leading to the Stokes problem

We consider the general three-level scheme presented in Section 4.4.1 and already discussed (Section 6.3.1.b) in the case of the vorticity-streamfunction equations. Among the schemes (4.46) we select the subclass defined by  $\theta_1 = \theta = 1 - \theta_2$ , so that the family of schemes depends only on the two parameters  $\varepsilon$  and  $\theta$  when applied to the time-dependent Stokes equations. The semidiscrete equations are then

$$\begin{aligned} & \frac{(1 + \varepsilon) \hat{u}_k^{n+1} - 2\varepsilon \hat{u}_k^n - (1 - \varepsilon) \hat{u}_k^{n-1}}{2\Delta t} - \nu (\partial_{yy} - k^2) [\theta \hat{u}_k^{n+1} + (1 - \theta) \hat{u}_k^n] \\ & + \underline{i}k \hat{p}_k^{n+\tau} = \theta \hat{f}_{u,k}^{n+1} + (1 - \theta) \hat{f}_{u,k}^n \end{aligned} \quad (7.29)$$



$$\begin{aligned} & \frac{(1+\varepsilon)\hat{v}_k^{n+1} - 2\varepsilon\hat{v}_k^n - (1-\varepsilon)\hat{v}_k^{n-1}}{2\Delta t} - \nu(\partial_{yy} - k^2)[\theta\hat{v}_k^{n+1} + (1-\theta)\hat{v}_k^n] \\ & + \partial_y\hat{p}_k^{n+\tau} = \theta\hat{f}_{v,k}^{n+1} + (1-\theta)\hat{f}_{v,k}^n \end{aligned} \quad (7.30)$$

$$ik\hat{u}_k^{n+1} + \partial_y\hat{v}_k^{n+1} = 0 \quad (7.31)$$

$$\hat{u}_k^{n+1}(\pm 1) = \left(\hat{u}_{\Gamma,k}^{\pm}\right)^{n+1} \quad (7.32)$$

$$\hat{v}_k^{n+1}(\pm 1) = \left(\hat{v}_{\Gamma,k}^{\pm}\right)^{n+1}, \quad (7.33)$$

where  $\hat{u}_k^n \cong \hat{u}_k(y, n\Delta t)$ ,  $\hat{v}_k^n \cong \hat{v}_k(y, n\Delta t)$ , and  $\hat{p}_k^{n+\tau} \cong \hat{p}_k[y, (n+\tau)\Delta t]$ . The number  $\tau$  which characterizes the time  $(n+\tau)\Delta t$  with  $\tau > 0$  is arbitrary and must be chosen in order to ensure the better truncation error. This is due to the fact that no time derivative of the pressure appears in the equations so that the pressure is defined within an arbitrary function of time. As a matter of fact, it is the pressure gradient  $\nabla p$  itself which should be fixed at the time-level  $n+\tau$ . In the above equations, the truncation error analysis shows that  $\tau$  can be taken equal to  $\theta$ , except in the case  $\theta = 0$  (first-order scheme) where  $\tau$  is arbitrary ( $\tau > 0$ ) and, generally, is taken equal to 1. It must be noted that for  $k = 0$ , the pressure term disappears from Eq.(7.29) and the mode  $\hat{p}_0$  appears only through its  $y$ -derivative in Eq.(7.30).

Therefore, the equations for determining the unknowns  $(\hat{u}_k^{n+1}, \hat{v}_k^{n+1}, \hat{p}_k^{n+1})$  constitute a Stokes-type problem, also called a “generalized Stokes-problem”. The classical Stokes problem corresponds to  $\sigma = 0$ . We use the simpler nomenclature “Stokes problem” even if  $\sigma \neq 0$ .

Now the discretization of Eqs.(7.29)-(7.33) with respect to  $y$  makes use of the Chebyshev collocation approximation. Therefore, the Fourier coefficients of  $u$ ,  $v$ , and  $p$  at time  $n\Delta t$  are approximated with polynomials of degree at most  $N$  (namely,  $\hat{u}_{kN}^n$ ,  $\hat{v}_{kN}^n$ , and  $\hat{p}_{kN}^n$ ), and Eqs. (7.29)-(7.33) are discretized using the collocation method (see Section 3.4.2) based on the Gauss-Lobatto points  $y_j$  [Eq.(6.26)]. The derivatives at each collocation point  $y_j$  are expressed in terms of the grid values by the usual expressions developed in Section 3.3.4.

By prescribing the equations (7.29)-(7.31) to be satisfied at all inner points  $y_j$ ,  $j = 1, \dots, N$ , and adding the boundary conditions, we get a system of  $3N + 1$  complex equations for the  $3(N + 1)$  complex unknowns (including the boundary values of the velocity). Hence, two equations must be added. One possibility is to add the normal momentum equation written for  $j = 0$  and  $j = N$ . But, as shown by Orszag *et al.* (1986), such a technique may be subject to instabilities which might be avoided by using a special form of the diffusive term (see Section 7.4.2.c). Another way to close the system is to add the incompressibility equation (7.31) prescribed at the boundaries  $j = 0$  and  $j = N$ . We consider here this second option

which is in common use. Therefore, we get an algebraic system of  $3(N+1)$  equations for the  $3(N+1)$  unknowns  $\hat{u}_{kN}^{n+1}(y_j)$ ,  $\hat{v}_{kN}^{n+1}(y_j)$ , and  $\hat{p}_{kN}^{n+1}(y_j)$ ,  $j = 0, \dots, N$ .

For  $k \neq 0$ , it can be shown theoretically (Bernardi and Maday, 1992) and numerically (Bwemba, 1994) that the algebraic system admits a unique solution. For  $k = 0$ , the matrix of the system has two eigenvalues equal to zero. One corresponds to the fact that the pressure is defined up to a constant, but the other zero eigenvalue corresponds to a spurious pressure mode. This issue will be addressed at the end of the section, and up to that point we assume that  $k \neq 0$ .

The stability has been numerically studied by Bwemba (1994) in the case of the two-level scheme ( $\theta$ -scheme) defined by  $\varepsilon = 1$ . By considering the homogeneous problem ( $\hat{u}_{\Gamma,k}^{\pm} = \hat{v}_{\Gamma,k}^{\pm} = \hat{f}_{u,k} = \hat{f}_{v,k} = 0$ ), the algebraic system has the form

$$\mathcal{A}_1 \Phi^{n+1} + \mathcal{A}_0 \Phi^n = 0 \quad (7.34)$$

where  $\Phi^{n+1}$  is the  $(3N-1)$ -vector defined by

$$\begin{aligned} \Phi^{n+1} = & (\hat{u}_{kN}^{n+1}(y_1), \dots, \hat{u}_{kN}^{n+1}(y_{N-1}), \hat{v}_{kN}^{n+1}(y_1), \\ & \dots, \hat{v}_{kN}^{n+1}(y_{N-1}), \hat{p}_{kN}^{n+\tau}(y_0), \dots, \hat{p}_{kN}^{n+\tau}(y_N))^T. \end{aligned}$$

The expression for the  $(3N-1) \times (3N-1)$  matrices  $\mathcal{A}_0$  and  $\mathcal{A}_1$  is easily obtained from the Chebyshev differentiation matrix  $\mathcal{D}$ . A necessary condition for stability is that the spectral radius of  $\mathcal{E} = \mathcal{A}_1^{-1} \mathcal{A}_0$  is not larger than 1. The eigenvalues  $\mu$  of the matrix  $\mathcal{E}$  are the roots of the equation

$$\det(\mathcal{A}_0 + \mu \mathcal{A}_1) = 0 \quad (7.35)$$

numerically calculated by means of a computer routine for generalized eigenvalue problems. Because the pressure appears only in  $\mathcal{A}_1 \Phi^{n+1}$  (i.e., not in  $\mathcal{A}_0 \Phi^n$ ) and the divergence equation only at level  $n+1$ , the equation (7.35) admits only  $N-3$  nonzero roots. It is found that  $\rho(\mathcal{E}) \leq 1$  for any  $\Delta t$  if  $\theta \geq 1/2$ . For  $\theta < 1/2$ , there exists a critical time-step  $\Delta t_*$  such that  $\rho(\mathcal{E}) \leq 1$  if  $\Delta t \leq \Delta t_*$ . This result is in agreement with the usual result concerning the stability of the  $\theta$ -scheme applied to the diffusion equation (Section 4.3.2), contrary to the case of the vorticity-streamfunction equations (Section 6.3.2.e).

For several reasons already discussed (Sections 4.4.2.a, 4.5.1.b), the three-level scheme defined by  $\varepsilon = 2$  and  $\theta = \tau = 1$  is preferred to the two-level Crank-Nicolson scheme defined by  $\varepsilon = 1$  and  $\theta = \tau = 1/2$ . The apparent advantage of the Crank-Nicolson scheme concerning the number of time-levels involved disappears when considering the Navier-Stokes equations, since both schemes use a three-level Adams-Bashforth extrapolation for the nonlinear convective term. The solution of the algebraic system is addressed in the next section.

**(b) Solution of the Stokes problem by the Uzawa method**

As indicated, the implicit discretization of the time-dependent Stokes equations leads to the solution of a Stokes problem at each time-cycle. In this section, we discuss the solution of the discrete Stokes problem by means of the Uzawa method.

Let us consider Eqs. (7.29)-(7.33) in the case  $\varepsilon = 2$  and  $\theta = \tau = 1$ . Then, for the sake of simplification, we set

$$\begin{aligned} \hat{u}_k^{n+1} &= u, \quad \hat{v}_k^{n+1} = v, \quad \hat{p}_k^{n+1} = p, \quad \frac{3}{2\nu\Delta t} + k^2 = \sigma, \\ -\frac{i}{\nu}\hat{f}_{u,k}^{n+1} - i\frac{4\hat{u}_k^n - 3\hat{u}_k^{n-1}}{2\Delta t} &= f_u, \quad -\frac{1}{\nu}\hat{f}_{v,k}^{n+1} - i\frac{4\hat{v}_k^n - 3\hat{v}_k^{n-1}}{2\Delta t} = f_v, \\ i\left(\hat{u}_{\Gamma,k}^\pm\right)^{n+1} &= \gamma_\pm, \quad i\left(\hat{v}_{\Gamma,k}^\pm\right)^{n+1} = g_\pm. \end{aligned}$$

Hence, the Stokes problem to be solved for each Fourier mode and at each time-cycle is

$$u'' - \sigma u + k p = f_u, \quad -1 < y < 1 \quad (7.36)$$

$$v'' - \sigma v - p' = f_v, \quad -1 < y < 1 \quad (7.37)$$

$$k u + v' = 0, \quad -1 < y < 1 \quad (7.38)$$

$$u(-1) = \gamma_-, \quad u(1) = \gamma_+ \quad (7.39)$$

$$v(-1) = g_-, \quad v(1) = g_+. \quad (7.40)$$

In the Chebyshev collocation method considered here, the momentum equations (7.36) and (7.37) are enforced at the inner collocation points  $y_j$ ,  $j = 1, \dots, N-1$  [see Eq.(6.26)], while the incompressibility equation (7.38) is enforced at all collocation points, including the boundaries  $y_0 = 1$  and  $y_N = -1$ . The velocity  $(u, v)$  and the pressure  $p$  are approximated with polynomials of degree at most  $N$ , namely,  $u \cong u_N \in \mathbb{P}_N$ ,  $v \cong v_N \in \mathbb{P}_N$ , and  $p \cong p_N \in \mathbb{P}_N$ . Such an approximation, where velocity and pressure are both approximated in  $\mathbb{P}_N$ , is usually denoted by  $\mathbb{P}_N - \mathbb{P}_N$ , in contrast with the method where the pressure is approximated with a polynomial of degree  $N-2$  ( $\mathbb{P}_N - \mathbb{P}_{N-2}$  approximation). This point will be discussed below and especially in Section 7.4.2.b.

We set  $u_j = u_N(y_j)$ ,  $v_j = v_N(y_j)$ ,  $p_j = p_N(y_j)$ , and we introduce the notations

$$D\phi_j = \sum_{l=0}^N d_{j,l}^{(1)} \phi_l, \quad D^2\phi_j = \sum_{l=0}^N d_{j,l}^{(2)} \phi_l. \quad (7.41)$$

The system (7.36)-(7.40) is discretized according to

$$(D^2 - \sigma) u_j + k p_j = f_{u,j}, \quad j = 1, \dots, N-1, \quad (7.42)$$

$$(D^2 - \sigma) v_j - D p_j = f_{v,j}, \quad j = 1, \dots, N-1, \quad (7.43)$$

$$k u_j + D v_j = 0, \quad j = 0, \dots, N, \quad (7.44)$$

$$u_0 = \gamma_+, \quad u_N = \gamma_-, \quad (7.45)$$

$$v_0 = g_+, \quad v_N = g_-. \quad (7.46)$$

It is convenient to write these equations in vector form. We introduce the vectors of the unknowns

$$U_\star = \begin{pmatrix} u_1 \\ \vdots \\ u_{N-1} \end{pmatrix}, \quad V_\star = \begin{pmatrix} v_1 \\ \vdots \\ v_{N-1} \end{pmatrix}, \quad P = \begin{pmatrix} p_0 \\ \vdots \\ p_N \end{pmatrix}$$

such that Eqs (7.42)-(7.44) become

$$\mathcal{H} U_\star + k \mathcal{I}_1 P = S_u^\star \quad (7.47)$$

$$\mathcal{H} V_\star - \mathcal{D}_5 P = S_v^\star \quad (7.48)$$

$$k \mathcal{I}_3 U_\star + \mathcal{D}_6 V_\star = S_Q, \quad (7.49)$$

with  $\mathcal{H}$  defined by

$$\mathcal{H} = \mathcal{D}_2 - \sigma \mathcal{I},$$

where  $\mathcal{D}_2$  is the  $(N-1) \times (N-1)$  matrix defined in Section 4.4.2, namely,

$$\mathcal{D}_2 = \left[ d_{i,j}^{(2)} \right], \quad i = 1, \dots, N-1, \quad j = 1, \dots, N-1. \quad (7.50)$$

The  $(N-1) \times (N+1)$  matrix  $\mathcal{D}_5$  is defined by

$$\mathcal{D}_5 = \left[ d_{i,j}^{(1)} \right], \quad i = 1, \dots, N-1, \quad j = 0, \dots, N, \quad (7.51)$$

and the  $(N+1) \times (N-1)$  matrix  $\mathcal{D}_6$  by

$$\mathcal{D}_6 = \left[ d_{i,j}^{(1)} \right], \quad i = 0, \dots, N, \quad j = 1, \dots, N-1. \quad (7.52)$$

Then  $\mathcal{I}$  is the  $(N-1) \times (N-1)$  identity matrix,  $\mathcal{I}_1$  is the  $(N-1) \times (N+1)$  matrix defined in Section 6.3.2.b [Eq.(6.33)], and  $\mathcal{I}_3$  is the  $(N+1) \times (N-1)$  matrix defined by

$$\mathcal{I}_3 = [\delta_{i-1,j}], \quad i = 0, \dots, N, \quad j = 1, \dots, N-1, \quad (7.53)$$

where  $\delta_{i,j}$  is the Kronecker delta.

Lastly, the  $(N-1)$ -vectors,  $S_u^\star = [s_{u,j}]$  and  $S_v^\star = [s_{v,j}]$ ,  $j = 1, \dots, N-1$ , are, respectively, defined by

$$s_{u,j} = f_{u,j} - d_{j,0}^{(2)} \gamma_+ - d_{j,N}^{(2)} \gamma_-,$$

$$s_{v,j} = f_{v,j} - d_{j,0}^{(2)} g_+ - d_{j,N}^{(2)} g_- ,$$

while the  $(N + 1)$ -vector  $S_Q = [s_{Q,j}]$ ,  $j = 0, \dots, N$ , is defined by

$$\begin{aligned} s_{Q,0} &= -k \gamma_+ - d_{0,0}^{(1)} g_+ - d_{0,N}^{(1)} g_- , \\ s_{Q,j} &= -d_{j,0}^{(1)} g_+ - d_{j,N}^{(1)} g_- , \quad j = 1, \dots, N-1 , \\ s_{Q,N} &= -k \gamma_- - d_{N,0}^{(1)} g_+ - d_{N,N}^{(1)} g_- . \end{aligned}$$

The solution of (7.47)-(7.49) is obtained by the Uzawa method. The method consists of formally solving (7.47) and (7.48) with respect to  $U_\star$  and  $V_\star$  :

$$U_\star = \mathcal{H}^{-1} (S_u^\star - k \mathcal{I}_1 P)$$

$$V_\star = \mathcal{H}^{-1} (S_v^\star + \mathcal{D}_5 P) ,$$

which are then brought into Eq.(7.49) to give the equation

$$\mathcal{U} P = G , \tag{7.54}$$

where  $\mathcal{U}$  is the Uzawa operator defined by

$$\mathcal{U} = \mathcal{D}_6 \mathcal{H}^{-1} \mathcal{D}_5 - k^2 \mathcal{I}_3 \mathcal{H}^{-1} \mathcal{I}_1 \tag{7.55}$$

and

$$G = S_Q - k \mathcal{I}_3 \mathcal{H}^{-1} S_u^\star - \mathcal{D}_6 \mathcal{H}^{-1} S_v^\star .$$

Therefore, the direct Uzawa algorithm is as follows :

1. Calculate  $P$  by solving Eq.(7.54).
2. Calculate  $U_\star$  and  $V_\star$  by, respectively, solving Eqs.(7.47) and (7.48).

The implementation of the algorithm involves the solution of two algebraic systems with matrices  $\mathcal{U}$  and  $\mathcal{H}$ , respectively. The eigenvalues of  $\mathcal{H}$  are real and negative since this matrix is the discrete approximation to the Helmholtz equations with Dirichlet conditions (see Section 3.7.1). The matrix  $\mathcal{U}$  has real positive eigenvalues except for  $k = 0$ , in which case two eigenvalues are zero (the case  $k = 0$  is discussed below). For fixed  $N$  and  $k (\neq 0)$ , the condition number  $\kappa(\sigma)$  of  $\mathcal{U}$  is a slowly varying function of  $\sigma$ . For example, for  $N = 16$ ,  $k = 1$ , we find  $\kappa(10) = 1.77 \times 10^{-3}$  and  $\kappa(10^4) = 2.54 \times 10^{-4}$ .

In the present one-dimensional case, the matrices  $\mathcal{U}$  and  $\mathcal{H}$  can be inverted and their inverse stored. Thus, at each time-step the above algorithm reduces to three matrix-vector products. On the other hand, in the multidimensional case, the inversions cannot be considered and the solution must be iterative. The crude iterative Uzawa procedure for solving the system (7.47)-(7.49) is the following :

$$P^{m+1} - P^m = -\alpha (k \mathcal{I}_3 U_\star^m + \mathcal{D}_6 V_\star^m - S_Q) \tag{7.56}$$

$$\mathcal{H}U_{\star}^{m+1} = S_u^* - k\mathcal{I}_1 P^{m+1} \quad (7.57)$$

$$\mathcal{H}V_{\star}^{m+1} = S_v^* + \mathcal{D}_5 P^{m+1}, \quad (7.58)$$

where  $m$  refers to the iteration and  $\alpha$  is a relaxation parameter. The convergence of such an iterative procedure is very poor and must be improved by applying a preconditioning operator  $\mathcal{A}_0$  to  $P^{m+1} - P^m$  and calculating the relaxation parameter  $\alpha$  in a dynamic way as described in Section 3.8. The choice of the preconditioner  $\mathcal{A}_0$  is crucial and difficult. To our knowledge, the ideal preconditioner does not exist yet, mainly because of the poor properties of the Chebyshev collocation Uzawa operator.

Le Marec *et al.* (1996) have applied the preconditioner, originally proposed by Cahouet and Chabard (1988) for a finite-element method, to the Chebyshev collocation method. The idea is to examine what the Uzawa operator would be if the problem were also approximated in the  $y$ -direction by the Fourier rather than the Chebyshev method. For this it is convenient to come back to the continuous equations (7.36)-(7.3.21) and to consider their expressions in the Fourier space (with frequency  $\lambda$ ) corresponding to  $y$ . We recall that  $\sigma = k^2 + \sigma_0$  where  $k$  refers to the Fourier approximation in the  $x$ -direction and  $\sigma_0 = 2/(3\nu\Delta t)$ . Thus, the expression of the Uzawa operator in the  $(k, \lambda)$  Fourier space is

$$\frac{k^2 + \lambda^2}{\sigma_0 + k^2 + \lambda^2}$$

which may be considered as the Fourier approximation to the operator  $A$  such that

$$A_0 = \left[ I - \sigma_0 (\partial_{xx} + \partial_{yy})^{-1} \right]^{-1}.$$

This operator, supplemented with homogeneous Neumann boundary conditions, is proposed as a preconditioner by Cahouet and Chabard (1988). In the present case, the discrete form of  $A$  is the matrix

$$\mathcal{A}_0 = \left[ \mathcal{I}_0 - \sigma_0 (\mathcal{D}_7 - k^2 \mathcal{I}_0)^{-1} \right]^{-1},$$

where  $\mathcal{I}_0$  is the  $(N+1) \times (N+1)$  identity matrix and  $\mathcal{D}_7$  is the  $(N+1) \times (N+1)$  matrix approximating the second-order  $y$ -derivative operator with Neumann boundary conditions, that is,

$$\mathcal{D}_7 = \begin{pmatrix} d_{0,0}^{(1)} & \cdots & d_{0,N}^{(1)} \\ \vdots & \mathcal{D}_4 & \vdots \\ d_{N,0}^{(1)} & \cdots & d_{N,N}^{(1)} \end{pmatrix}, \quad (7.59)$$

where  $\mathcal{D}_4$  is the  $(N-1) \times (N+1)$  matrix defined in Eq.(6.36). Therefore, Eq.(7.56) is replaced by

$$\mathcal{A}_0 (P^{m+1} - P^m) = -\alpha_m (k\mathcal{I}_3 U_{\star}^m + \mathcal{D}_6 V_{\star}^m - S_Q), \quad (7.60)$$

where  $\alpha_m$  is the relaxation parameter whose expression is specified below. The system consisting of Eqs.(7.57), (7.58) and (7.60) is solved iteratively according to the following algorithm :

1. *Initialization*

$P^0$  is given,

$$\begin{aligned}\mathcal{H}U_{\star}^0 &= S_u^{\star} - k\mathcal{I}_1 P^0 \\ \mathcal{H}V_{\star}^0 &= S_v^{\star} + \mathcal{D}_5 P^0 \\ R^0 &= k\mathcal{I}_3 U_{\star}^0 + \mathcal{D}_6 V_{\star}^0 - S_Q,\end{aligned}$$

and

$$\begin{aligned}-(\mathcal{D}_7 - k^2 \mathcal{I}_0) \Phi^0 &= R^0 \\ W^0 &= R^0 + \sigma_0 \Phi^0.\end{aligned}$$

2. *Current iteration* ( $m \geq 0$ )

$P^m$ ,  $U_{\star}^m$ ,  $V_{\star}^m$ ,  $R^m$ , and  $W^m$  are known,

$$\begin{aligned}\mathcal{H}\bar{U}_{\star}^m &= -k\mathcal{I}_1 W^m \\ \mathcal{H}\bar{V}_{\star}^m &= \mathcal{D}_5 W^m \\ \bar{R}^m &= k\mathcal{I}_3 \bar{U}_{\star}^m + \mathcal{D}_6 \bar{V}_{\star}^m\end{aligned}$$

then

$$\begin{aligned}P^{m+1} &= P^m - \alpha_m W^m \\ U_{\star}^{m+1} &= U_{\star}^m - \alpha_m \bar{U}_{\star}^m \\ V_{\star}^{m+1} &= V_{\star}^m - \alpha_m \bar{V}_{\star}^m \\ R^{m+1} &= R^m - \alpha_m \bar{R}^m,\end{aligned}$$

and

$$\begin{aligned}-(\mathcal{D}_7 - k^2 \mathcal{I}_0) \bar{\Phi}^m &= \bar{R}^m \\ W^{m+1} &= W^m - \alpha_m (\bar{R}^m + \sigma_0 \bar{\Phi}^m).\end{aligned}$$

The algorithm is stopped when the divergence measured by  $R^m$  is sufficiently small, as well as  $\bar{U}_{\star}^m$  and  $\bar{V}_{\star}^m$ .

Le Marec *et al.* (1996) have tested two types of relaxation parameters. The parameter associated with the steepest descent method

$$\alpha_m = \frac{(R^m, W^m)}{(\bar{R}^m, W^m)}$$

and the one corresponding to the minimal residual method

$$\alpha_m = \frac{(R^m, \bar{R}^m)}{(\bar{R}^m, \bar{R}^m)},$$

where  $(\cdot, \cdot)$  designates the Euclidian scalar product. Both parameters give roughly the same rate of convergence  $\tau$ . This rate of convergence is defined as

$$\tau = \left( \frac{\|\nabla \cdot \vec{V}^m\|}{\|\nabla \cdot \vec{V}^0\|} \right)^{1/m},$$

where  $\|\cdot\|$  is the discrete  $L^\infty$ -norm. Numerical experiments performed by Le Marec *et al.* (1996) show that  $\tau \simeq 0.52 - 0.68$  for two- and three-dimensional Cartesian computations. On the other hand, the rate deteriorates for three-dimensional axisymmetric computations since it is about  $0.69 - 0.75$  according to the values of  $\sigma_0$ . These results show that, at least in the Cartesian case, the number of iterations needed to reach a reasonable convergence may be relatively small if the initial field is not too far from solenoidal. This happens in a time-integration process when the iterative Uzawa procedure is initialized with the solution at the previous time-cycle.

The above Uzawa algorithm needs only the solution of Helmholtz or Poisson problems with Dirichlet or Neumann conditions. In the multidimensional case, the algebraic systems can be solved by the matrix-diagonalization method described in Section 3.7.

Now we consider the special case  $k = 0$ . For  $k = 0$ , the matrix  $\mathcal{U}$  is found to have two eigenvalues equal to zero. This behaviour was already mentioned. One of the zero eigenvalues expresses the fact that the pressure is defined up to a constant. The second null eigenvalue corresponds to the so-called spurious mode of pressure associated with the collocation discretization. This comes from the fact that, when  $k = 0$ , the pressure appears in Eqs.(7.36)-(7.40) only through its derivative  $p'(y)$  and that it is the same in the approximate problem. More precisely, let us consider the Chebyshev approximation  $p_N(y)$  to the pressure  $p(y)$ , namely,

$$p_N(y) = \hat{p}_0 T_0 + \hat{p}_N T_N(y) + \sum_{k=1}^{N-1} \hat{p}_k T_k(y)$$

with  $T_0 = 1$ . Then, by differentiation, we get

$$p'_N(y) = \hat{p}_N T'_N(y) + \sum_{k=1}^{N-1} \hat{p}_k T'_k(y).$$

Therefore, the mode  $\hat{p}_0$  does not appear in the discrete derivative. This mode is not spurious since it has a physical meaning. On the other hand,



since  $T'_N(y_j) = 0$  for  $j = 1, \dots, N-1$ , the mode  $\hat{p}_N$  does not also appear in the discrete Stokes problem, which does not involve the pressure nor the boundary values of its derivative since the discrete system is closed by prescribing the incompressibility equation (7.39) at the boundaries. Therefore, the mode  $\hat{p}_N$ , which is not controlled by the problem is spurious, its value remains undetermined. If we assume the algebraic system to be solved by a special technique (Section 7.4.1.b), the resulting pressure remains contaminated by the spurious mode and may exhibit oscillations. The velocity field, however, is not contaminated and is correctly computed. Hence, if the pressure field is not of physical interest, the presence of spurious modes is unimportant. If not, the possible remedies are :

- (i) filter the spurious mode,
- (ii) calculate the correct pressure from the Poisson equation,
- (iii) use a staggered mesh,
- (iv) use the single-grid  $\mathbb{P}_N - \mathbb{P}_{N-2}$  approximation.

This question of spurious modes will again be discussed in the two-dimensional case (Section 7.4) where the problem is more delicate than here. In fact, for  $k = 0$ , the Stokes problem (7.36)-(7.40) considerably simplifies and becomes

$$u'' - \sigma u = f_u, \quad -1 < y < 1 \quad (7.61)$$

$$u(-1) = \gamma_-, \quad u(1) = \gamma_+, \quad (7.62)$$

and

$$v'' - \sigma v - p' = f_v, \quad -1 < y < 1 \quad (7.63)$$

$$v' = 0, \quad -1 < y < 1 \quad (7.64)$$

$$v(-1) = g_-, \quad v(1) = g_+. \quad (7.65)$$

This problem is easily solved. The first two equations give  $u$ . The conservation of the total flow rate imposes  $g_+ = g_- = g$ , so that  $v(y) = g$ ; then the  $v$ -equation gives, up to a constant,

$$p = -\sigma g y - \int^y f_v(\eta) d\eta.$$

Note that, for  $k = 0$ , the stability of the time-discretization scheme (7.29)-(7.33) reduces to the stability of the diffusion equation for  $\hat{u}_k^{n+1}$  and  $\hat{v}_k^{n+1}$ .

To close this section, we mention that the consideration of the stress-free boundary conditions (6.14) does not change the algorithm described above. As done in Section 3.7.1, the value of the tangential velocity at the boundary  $y = 1$ , namely,  $u_0$ , is eliminated thanks to the boundary condition  $Du_0 = 0$ . In this way, we again obtain Eqs. (7.47)-(7.49) with a redefinition of  $\mathcal{H}$ ,  $\mathcal{I}_3$ ,  $S_u^*$  and  $S_Q$ .

### (c) Solution of the Stokes problem by the influence matrix method

The Uzawa method, just described, loses some efficiency in the multidimensional case because of the necessity to employ an iterative procedure whose convergence may be slow. In the context of direct method, an alternative is constituted by the influence matrix method. The method is commonly used in two-dimensional problems, but its application to the three-dimensional case cannot be considered due to the huge size of the influence matrix, whose order is determined by the number of collocation points belonging to the domain boundary.

The influence matrix method for solving the Stokes problem in primitive variables has been introduced by Kleiser and Schumann (1980). They were interested in the calculation of three-dimensional channel flows with two periodic directions. In the third direction, associated with no-slip boundary conditions, Kleiser and Schumann derive a Poisson equation for the pressure and determine the boundary values of the pressure ensuring that the velocity divergence is zero. This problem is solved by means of the Chebyshev tau method. This method extended to the two-dimensional configuration has been developed by Le Quéré and Alziary de Roquefort (1985), then it was improved by Tuckerman (1989). This improvement ensures that the polynomial approximating the divergence of the velocity is identically zero, as obtained, in the one-dimensional case, by Kleiser and Schumann (1980). The extension of the method to the Chebyshev collocation approximation was proposed by Canuto *et al.* (1988) in the one-dimensional case and by Madabhushi *et al.* (1993) in the two-dimensional case. The mathematical analysis of the influence matrix technique associated with the tau method has been done by Canuto and Sacchi-Landriani (1986) for the Legendre polynomial approximation and by Sacchi-Landriani (1986) in the Chebyshev polynomial case.

To begin with, we present the influence matrix method in its simplified version, which does not ensure the velocity divergence to be exactly zero. The reason for this deficiency will be discussed, and the remedy resulting from this discussion will be presented.

#### c1. The “simplified” influence matrix method

As commonly done when dealing with the Stokes or Navier-Stokes equations, we begin to construct a Poisson equation for the pressure. As a matter of fact, in Fourier space, as considered in Eqs.(7.36)-(7.40), the pressure is a solution of a Helmholtz equation. This equation is obtained by considering the linear combination of Eq.(7.36) multiplied by  $k$ , and Eq.(7.37) differentiated with respect to  $y$ , in order to make apparent the divergence of the velocity  $Q = k u + v'$ . We obtain

$$(Q'' - \sigma Q) - (p'' - k^2 p) = k f_u + f'_v.$$

Then, from Eq.(7.38), we have simply

$$p'' - k^2 p = f_p, \quad (7.66)$$

where

$$f_p = -k f_u - f'_v. \quad (7.67)$$

Therefore, if  $p$  satisfies Eq.(7.66) and if  $u$  and  $v$  are, respectively, determined by Eq.(7.36) and (7.37), the divergence  $Q$  satisfies the Helmholtz equation

$$Q'' - \sigma Q = 0, \quad -1 < y < 1. \quad (7.68)$$

Consequently  $Q = 0$  everywhere in  $-1 \leq y \leq 1$  if  $Q$  satisfies the conditions

$$Q(-1) = 0, \quad Q(1) = 0. \quad (7.69)$$

Therefore, the  $\mathcal{P}$ -Problem to be solved is decomposed into two problems. The first one determines  $p$  and  $v$  :

$$p'' - k^2 p = f_p, \quad -1 < y < 1 \quad (7.70)$$

$$v'' - \sigma v - p' = f_v, \quad -1 < y < 1 \quad (7.71)$$

$$v(\pm 1) = g_{\pm}, \quad (7.72)$$

$$v'(\pm 1) = -k g_{\pm} = h_{\pm}, \quad (7.73)$$

and the second one determines  $u$  once  $p$  is known

$$u'' - \sigma u = f_u - k p, \quad -1 < y < 1, \quad (7.74)$$

$$u(\pm 1) = \gamma_{\pm}. \quad (7.75)$$

This simple Helmholtz problem is easily solved.

One may observe that the problem for  $p$  and  $v$  is similar to the one previously encountered for  $\omega$  and  $\psi$  in Section 6.3, so that the solution algorithm for calculating  $(p, v)$  will follow the same outlines. This leads to inevitable repetitions, but is necessary for an understanding of the developments which follow.

The Chebyshev collocation method, applied to the solution of the two problems above, leads to algebraic systems determining the unknowns

$$u(y_j) \cong u_N(y_j) = u_j, \quad v(y_j) \cong v_N(y_j) = v_j, \quad p(y_j) \cong p_N(y_j) = p_j,$$

where  $u_N$ ,  $v_N$ , and  $p_N$  are polynomials of degree at most  $N$ , and  $y_j$ ,  $j = 0, \dots, N$ , are the Gauss-Lobatto points (6.26).

The system determining  $(p_j, v_j)$ ,  $j = 0, \dots, N$ , is

$$L_{p,j} \equiv (D^2 - k^2) p_j - f_{p,j} = 0, \quad j = 1, \dots, N-1 \quad (7.76)$$

$$L_{v,j} \equiv (D^2 - \sigma) v_j - D p_j - f_{v,j} = 0, \quad j = 1, \dots, N-1 \quad (7.77)$$

$$v_0 = g_+ \quad (7.78)$$

$$v_N = g_- \quad (7.79)$$

$$D v_0 = h_+ \quad (7.80)$$

$$D v_N = h_-, \quad (7.81)$$

and the system determining  $u_j$ ,  $j = 0, \dots, N$ , is

$$L_{u,j} \equiv (D^2 - \sigma) u_j + k p_j - f_{u,j} = 0, \quad j = 1, \dots, N-1 \quad (7.82)$$

$$u_0 = \gamma_+ \quad (7.83)$$

$$u_N = \gamma_-. \quad (7.84)$$

This last system is easily solved once as  $p_j$  is known from the first system. It is convenient to write this first system in vector form. Let us consider the vectors  $P$  and  $V$  containing the unknowns, and the  $2(N+1)$ -vector  $\Phi$ , respectively, defined by

$$P = \begin{pmatrix} p_0 \\ \vdots \\ p_N \end{pmatrix}, \quad V = \begin{pmatrix} v_0 \\ \vdots \\ v_N \end{pmatrix}, \quad \Phi = \begin{pmatrix} P \\ V \end{pmatrix},$$

then the vector form of the system (7.76)-(7.81) is

$$\mathcal{A} \Phi = S. \quad (7.85)$$

The  $2(N+1) \times 2(N+1)$  matrix  $\mathcal{A}$  is

$$\mathcal{A} = \begin{pmatrix} \mathcal{B}_2 & \mathcal{O}_1 \\ -\mathcal{D}_5 & \mathcal{B}_1 \\ \mathcal{O}_2 & \mathcal{I}_2 \\ \mathcal{O}_2 & \mathcal{D}_3 \end{pmatrix},$$

where  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are  $(N-1) \times (N+1)$  matrices such that

$$\mathcal{B}_1 = \mathcal{D}_4 - \sigma \mathcal{I}_1, \quad \mathcal{B}_2 = \mathcal{D}_4 - k^2 \mathcal{I}_1$$

with  $\mathcal{D}_4$  and  $\mathcal{I}_1$  defined in Section 6.3.2.b by Eqs.(6.36) and (6.33), respectively. The matrices  $\mathcal{O}_1$ ,  $\mathcal{O}_2$ , and  $\mathcal{I}_2$  were also defined in the same section. The matrix  $\mathcal{D}_5$  is defined by Eq.(7.51). Finally, the  $2(N+1)$ -vector  $S$  is

$$S = (f_{p,1}, \dots, f_{p,N-1}, f_{v,1}, \dots, f_{v,N-1}, g_+, g_-, h_+, h_-)^T.$$

As for the  $\omega$ - $\psi$  problem, the large  $2(N+1) \times 2(N+1)$  system (7.85) is solved by an algorithm leading to the solution of six smaller  $(N+1) \times (N+1)$  systems. This algorithm is the discrete form of the influence matrix

technique usually presented on the continuous problem, and which will be described later.

The solution of (7.85) is sought in the form

$$\Phi = \tilde{\Phi} + \xi_1 \bar{\Phi}_1 + \xi_2 \bar{\Phi}_2, \quad (7.86)$$

where  $\xi_1$  and  $\xi_2$  are scalars and  $\tilde{\Phi}$ ,  $\bar{\Phi}_1$  and  $\bar{\Phi}_2$  are respectively the vector solution of

$$\hat{\mathcal{A}} \tilde{\Phi} = \tilde{S} \quad (7.87)$$

$$\hat{\mathcal{A}} \bar{\Phi}_1 = \bar{S}_1 \quad (7.88)$$

$$\hat{\mathcal{A}} \bar{\Phi}_2 = \bar{S}_2, \quad (7.89)$$

where

$$\hat{\mathcal{A}} = \begin{pmatrix} \mathcal{B}_2 & \mathcal{O}_1 \\ -\mathcal{D}_5 & \mathcal{B}_1 \\ \mathcal{O}_2 & \mathcal{I}_2 \\ \mathcal{I}_2 & \mathcal{O}_2 \end{pmatrix}.$$

Therefore,  $\hat{\mathcal{A}}$  is the matrix which would result from the discrete problem (7.76)-(7.81) if the Neumann conditions on  $v$  [Eqs.(7.80) and (7.81)] were replaced by Dirichlet conditions on  $p$ . These Dirichlet conditions are homogeneous so that the vector  $\tilde{S}$  has the form

$$\tilde{S} = (f_{p,1}, \dots, f_{p,N-1}, f_{v,1}, \dots, f_{v,N-1}, g_+, g_-, 0, 0)^T.$$

Then, the vectors  $\bar{S}_1$  and  $\bar{S}_2$  are

$$\bar{S}_1 = (0, \dots, 0, 0, 1)^T, \quad \bar{S}_2 = (0, \dots, 0, 1, 1)^T.$$

From this it results that  $\Phi$ , as defined by Eq.(7.86), is the solution to

$$\hat{\mathcal{A}} \Phi = S_\star \quad (7.90)$$

with

$$S_\star = (f_{p,1}, \dots, f_{p,N-1}, f_{v,1}, \dots, f_{v,N-1}, g_+, g_-, \xi_2, \xi_1)^T.$$

Now it is easy to check that the first  $2N$  equations of system (7.90) coincide with the  $2N$  equations (7.76)-(7.79). In other words, the vector  $\Phi$  defined by Eq.(7.86) satisfies these  $2N$  equations whatever the constants  $\xi_1$  and  $\xi_2$ . These constants are then determined such that the last two equations (7.80) and (7.81), namely, the Neumann conditions on  $v$ , are satisfied. Thus, we impose

$$\mathcal{C} (\tilde{\Phi} + \xi_1 \bar{\Phi}_1 + \xi_2 \bar{\Phi}_2) = E \quad (7.91)$$

where the  $2 \times 2(N+1)$  matrix  $\mathcal{C}$  and the vector  $E$  are, respectively, defined by

$$\mathcal{C} = [\mathcal{O}_2 \quad \mathcal{D}_3], \quad E = (h_+, h_-)^T.$$

Now we consider the vectors  $\tilde{V}$ ,  $\bar{V}_1$ , and  $\bar{V}_2$  such that

$$\tilde{\Phi} = \begin{pmatrix} \tilde{P} \\ \tilde{V} \end{pmatrix}, \quad \bar{\Phi}_l = \begin{pmatrix} \bar{P}_l \\ \bar{V}_l \end{pmatrix}, \quad l = 1, 2$$

with

$$\tilde{P} = (\tilde{p}_0, \dots, \tilde{p}_N)^T, \quad \bar{P}_l = (\bar{p}_{l,0}, \dots, \bar{p}_{l,N})^T, \quad l = 1, 2$$

$$\tilde{V} = (\tilde{v}_0, \dots, \tilde{v}_N)^T, \quad \bar{V}_l = (\bar{v}_{l,0}, \dots, \bar{v}_{l,N})^T, \quad l = 1, 2.$$

Because of the special structure of  $\mathcal{C}$ , the  $2 \times 2$  system (7.91) reduces to

$$\xi_1 \mathcal{D}_3 \bar{V}_1 + \xi_2 \mathcal{D}_3 \bar{V}_2 = \tilde{E}$$

where

$$\tilde{E} = E - \mathcal{D}_3 \tilde{V}. \quad (7.92)$$

This system is written as

$$\mathcal{M} \Xi = \tilde{E} \quad (7.93)$$

where  $\Xi = (\xi_1, \xi_2)^T$  and the influence (or “capacitance”) matrix  $\mathcal{M}$  is the  $2 \times 2$  matrix

$$\mathcal{M} = [\mathcal{D}_3 \bar{V}_1 \quad \mathcal{D}_3 \bar{V}_2] = [m_{i,j}], \quad i = 0, N, \quad j = 1, 2$$

such that

$$m_{i,j} = \sum_{l=0}^N d_{i,l}^{(1)} \bar{v}_{j,l}.$$

The solution of the system (7.93) gives the constants  $\xi_1$  and  $\xi_2$ , so that the solution  $\Phi$  is completely determined. The matrix  $\mathcal{M}$  is invertible (as discussed later) except for  $k = 0$ . For this value of  $k$ , the matrix  $\mathcal{M}$  has a null eigenvalue. This corresponds to the fact that the pressure is determined up to a constant. In the present one-dimensional case, the mode  $k = 0$  is easily calculated as explained in the preceding section.

Now the important point is that the decomposition described above allows us to replace the solution of the large initial  $2(N+1) \times 2(N+1)$  system by that of the smaller  $(N+1) \times (N+1)$  systems. This is obtained thanks to the special structure of the matrix  $\hat{\mathcal{A}}$ . Thus, the system (7.87) is split into the following two systems. The first one determines  $\tilde{P}$ :

$$\hat{\mathcal{A}}_1 \tilde{P} = \tilde{S}_1 \quad (7.94)$$

where

$$\hat{\mathcal{A}}_1 = \begin{pmatrix} \mathcal{B}_2 \\ \mathcal{I}_2 \end{pmatrix}$$

and

$$\tilde{S}_1 = (f_{p,1}, \dots, f_{p,N-1}, 0, 0)^T.$$

Then  $\tilde{P}$  having been calculated, the second system determines  $\tilde{V}$ , namely

$$\hat{\mathcal{A}}_2 \tilde{V} = \tilde{S}_2 + \hat{\mathcal{A}}_3 \tilde{P} \quad (7.95)$$

with

$$\hat{\mathcal{A}}_2 = \begin{pmatrix} \mathcal{B}_1 \\ \mathcal{I}_2 \end{pmatrix}, \quad \hat{\mathcal{A}}_3 = \begin{pmatrix} \mathcal{D}_5 \\ \mathcal{O}_2 \end{pmatrix}$$

and

$$\tilde{S}_2 = (f_{v,1}, \dots, f_{v,N-1}, g_+, g_-)^T.$$

In the same way, each of the systems (7.88) and (7.89) is split into two smaller systems. The first system determines  $\bar{P}_l$ ,  $l = 1, 2$ :

$$\hat{\mathcal{A}}_1 \bar{P}_l = \bar{S}_{l,1}, \quad l = 1, 2 \quad (7.96)$$

with

$$\bar{S}_{1,1} = (0, \dots, 0, 0, 1)^T, \quad \bar{S}_{2,1} = (0, \dots, 0, 1, 0)^T$$

and the second system determines  $\bar{V}_l$ ,  $l = 1, 2$ , namely

$$\hat{\mathcal{A}}_2 \bar{V}_l = \hat{\mathcal{A}}_3 \bar{P}_l, \quad l = 1, 2. \quad (7.97)$$

The dimension of each of these smaller systems is  $(N+1) \times (N+1)$ . The matrices  $\hat{\mathcal{A}}_l$ ,  $l = 1, 2$ , are inverted once and for all before the start of the time-integration and their inverses are stored. In the same way, the elementary solutions  $\bar{P}_l$  and  $\bar{V}_l$ ,  $l = 1, 2$ , are calculated in the preprocessing stage, as well as the matrix  $\mathcal{M}$  and its inverse  $\mathcal{M}^{-1}$ . The elementary solutions and the matrix  $\mathcal{M}^{-1}$  are stored, so that the solution at each time-cycle is obtained through *Algorithm A*:

1. Calculate  $\tilde{P}$  and  $\tilde{V}$  by

$$\tilde{P} = \hat{\mathcal{A}}_1^{-1} \tilde{S}_1, \quad \tilde{V} = \hat{\mathcal{A}}_2^{-1} (\tilde{S}_2 + \hat{\mathcal{A}}_3 \tilde{P}).$$

2. Calculate  $\tilde{E}$  by Eq.(7.92).
3. Calculate  $\Xi$  by  $\Xi = \mathcal{M}^{-1} \tilde{E}$ .
4. Calculate  $P$  and  $V$  by Eq.(7.86).
5. Calculate  $U = (u_0, \dots, u_N)^T$  by solving the system (7.82)-(7.84).

We point out that the resolvability of the initial system (7.85) (for  $k \neq 0$ ) is a consequence of the resolvability of the various systems (7.94)-(7.97) corresponding to Dirichlet problems for Helmholtz equations and the resolvability of the system (7.93).

As for the vorticity-streamfunction equations, the algorithm just described for solving the discrete system (7.76)-(7.81) can also be obtained from the Chebyshev collocation approximation of a set of differential problems derived from Eqs.(7.70)-(7.73). As a matter of fact, this is the usual way to introduce the influence matrix method, as done by Kleiser and Schumann (1980). This method consists of searching the boundary values of the pressure ensuring that the divergence of the velocity at the boundary is zero. From a more general point of view, the advantage of the influence matrix method has already been discussed in Sections 3.7.2 (Remark 1) and 6.3.2.c.

Returning to problem (7.70)-(7.73), we look for its solution according to the decomposition

$$p = \tilde{p} + \bar{p}, \quad v = \tilde{v} + \bar{v}. \quad (7.98)$$

The pair  $(\tilde{p}, \tilde{v})$  is the solution of the  $\tilde{\mathcal{P}}$ -Problem defined by

$$\tilde{p}'' - k^2 \tilde{p} = f_p, \quad -1 < y < 1$$

$$\tilde{p}(-1) = 0, \quad \tilde{p}(1) = 0,$$

and

$$\tilde{v}'' - \sigma \tilde{v} = f_v + \tilde{p}', \quad -1 < y < 1$$

$$\tilde{v}(-1) = g_-, \quad \tilde{v}(1) = g_+.$$

Thus, the determination of  $(\tilde{p}, \tilde{v})$  reduces to the successive solution of two Helmholtz equations with Dirichlet conditions.

Then the pair  $(\bar{p}, \bar{v})$  is the solution of the  $\bar{\mathcal{P}}$ -Problem :

$$\bar{p}'' - k^2 \bar{p} = 0, \quad -1 < y < 1 \quad (7.99)$$

$$\bar{v}'' - \sigma \bar{v} - \bar{p}' = 0, \quad -1 < y < 1 \quad (7.100)$$

$$\bar{v}(-1) = 0, \quad \bar{v}(1) = 0 \quad (7.101)$$

$$\bar{v}'(-1) = h_- - \tilde{v}'(-1), \quad \bar{v}'(1) = h_+ - \tilde{v}'(1). \quad (7.102)$$

Now this problem coupling  $\bar{p}$  and  $\bar{v}$  is transformed into a problem in which the boundary values  $\bar{p}(\pm 1)$  must be determined such that the Neumann condition on  $\bar{v}$  is satisfied. This defines the  $\bar{\mathcal{P}}_0$ -Problem :

$$\bar{p}'' - k^2 \bar{p} = 0, \quad -1 < y < 1$$

$$\bar{v}'' - \sigma \bar{v} - \bar{p}' = 0, \quad -1 < y < 1$$

$$\bar{p}(-1) = \xi_1, \quad \bar{p}(1) = \xi_2$$

$$\bar{v}(-1) = 0, \quad \bar{v}(1) = 0,$$



where the unknown quantities  $\xi_1$  and  $\xi_2$  are determined such that the boundary conditions on  $\bar{v}'(\pm 1)$  are satisfied. Now we take advantage of the linearity of the  $\bar{\mathcal{P}}_0$ -problem to express its solution as the linear combination

$$\bar{p} = \xi_1 \bar{p}_1 + \xi_2 \bar{p}_2, \quad \bar{v} = \xi_1 \bar{v}_1 + \xi_2 \bar{v}_2, \quad (7.103)$$

where the elementary solutions  $(\bar{p}_l, \bar{v}_l)$ ,  $l = 1, 2$ , satisfy the  $\bar{\mathcal{P}}_l$ -Problems, namely,

$\bar{\mathcal{P}}_1$ -Problem :

$$\bar{p}_1'' - k^2 \bar{p}_1 = 0, \quad -1 < y < 1$$

$$\bar{p}_1(-1) = 1, \quad \bar{p}_1(1) = 0,$$

and

$$\bar{v}_1'' - \sigma \bar{v}_1 = \bar{p}_1', \quad -1 < y < 1$$

$$\bar{v}_1(-1) = 0, \quad \bar{v}_1(1) = 0.$$

$\bar{\mathcal{P}}_2$ -Problem :

$$\bar{p}_2'' - k^2 \bar{p}_2 = 0, \quad -1 < y < 1$$

$$\bar{p}_2(-1) = 0, \quad \bar{p}_2(1) = 1,$$

and

$$\bar{v}_2'' - \sigma \bar{v}_2 = \bar{p}_2', \quad -1 < y < 1$$

$$\bar{v}_2(-1) = 0, \quad \bar{v}_2(1) = 0.$$

Therefore, the linear combination (7.103) satisfies the equations (7.99), (7.100), and the boundary conditions (7.101) whatever  $\xi_1$  and  $\xi_2$ . These quantities are then determined by imposing the conditions (7.102), namely,

$$\bar{v}_1'(-1) \xi_1 + \bar{v}_2'(-1) \xi_2 = h_- - \tilde{v}'(-1)$$

$$\bar{v}_1'(1) \xi_1 + \bar{v}_2'(1) \xi_2 = h_+ - \tilde{v}'(1).$$

This system is written

$$\mathcal{M} \Xi = \tilde{E} \quad (7.104)$$

where  $\Xi = (\xi_1, \xi_2)^T$ ,  $\tilde{E} = (h_- - \tilde{v}'(-1), h_+ - \tilde{v}'(1))^T$ , and

$$\mathcal{M} = \begin{pmatrix} \bar{v}_1'(-1) & \bar{v}_2'(-1) \\ \bar{v}_1'(1) & \bar{v}_2'(1) \end{pmatrix} \quad (7.105)$$

is the influence (or “capacitance”) matrix. Thus, the knowledge of  $\xi_1$  and  $\xi_2$  determines the solution  $p$ ,  $v$  and, finally, the determination of  $u$  by Eqs.(7.74)-(7.75) achieves the solution of the Stokes problem.

The various Dirichlet ( $\tilde{\mathcal{P}}$ ,  $\overline{\mathcal{P}}_1$ , and  $\overline{\mathcal{P}}_2$ ) problems are solved using the Chebyshev (collocation or tau) method. Considering the collocation approximation, it is easy to see that the present method is identical to the algorithm above for solving the algebraic system. More precisely, the discretized  $\tilde{\mathcal{P}}$ -problem identifies with systems (7.94) and (7.95) ; then the discretized  $\overline{\mathcal{P}}_l$ -problems,  $l = 1, 2$ , identify with systems (7.96) and (7.97). Finally, the system (7.104) identifies with (7.93). The continuous formulation of the influence matrix method, as just described, is much easier to conceive than the preceding algebraic algorithm and, in fact, it constitutes the guide to construct this algorithm.

The  $\overline{\mathcal{P}}_l$ -problems,  $l = 1, 2$ , are time-independent and can be solved once and for all at a stage performed before the start of the time-integration. Moreover, the elementary solutions possess the symmetry properties

$$\overline{p}_2(y) = \overline{p}_1(-y) , \quad \overline{v}_2(y) = -\overline{v}_1(-y) ,$$

so that only the  $\overline{\mathcal{P}}_1$ -problem has to be solved, and only the elementary solution  $(\overline{p}_1, \overline{v}_1)$  is stored. Also the influence matrix  $\mathcal{M}$  is calculated, inverted and stored. Hence, at each time-cycle, the solution of the Stokes problem (7.70)-(7.75) is obtained through *Algorithm B* :

1. Solve the  $\tilde{\mathcal{P}}$ -problem.
2. Calculate  $\tilde{E}$ .
3. Calculate  $\Xi$  by  $\Xi = \mathcal{M}^{-1} \tilde{E}$ .
4. Calculate  $(p, v)$  by Eq.(7.98).
5. Calculate  $u$  by solving Eqs.(7.74)-(7.75).

Since the  $\overline{\mathcal{P}}_1$ -problem is time-independent and very simple, one may be tempted to solve it analytically. This is not recommended and leads to a serious loss of accuracy. The reason is that, in such a situation, the equivalence between the Chebyshev collocation approximation to the sets of differential  $\tilde{\mathcal{P}}$ - and  $\overline{\mathcal{P}}$ -problems and the algebraic system (7.85) is lost. As a matter of fact, the exact solution of the  $\overline{\mathcal{P}}_l$ -problems exhibits a boundary layer behaviour when  $\sigma$  and  $k$  are large, so that the polynomial approximation to  $(\overline{p}_l, \overline{v}_l)$  is very far from the exact solution, unless  $N$  is sufficiently large. However, the accuracy of the polynomial representation of  $(\overline{p}_l, \overline{v}_l)$  is unnecessary, since it constitutes only an intermediate step in the procedure, and the real accuracy [that of the algebraic system (7.85)] is recovered when the final combination (7.98) is reconstituted. Note that the error obtained by the use of the analytical solution of  $\overline{\mathcal{P}}_l$  is diminished if  $N$  is sufficiently large to represent correctly the boundary layer behaviour of  $(\overline{p}_l, \overline{v}_l)$ . However, this could necessitate a much larger value of  $N$  than effectively needed to represent the final solution.

Thanks to the property  $\overline{v}_2(y) = -\overline{v}_1(-y)$ , the matrix  $\mathcal{M}$  can be expressed in terms of  $\overline{v}'_1(\pm 1)$  only, and its eigenvalues are  $\lambda_{\pm} = \overline{v}'_1(-1) \pm \overline{v}'_1(1)$ . From the exact calculation of  $\overline{v}_1$ , it is found that these eigenvalues

are nonzero except for the case  $k = 0$  where  $\lambda_+ = 0$ . This is a consequence of the property of the pressure which is determined to within a constant. This result, concerning the invertibility of  $\mathcal{M}$ , however, does not apply to the approximate problem for any value of  $N$ . As explained above, whatever the Chebyshev method, collocation or tau, as discussed above, the accuracy of the polynomial approximation to the elementary solutions  $(\bar{p}_l, \bar{v}_l)$ ,  $l = 1, 2$ , is very poor, except if  $N$  is very large. Therefore, the reasoning based on the continuous differential problem gives information only for large  $N$ . An analysis of the eigenvalues of  $\mathcal{M}$ , in the Chebyshev tau approximation, was done by Sacchi-Landriani (1986).

Finally, taking into account that  $\xi_1$  and  $\xi_2$  are the boundary values ensuring that the Neumann condition is satisfied, we may avoid the storage of the  $K$  solutions  $(\bar{p}_1, \bar{v}_1)$ , where  $K$  is the cut-off frequency of the Fourier series. This is obtained by replacing *step 4* of Algorithm B by the successive solution of the following problems, deduced from the  $\bar{\mathcal{P}}_0$ -Problem :

$$p'' - k^2 p = f_p, \quad -1 < y < 1$$

$$p(-1) = \xi_1, \quad p(1) = \xi_2,$$

and

$$v'' - \sigma v = f_v + p', \quad -1 < y < 1$$

$$v(-1) = g_-, \quad v(1) = g_+,$$

which determines the final solution  $(p, v)$ . This variant is not really necessary in the one-dimensional case, but is currently used in two-dimensional problems because of memory requirement.

Of course, this variant has its counterpart in the algorithm developed earlier for the solution of the full algebraic system. Hence, *step 4* of Algorithm A is replaced by

$$\hat{\mathcal{A}}_1 P = G,$$

where

$$G = (f_{p,1}, \dots, f_{p,N-1}, \xi_2, \xi_1)^T$$

and

$$\hat{\mathcal{A}}_1 V = H$$

with

$$H = (f_{v,1}, \dots, f_{v,N-1}, g_+, g_-)^T.$$

Now we discuss the important point concerning the constraint on the divergence of the velocity. From the continuous point of view, the divergence  $Q = k u + v'$  is identically zero since it satisfies the homogeneous Helmholtz equation (7.68) with the homogeneous Dirichlet conditions (7.69). But, in

the discrete case, the fact that the momentum equation (7.71) is not satisfied at the boundaries  $y = \pm 1$  prevents the grid values of the approximate divergence

$$Q_j = k u_j + D v_j \quad (7.106)$$

to be exactly zero at the inner collocation points (it is prescribed zero at the boundaries). This can be seen in the following two equivalent ways.

First, consider the equations (7.70)-(7.75) approximated with the Chebyshev collocation method. It has been shown (Section 3.4.3) that the polynomial approximation to the solution of a differential problem satisfies, in fact, a modified equation called the “error equation” with the same boundary conditions. More precisely, the polynomials  $u_N$ ,  $v_N$ , and  $p_N$  are solutions to

$$u_N'' - \sigma u_N + k p_N = f_{uN} + e_{uN}, \quad -1 < y < 1$$

$$v_N'' - \sigma v_N - k p_N' = f_{vN} + e_{vN}, \quad -1 < y < 1$$

$$p_N'' - k^2 p_N = f_{pN} + e_{pN}, \quad -1 < y < 1$$

$$u(\pm 1) = \gamma_{\pm}, \quad v(\pm 1) = g_{\pm}, \quad v'(\pm 1) = -k g_{\pm} = h_{\pm},$$

where  $f_{uN}$ ,  $f_{vN}$ , and  $f_{pN}$  are the polynomials interpolating, respectively,  $f_u$ ,  $f_v$ , and  $f_p$  at the collocation points. The polynomials (of degree  $N$  at most)  $e_{uN}$ ,  $e_{vN}$ , and  $e_{pN}$  are of the general form

$$e_N = (\lambda y + \mu) T_N'(y),$$

where  $\lambda$  and  $\mu$  are directly connected to the residuals of the corresponding equation at the boundaries [see Eqs.(3.93) and (3.94)]. Now, if we construct, from the above equations, the Helmholtz equation satisfied by the polynomial  $Q_N$  approximating the divergence  $Q$  we obtain the problem

$$Q_N'' - \sigma Q_N = k e_{uN} + e_{vN}' - e_{pN} \equiv E_N \quad (7.107)$$

$$Q_N(-1) = 0, \quad Q_N(1) = 0. \quad (7.108)$$

Taking the expressions of  $e_{uN}$ ,  $e_{vN}$ , and  $e_{pN}$  into account, it is easy to see that the polynomial  $E_N$  at the inner collocation points

$$E_N(y_j) = e_{vN}'(y_j) = (\lambda_v y_j + \mu_v) T_N''(y_j)$$

is not zero since  $T_N''(y_j) = (-1)^{j+1} N^2 / (1 - y_j^2) \neq 0$  for  $j = 1, \dots, N-1$ . Thus, the Helmholtz equation satisfied by  $Q_N$  is not homogeneous and, consequently,  $Q_N$  cannot be zero, even at the collocation points. From Eqs.(3.93) and (3.94), the magnitude of  $\lambda_v$  and  $\mu_v$  and, consequently of  $Q_N(y_j)$  depend on the way in which the  $v$ -equation is satisfied at the boundaries  $y = \pm 1$ . Thus,  $Q_N(y_j) \rightarrow 0$  when  $N \rightarrow \infty$  but is not identically

zero for arbitrary  $N$ . Similar behaviour exists in the case of the Chebyshev tau method (Kleiser and Schumann, 1980), the error coming from the last two Galerkin equations that are not satisfied (Section 3.4.3.a).

The second way consists of directly analyzing the discrete problem (7.76)-(7.84). We construct the algebraic system satisfied by the grid values  $Q_j = Q_N(y_j)$  of the velocity divergence defined by Eq.(7.106). For that we combine Eqs.(7.82) and (7.77) according to

$$k L_{u,j} + \sum_{l=1}^{N-1} d_{j,l}^{(1)} L_{v,l} = 0, \quad j = 1, \dots, N-1. \quad (7.109)$$

To make apparent the velocity divergence  $Q_j$  in this equation, we must commute the summations in the second term. This necessitates having the same summation limits, namely,  $l = 0$  to  $l = N$ , in both sums. Consequently, we add and subtract the terms corresponding to  $l = 0$  and  $l = N$  in the second term of Eq.(7.109). We get

$$\begin{aligned} (D^2 - \sigma) Q_j - d_{j,0}^{(1)} L_{v,0} - d_{j,N}^{(1)} L_{v,N} - [(D^2 - k^2) p_j + k f_{u,j} + D f_{v,j}] &= 0 \\ j = 1, \dots, N-1. \end{aligned} \quad (7.110)$$

Now, taking into account Eq.(7.67) expressing that

$$k f_{u,j} + D f_{v,j} = -f_{p,j},$$

we observe that the term in square brackets is nothing other than  $L_{p,j}$  which is zero as prescribed by Eq.(7.76). Therefore, the algebraic system satisfied by  $Q_j$  is

$$(D^2 - \sigma) Q_j = d_{j,0}^{(1)} L_{v,0} + d_{j,N}^{(1)} L_{v,N}, \quad j = 1, \dots, N-1 \quad (7.111)$$

$$Q_0 = 0, \quad Q_N = 0. \quad (7.112)$$

Since  $L_{v,0}$  and  $L_{v,N}$  are not zero, the above system is not homogeneous and its solution is not identically zero. It is easy to check that the right-hand-side of Eq.(7.111) coincides with  $E_N(y_j)$  determined above.

From the analysis above, it can be deduced that the divergence  $Q_j$  could be made equal to zero for  $j = 0, \dots, N$  if the pressure equation

$$L_{p,j} = 0, \quad j = 1, \dots, N-1$$

was replaced by the corrected equation

$$L_{p,j} = - \left( d_{j,0}^{(1)} L_{v,0} + d_{j,N}^{(1)} L_{v,N} \right), \quad j = 1, \dots, N-1 \quad (7.113)$$

so that Eq. (7.111) becomes an homogeneous equation ensuring, with the boundary conditions (7.112), that  $Q_j = 0$  for  $j = 0, \dots, N$ .

Since the correction term in Eq.(7.113) depends on the solution itself, it remains unknown and must be determined as part of the global solution. This may be obtained through a correction method which is now described.

*c2. The “rigorous” influence matrix method*

This method is presented by Canuto *et al.* (1988) for the one-dimensional collocation approximation. Its application to the two-dimensional case has been developed by Madabhushi *et al.* (1993) and by Sabbah and Pasquetti (1998). The method is called the “extended influence matrix method” in the last two works. We prefer the designation “rigorous influence matrix method,” or simply “influence matrix method,” because it constitutes the correct way to understand and apply the method, as was done originally by Kleiser and Schumann (1980) in the case of the tau method.

Since the source of the difficulty is of a discrete nature, it is advisable to discuss the correction method in the discrete framework.

By introducing the notations

$$A = D^2 - k^2, \quad B = D^2 - \sigma^2,$$

we write the discrete  $\mathcal{P}$ -Problem to be solved as

$$A p_j = f_{p,j} - D b_j, \quad j = 1, \dots, N-1 \quad (7.114)$$

$$B v_j - D p_j = f_{v,j} + b_j, \quad j = 0, \dots, N \quad (7.115)$$

$$v_0 = g_+, \quad v_N = g_- \quad (7.116)$$

$$D v_0 = h_+, \quad D v_N = h_-, \quad (7.117)$$

where the correction term  $b_j$  (which is determined as a part of the solution) is zero except at the boundaries  $y = \pm 1$ , namely ,

$$b_j = 0, \quad j = 1, \dots, N-1.$$

It is clear that the correction term  $-D b_j$  in the pressure equation (7.114) is nothing other than the right-hand side of Eq.(7.113).

As previously, the solution is decomposed according to

$$p_j = \tilde{p}_j + \bar{p}_j, \quad v_j = \tilde{v}_j + \bar{v}_j. \quad (7.118)$$

The couple  $(\tilde{p}_j, \tilde{v}_j)$ ,  $j = 0, \dots, N$ , is determined as previously by the  $\tilde{\mathcal{P}}$ -Problem defined by

$$A \tilde{p}_j = f_{p,j}, \quad j = 1, \dots, N-1$$

$$\tilde{p}_0 = 0, \quad \tilde{p}_N = 0,$$

and

$$B \tilde{v}_j = f_{v,j} + D \tilde{p}_j, \quad j = 1, \dots, N-1$$

$$\tilde{v}_0 = g_+, \quad \tilde{v}_N = g_-.$$

$l$	$b_{l,0}$	$b_{l,N}$	$q_{l,0}$	$q_{l,N}$
1	0	0	0	1
2	0	0	1	0
3	0	1	0	0
4	1	0	0	0

Table 7.1. Boundary values  $b_{l,j}$  and  $q_{l,j}$ .

When  $\tilde{p}_j$  and  $\tilde{v}_j$  have been calculated, the residual of the  $\tilde{v}$ -equation at the boundaries may be determined, that is,

$$\tilde{b}_j = B \tilde{v}_j - D \tilde{p}_j - f_{v,j}, \quad j = 0, N.$$

Then the couple  $(\bar{p}_j, \bar{v}_j)$ ,  $j = 0, \dots, N$ , together with  $b_0$  and  $b_N$ , are determined by the  $\overline{\mathcal{P}}$ -Problem :

$$A \bar{p}_j = -D b_j, \quad j = 1, \dots, N-1 \quad (7.119)$$

$$B \bar{v}_j - D \bar{p}_j = 0, \quad j = 1, \dots, N-1 \quad (7.120)$$

$$\bar{v}_0 = 0, \quad \bar{v}_N = 0 \quad (7.121)$$

$$D \bar{v}_0 = h_+ - D \tilde{v}_0, \quad D \bar{v}_N = h_- - D \tilde{v}_N \quad (7.122)$$

$$b_0 = \tilde{b}_0 + \bar{b}_0, \quad b_N = \tilde{b}_N + \bar{b}_N, \quad (7.123)$$

where

$$\bar{b}_j = B \bar{v}_j - D \bar{p}_j, \quad j = 0, N.$$

Now the  $\overline{\mathcal{P}}$ -problem is solved by means of the superposition of elementary solutions of the form

$$\phi_j = \sum_{l=1}^4 \xi_l \phi_{l,j} \quad (7.124)$$

with  $\phi_j = \bar{p}_j, \bar{v}_j, \bar{b}_j, b_j$ . Therefore, we begin by solving the  $\overline{\mathcal{P}}_l$ -Problems,  $l = 1, \dots, 4$ , defined by

$$A \bar{p}_{l,j} = -D b_{l,j}, \quad j = 1, \dots, N-1$$

$$\bar{p}_{l,0} = q_{l,0}, \quad \bar{p}_{l,N} = q_{l,N}$$

and

$$B \bar{v}_{l,j} - D \bar{p}_{l,j} = 0, \quad j = 1, \dots, N-1$$

$$\bar{v}_{l,0} = 0, \quad \bar{v}_{l,N} = 0,$$

where again  $b_{j,l} = 0$  for  $j = 1, \dots, N-1$ , and every  $l$ . The boundary values  $b_{l,j}$  and  $q_{l,j}$ , for  $j = 0, N$ , required for the solution of the  $\overline{\mathcal{P}}_l$ -problems, are given in Table 7.1.

After  $(\bar{p}_{l,j}, \bar{v}_{l,j})$ ,  $l = 1, \dots, 4$ ,  $j = 0, \dots, N$ , have been calculated, the residuals  $\bar{b}_{l,0}$  and  $\bar{b}_{l,N}$  are determined by

$$\bar{b}_{l,j} = A \bar{v}_{l,j} - D \bar{p}_{l,j}, \quad l = 1, \dots, 4, \quad j = 0, N.$$

It is easy to see that the combination (7.124) satisfies the first three equations (7.119)-(7.121) of the  $\bar{\mathcal{P}}$ -problem, whatever the constants  $\xi_l$ ,  $l = 1, \dots, 4$ . Therefore, these constants are determined by requiring the last equations (7.122) and (7.123) to be satisfied by the combination (7.124), that is

$$\begin{aligned} \sum_{l=1}^4 \xi_l D \bar{v}_{l,0} &= \gamma_+ - D \tilde{v}_0 \\ \sum_{l=1}^4 \xi_l D \bar{v}_{l,N} &= \gamma_- - D \tilde{v}_N \\ \sum_{l=1}^4 \xi_l (b_{l,0} - \bar{b}_{l,0}) &= \tilde{b}_0 \\ \sum_{l=1}^4 \xi_l (b_{l,N} - \bar{b}_{l,N}) &= \tilde{b}_N. \end{aligned}$$

The  $4 \times 4$  matrix  $\mathcal{M}$  of this system is the “extended influence matrix”. It is invertible except for  $k = 0$  (Madabhushi *et al.*, 1993). For  $k = 0$ , the matrix  $\mathcal{M}$  has two eigenvalues equal to zero. This behaviour is identical to that encountered previously for the Uzawa operator  $\mathcal{U}$ . This result is easily understood since the addition of the correction term  $b$ , to ensure the solenoidal character of the approximate velocity field, effectively recovers the original algebraic system (7.42)-(7.46). Note that, for  $k = 0$ , the solution of the Stokes problem is easily calculated, as shown at the end of Section 7.3.2.b.

The  $\bar{\mathcal{P}}$ -problems are time-independent. Hence, they can be solved once and for all in the preprocessing stage of the time-integration process, and their solutions are stored. Also, the matrix  $\mathcal{M}$  is calculated and inverted at this stage and its inverse is stored. Therefore, the solution of the  $\bar{\mathcal{P}}$ -problem [Eqs.(7.114)-(7.117)] at each time-cycle is obtained through an algorithm similar to the above Algorithm B.

### c3. Shear-stress-free boundary

We close this section by considering the case where the no-slip condition at  $y = 1$  is replaced by the stress-free boundary condition (6.14). In this case, the unknowns  $p$  and  $v$  are coupled to  $u$  through the incompressibility equation at  $y = 1$ , since Eq.(7.75) is replaced by

$$u(-1) = 0, \quad u'(1) = 0$$



and Eq.(7.73) by

$$v'(-1) = 0, \quad k u(1) + v'(1) = 0.$$

The influence matrix method applies with a small change due to this coupling. The decomposition (7.98) is also done for  $u$ , namely,

$$u = \tilde{u} + \bar{u}.$$

In the  $\tilde{\mathcal{P}}$ -problem the velocity  $\tilde{u}$  is calculated from the knowledge of  $\tilde{p}$  as it is for  $\tilde{v}$ . On the other hand, the  $\overline{\mathcal{P}}$ -Problem in the simplified version of the method is now

$$\bar{u}'' - \sigma \bar{u} + k \bar{p} = 0, \quad -1 < y < 1$$

$$\bar{v}'' - \sigma \bar{v} - \bar{p}' = 0, \quad -1 < y < 1$$

$$\bar{p}'' - k^2 \bar{p} = 0, \quad -1 < y < 1$$

$$\bar{u}(-1) = 0, \quad \bar{u}'(1) = 0,$$

$$\bar{v}(-1) = 0, \quad \bar{v}(1) = 0,$$

$$\bar{v}'(-1) = -\tilde{v}'(-1), \quad k \bar{u}(1) + \bar{v}'(1) = -k \tilde{u}(1) - \tilde{v}'(1).$$

Then the algorithm follows the same lines, except that  $\bar{u}$  is also decomposed according to Eq.(7.103) and  $\bar{u}_1$  and  $\bar{u}_2$  have to be determined in order to construct the influence matrix on the same principles as that above. Note that such changes in the algorithm also apply for the rigorous influence matrix method.

#### (d) Time-schemes leading to the Darcy problem

The time-discretization considered here belongs to the general class of splitting (or fractional step) methods. At the first step, a provisional value  $\tilde{\mathbf{V}}^{n+1}$  of the velocity  $\mathbf{V}$ , that does not satisfy the incompressibility equation, is calculated. Then, at the second step, this velocity  $\tilde{\mathbf{V}}^{n+1}$  is corrected in such a way that the final velocity  $\mathbf{V}^{n+1}$  and the pressure  $p^{n+1}$  satisfy the complete Navier-Stokes equations. This is the basic principle of the projection method introduced by Chorin (1968) and Temam (1969). Since then, the method has largely been used in association with various spatial approximations. At the same time, some improvements have been introduced in order to increase the accuracy of the method.

To introduce time-splitting in a general way it is convenient to consider the time-dependent Stokes equations written in vector form

$$\partial_t \mathbf{V} - \nu \nabla^2 \mathbf{V} + \nabla p = \mathbf{f} \quad \text{in } \Omega \quad (7.125)$$

$$\nabla \cdot \mathbf{V} = 0 \quad \text{in } \Omega, \quad (7.126)$$

associated with the initial condition  $\mathbf{V} = \mathbf{V}_0$  and the boundary condition

$$\mathbf{V} = \mathbf{V}_\Gamma \quad \text{on } \Gamma = \partial\Omega. \quad (7.127)$$

In the *first step* (“diffusion step”), the provisional velocity  $\tilde{\mathbf{V}}^{n+1}$  is calculated according to the three-level scheme :

$$\begin{aligned} & \frac{(1 + \varepsilon) \tilde{\mathbf{V}}^{n+1} - 2\varepsilon \mathbf{V}^n - (1 - \varepsilon) \mathbf{V}^{n-1}}{2\Delta t} - \nu \nabla^2 [\theta \tilde{\mathbf{V}}^{n+1} + (1 - \theta) \mathbf{V}^n] \\ & + \nabla (\lambda_1 p^n) = \theta \mathbf{f}^{n+1} + (1 - \theta) \mathbf{f}^n \quad \text{in } \Omega. \end{aligned} \quad (7.128)$$

The associated boundary condition is generally

$$\tilde{\mathbf{V}}^{n+1} = \mathbf{V}_\Gamma^{n+1} \quad \text{on } \Gamma \quad (7.129)$$

since  $\tilde{\mathbf{V}}^{n+1}$ , as given by Eq.(7.128) with  $\lambda_1 = 1$ , may be seen as an approximation to  $\mathbf{V}^{n+1}$ . However, if  $\lambda_1 = 0$ , as it was in the original projection method, the equation (7.128) is no longer consistent with (7.125) and one may question the best boundary value of  $\tilde{\mathbf{V}}^{n+1}$  to improve the accuracy (see, e.g., Fortin *et al.*, 1971 ; Kim and Moin, 1985).

The *second step* (“projection step”) determines the solution  $(\mathbf{V}^{n+1}, p^{n+1})$  by

$$(1 + \varepsilon) \frac{\mathbf{V}^{n+1} - \tilde{\mathbf{V}}^{n+1}}{2\Delta t} + \nabla (\lambda_2 p^{n+1} + \lambda_3 p^n) = \mathbf{0} \quad \text{in } \Omega \quad (7.130)$$

$$\nabla \cdot \mathbf{V}^{n+1} = 0 \quad \text{in } \Omega \quad (7.131)$$

$$\mathbf{V}^{n+1} \cdot \mathbf{n} = \mathbf{V}_\Gamma^{n+1} \cdot \mathbf{n} \quad \text{on } \Gamma. \quad (7.132)$$

The parameters  $\varepsilon$ ,  $\theta$ ,  $\lambda_1$ ,  $\lambda_2$ , and  $\lambda_3$  characterize the scheme and determine its accuracy. Note that  $\lambda_2 \neq 0$  is required to ensure the presence of the pressure at level  $n + 1$ . The elimination of  $\tilde{\mathbf{V}}^{n+1}$  from (7.128) and (7.130) gives the semidiscrete equation actually approximating the momentum equation (7.125), that is,

$$\begin{aligned} & \frac{(1 + \varepsilon) \mathbf{V}^{n+1} - 2\varepsilon \mathbf{V}^n - (1 - \varepsilon) \mathbf{V}^{n-1}}{2\Delta t} - \nu \nabla^2 [\theta \mathbf{V}^{n+1} + (1 - \theta) \mathbf{V}^n] \\ & + \nabla [\lambda_2 p^{n+1} + (\lambda_1 + \lambda_3) p^n] - \frac{2\theta \Delta t}{1 + \varepsilon} \nu \nabla^2 [\nabla (\lambda_2 p^{n+1} + \lambda_3 p^n)] \\ & = \theta \mathbf{f}^{n+1} + (1 - \theta) \mathbf{f}^n. \end{aligned} \quad (7.133)$$

As usual, the consistency and accuracy are analyzed by Taylor's expansion with respect to time. The scheme (7.133) is consistent with the momentum equation (7.125) and is accurate to  $O(\Delta t)$  if the condition

$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$

is satisfied. It is accurate to second order  $O(\Delta t^2)$  if, moreover, the following conditions are satisfied

$$\frac{\varepsilon}{2} = \theta = 1 - (\lambda_1 + \lambda_3) ,$$

$$\frac{\theta}{1 + \varepsilon} (\lambda_2 + \lambda_3) = 0 .$$

In the original projection method (Chorin, 1968 ; Temam, 1969) the pressure  $p^n$  does not appear in the first step, namely,  $\lambda_1 = 0$ . From the above conditions, we easily see that such a splitting scheme cannot be  $O(\Delta t^2)$ . The first-order explicit scheme ( $\varepsilon = 1$ ,  $\theta = 0$ ,  $\lambda_1 = 0$ ,  $\lambda_2 = 1$ ,  $\lambda_3 = 0$ ) associated with the spatial approximation, referred to as Option (2) below, has been studied and applied to flow computations by Phillips and Roberts (1993), who examined the constraint on the time-step required for stability.

In the incremental projection method (Goda, 1979 ; Van Kan, 1986) the pressure  $p^n$  is involved in the first step, namely,  $\lambda_1 = 1$ . The resulting second-order schemes are then defined by  $\lambda_3 = -\lambda_2$  and  $\varepsilon/2 = \theta = \lambda_2$ . The Crank-Nicolson (CN) scheme corresponds to  $\varepsilon = 1$ ,  $\theta = 1/2$ , and  $\lambda_2 = -\lambda_3 = 1/2$ . The Backward-Differentiation BDI2 scheme (see Section 4.4) is defined by  $\varepsilon = 2$ ,  $\theta = 1$ , and  $\lambda_2 = -\lambda_3 = 1$ . As discussed several times, the BDI2 scheme is preferred to the CN scheme. Therefore, the BDI2 scheme is considered in the sequel.

The determination of  $\tilde{\mathbf{V}}^{n+1}$  at the first step requires the solution of two non coupled Helmholtz equations with Dirichlet conditions for the components  $\tilde{u}$  and  $\tilde{v}$  of  $\tilde{\mathbf{V}}^{n+1}$ . This is done by using the Chebyshev collocation method based on Gauss-Lobatto points (Sections 3.4. and 3.7).

The second step (7.130)-(7.132) corresponds to the Darcy problem, also called the "div-grad problem". Note that this problem is well-posed (see, e.g., Azaiez *et al.*, 1994a) if the normal velocity only is prescribed at the boundary  $\Gamma$ . The tangential velocity is then determined from the global solution. The effective approximate solution of problem (7.130)-(7.132) strongly depends on the closure of the discrete problem. This will be made more precise later.

Up to now, we have discussed the accuracy of the splitting method by considering only the discretization with respect to time of the equations of motion. As a matter of fact, the spatial approximation technique and especially the treatment of boundary conditions play a fundamental role in the overall accuracy of the discrete method. Therefore, we are discussing now the full discrete problem by returning to the one-dimensional equations

(7.23)-(7.25) satisfied by the Fourier components  $\hat{u}_k, \hat{v}_k, \hat{p}_k$  of the solution. By setting

$$\hat{u}_k = u, \quad \hat{v}_k = v, \quad \hat{p}_k = p, \quad \hat{f}_{u,k} = f_u, \quad \hat{f}_{v,k} = f_v,$$

equations (7.23)-(7.25) are

$$\partial_t u - \nu (\partial_{yy} - k^2) u - k p = f_u, \quad (7.134)$$

$$\partial_t v - \nu (\partial_{yy} - k^2) v + \partial_y p = f_v, \quad (7.135)$$

$$k u + \partial_y v = 0, \quad (7.136)$$

to be solved in  $-1 < y < 1$  with the initial condition

$$u(y, 0) = u_0(y), \quad v(y, 0) = v_0(y), \quad (7.137)$$

and the boundary conditions

$$u(\pm 1, t) = \gamma_{\pm}, \quad v(\pm 1, t) = g_{\pm}. \quad (7.138)$$

Then we introduce the time-discretization and polynomial approximation so that

$$u(y_j, n \Delta t) \cong u_N^n(y_j) = u_j^n,$$

$$v(y_j, n \Delta t) \cong v_N^n(y_j) = v_j^n,$$

$$p(y_j, n \Delta t) \cong p_N^n(y_j) = p_j^n$$

where  $y_j$  are the collocation points defined by Eq.(6.26). The  $y$ -derivatives are defined by Eq.(7.41). With these notations, the splitting scheme approximating the above equations is

*Step 1*

$$\begin{aligned} \frac{1}{2\Delta t} (3\tilde{u}_j^{n+1} - 4u_j^n + u_j^{n-1}) - \nu (D^2 - k^2) \tilde{u}_j^{n+1} - k p_j^n &= f_{u,j}^{n+1}, \\ j &= 1, \dots, N-1 \end{aligned} \quad (7.139)$$

$$\tilde{u}_0^{n+1} = \gamma_+^{n+1}, \quad \tilde{u}_N^{n+1} = \gamma_-^{n+1}, \quad (7.140)$$

$$\begin{aligned} \frac{1}{2\Delta t} (3\tilde{v}_j^{n+1} - 4v_j^n + v_j^{n-1}) - \nu (D^2 - k^2) \tilde{v}_j^{n+1} + D p_j^n &= f_{v,j}^{n+1}, \\ j &= 1, \dots, N-1 \end{aligned} \quad (7.141)$$

$$\tilde{v}_0^{n+1} = g_+^{n+1}, \quad \tilde{v}_N^{n+1} = g_-^{n+1}. \quad (7.142)$$

Each of the above systems is easily solved using classical algorithms. The start of the time-integration ( $n = 0$ ) requires the knowledge of  $u_j^{-1}, v_j^{-1}$

and  $p_j^0$ . Starting schemes for multistep methods have been discussed in Section 4.5.1.d and will again be considered for the Navier-Stokes equations in Section 7.4.3, in which the determination of the initial pressure  $p_j^0$  will also be addressed.

*Step 2*

$$\frac{3}{2\Delta t} (u_j^{n+1} - \tilde{u}_j^{n+1}) - k(p_j^{n+1} - p_j^n) = 0, \quad j = 1, \dots, N-1 \quad (7.143)$$

$$\frac{3}{2\Delta t} (v_j^{n+1} - \tilde{v}_j^{n+1}) + D(p_j^{n+1} - p_j^n) = 0, \quad j = 1, \dots, N-1 \quad (7.144)$$

$$k u_j^{n+1} + D v_j^{n+1} = 0, \quad j = 1, \dots, N-1 \quad (7.145)$$

$$v_0^{n+1} = g_+^{n+1}, \quad v_N^{n+1} = g_-^{n+1}. \quad (7.146)$$

Note that the above equations have been collocated only at the inner points,  $j = 1, \dots, N-1$ . This furnishes [with Eq.(7.3.124)]  $3N-1$  equations for the  $3N+3$  unknowns  $u_j^{n+1}$ ,  $v_j^{n+1}$ , and  $p_j^{n+1}$ ,  $j = 0, \dots, N$ . Consequently, four equations have to be added. We obtain two more equations by prescribing either Eq.(7.145) or Eq.(7.144) at the boundaries  $j = 0$  and  $j = N$ . As will be seen below, the first choice leads to an Uzawa operator to calculate the pressure, while the second choice leads to a Helmholtz operator. This being done, we need two more equations. One possibility is to prescribe Eq.(7.143) at  $j = 0, N$ . This choice is in agreement with the mathematical properties of the Darcy problem but introduces an error in the tangential velocity at the boundaries. However, if we consider that the above Darcy problem is not isolated, but results from the splitting of unsteady equations with given boundary conditions, we may contemplate the possibility of adding the exact boundary conditions

$$u_0^{n+1} = \gamma_+^{n+1}, \quad u_N^{n+1} = \gamma_-^{n+1}, \quad (7.147)$$

instead of Eq.(7.143) at  $j = 0, N$ .

To summarize, we have the choice between four options to close the system (7.143)-(7.146) :

- Option (1)* : Use Eq.(7.145) and Eq.(7.143) at  $j = 0, N$ .
- Option (2)* : Use Eq.(7.145) at  $j = 0, N$  and Eq.(7.147).
- Option (3)* : Use Eq.(7.144) and Eq.(7.143) at  $j = 0, N$ .
- Option (4)* : Use Eq.(7.144) at  $j = 0, N$  and Eq.(7.147).

Now, we analyze the truncation error associated with each of these options. For this purpose, we eliminate the intermediate quantities  $\tilde{u}_j^{n+1}$  and  $\tilde{v}_j^{n+1}$ , in order to derive the discrete equations effectively solved by  $(u^{n+1}, v^{n+1}, p^{n+1})$ . We successively consider the four options.

*Option (1).* We get the system

$$\begin{aligned} & \frac{1}{2\Delta t} (3u_j^{n+1} - 4u_j^n + u_j^{n-1}) \\ & -\nu (D^2 - k^2) u_j^{n+1} - k p_j^{n+1} + \Delta t E_{u,j}^{n+1} = f_{u,j}^{n+1}, \quad j = 1, \dots, N-1 \end{aligned} \quad (7.148)$$

$$\begin{aligned} & \frac{1}{2\Delta t} (3v_j^{n+1} - 4v_j^n + v_j^{n-1}) \\ & -\nu (D^2 - k^2) v_j^{n+1} + D p_j^{n+1} + \Delta t E_{v,j}^{n+1} = f_{v,j}^{n+1}, \quad j = 1, \dots, N-1 \end{aligned} \quad (7.149)$$

$$k u_j^{n+1} + D v_j^{n+1} = 0, \quad j = 0, \dots, N \quad (7.150)$$

$$\begin{aligned} u_0^{n+1} - \frac{2\Delta t}{3} k (p_0^{n+1} - p_0^n) &= \gamma_+^{n+1}, \quad u_N^{n+1} - \frac{2\Delta t}{3} k (p_N^{n+1} - p_N^n) = \gamma_-^{n+1} \\ v_0^{n+1} &= g_+^{n+1}, \quad v_N^{n+1} = g_-^{n+1}, \end{aligned} \quad (7.151)$$

$$(7.152)$$

where

$$E_{u,j}^{n+1} = \frac{2}{3} \nu k \left[ \sum_{l=0}^N d_{j,l}^{(2)} (p_l^{n+1} - p_l^n) - k^2 (p_j^{n+1} - p_j^n) \right], \quad (7.153)$$

$$E_{v,j}^{n+1} = -\frac{2}{3} \nu \left[ \sum_{l=1}^{N-1} d_{j,l}^{(2)} D (p_l^{n+1} - p_l^n) - k^2 D (p_j^{n+1} - p_j^n) \right]. \quad (7.154)$$

The terms  $\Delta t E_{u,j}^{n+1}$  and  $\Delta t E_{v,j}^{n+1}$  in Eqs.(7.148) and (7.149), respectively, are error terms due to splitting. With Taylor's expansion in time around  $(n+1)\Delta t$ , we evaluate these terms to be  $O(\Delta t^2)$ , that is, the same order of the whole discrete equations. Note that these splitting error terms vanish at steady state. The incompressibility equation (7.150) is obviously free of any temporal truncation error. On the other hand, the boundary equations (7.151) exhibit an error  $O(\Delta t^2)$  for the tangential velocity at the boundary since, by Taylor's expansion, we get

$$u_{0,N}^{n+1} = \gamma_{\pm}^{n+1} + \frac{2\Delta t^2}{3} k (\partial_t p)_{0,N}^{n+1} + \dots \quad (7.155)$$

The error term in  $\Delta t^2$  is the “slip velocity” associated with the splitting method.

The pressure solution of Eqs.(7.148)-(7.152) is exempt from spurious modes when  $k \neq 0$ , since the pressure appears in nondifferentiated form. On the other hand, for  $k = 0$ , the resulting system admits the spurious pressure mode  $T_N(y)$  (we recall that the constant mode  $T_0$  is physical and not spurious) and this for the same reason that was explained in Section 7.3.2.b. As a matter of fact, it is easy to see that the above system is similar

(except the error terms  $\Delta t E_{u,j}^{n+1}$  and  $\Delta t E_{v,j}^{n+1}$ ) to the system (7.42)-(7.46).

*Option (2).* We get the same equations (7.148)-(7.150) with

$$E_{u,j}^{n+1} = \frac{2\nu}{3} k \left[ \sum_{l=1}^{N-1} d_{j,l}^{(2)} (p_l^{n+1} - p_l^n) - k^2 (p_j^{n+1} - p_j^n) \right] \quad (7.156)$$

and  $E_{v,j}^{n+1}$  is again given by Eq.(7.154), that is, the inner truncation error of the momentum equations is again  $O(\Delta t^2)$ . On the other hand, the presence of the slip velocity in Eq.(7.155) is now avoided since Eq.(7.151) is replaced by Eq.(7.147). Finally, for  $k = 0$ , the system suffers from the same spurious mode as Option (1).

As already mentioned, a possible way to avoid the presence of spurious pressure modes is constituted by the  $P_N - P_{N-2}$  approximation method which, moreover, simplifies the question associated with the closure of the algebraic system. This method, of special interest to multidimensional problems, is described in Section 7.4.2.

*Option (3).* We get Eqs. (7.148) and (7.149) with  $E_{u,j}^{n+1}$  given by Eq.(7.153) and  $E_{v,j}^{n+1}$  by

$$E_{v,j}^{n+1} = -\frac{2\nu}{3} \left[ \sum_{l=0}^N d_{j,l}^{(2)} D(p_l^{n+1} - p_l^n) - k^2 D(p_j^{n+1} - p_j^n) \right]. \quad (7.157)$$

Therefore, the inner truncation error is  $O(\Delta t^2)$ . Of course, Eq.(7.150) is restricted to inner collocation points  $j = 1, \dots, N-1$ . Therefore the divergence of the approximate velocity is not the null polynomial as in Options (1) and (2). It is zero at the inner collocation points only. Equation (7.151) is considered in Option (3) as in Option (1), so that the tangential velocity at the boundaries is again subject to an erroneous slip velocity as shown by Eq.(7.155). The system is closed by the boundary equations (7.152) and the equations (7.144) prescribed at  $j = 0, N$ , that is,

$$v_0^{n+1} + \frac{2\Delta t}{3} D(p_0^{n+1} - p_0^n) = \tilde{v}_0^{n+1}, \quad v_N^{n+1} + \frac{2\Delta t}{3} D(p_N^{n+1} - p_N^n) = \tilde{v}_N^{n+1}. \quad (7.158)$$

Since  $\tilde{v}_0^{n+1} = v_0^{n+1} = g_+^{n+1}$  and  $\tilde{v}_N^{n+1} = v_N^{n+1} = g_-^{n+1}$ , these equations reduce to

$$D(p_j^{n+1} - p_j^n) = 0, \quad j = 0, N. \quad (7.159)$$

The Chebyshev collocation solution obtained in this Option (3) is free from spurious modes whatever  $k$  (the constant mode  $T_0$  is not considered as spurious). The reason lies in the fact that Eq. (7.159) involves the derivative  $T'_N(\pm 1)$  which is not zero, while  $T'_N(y_j) = 0$  for  $j = 1, \dots, N-1$  (see the discussion on spurious modes in Section 7.3.2.b).

Equation (7.159) means that

$$D p_j^{n+1} = D p_j^n = \dots = D p_j^0, \quad j = 0, N.$$

Thus, the normal pressure gradient at the boundary retains its initial value. Obviously, except for very special cases, this is not correct. Numerical experiments (Hugues and Randriamampianina, 1998 ; Raspo *et al.*, 2002) on the Chebyshev collocation approximation to the Navier-Stokes equations show that the errors on the pressure, as well as the velocity, are quite large (see Section 7.4.4). These errors prevent the use of such a strong formulation of this projection method as pointed out by Guermond and Quartapelle (1998) who recommend implementing the method in a weak framework.

Note that, in the original projection method ( $\lambda_1 = 0$ ), equation (7.159) is replaced by

$$D p_j^{n+1} = 0, \quad j = 0, N \quad (7.160)$$

which has no physical meaning either (see, e.g., Fortin *et al.*, 1971 ; Peyret and Taylor, 1983 ; Temam, 1991) and is subject to the same conclusion as above concerning its use in a strong framework. For theoretical results concerning the convergence of the projection method and its implementation we refer to Guermond (1996, 1999).

Hugues and Randriamampianina (1998) have proposed a Chebyshev collocation projection method that removes the drawback concerning the “incorrect” Neumann condition on the pressure. This method considers, as a first step, instead of  $p^n$ , a predicted pressure  $\bar{p}^{n+1}$  suitably defined. The method, which has been found very accurate, is described in Section 7.4.

*Option (4).* Equations (7.148) and (7.149) are again valid with  $E_{u,j}^{n+1}$  given by Eq.(7.156) and  $E_{v,j}^{n+1}$  given by Eq.(7.157). Therefore, the inner truncation error associated with the momentum equations is again  $O(\Delta t^2)$ . Equation (7.150) is restricted to the inner collocation points  $j = 1, \dots, N$  as in Option (3), therefore with the same result on the velocity divergence. Equation (7.151) is replaced by Eq.(7.147), so that the slip velocity is avoided. Lastly, the boundary equations (7.158) have to be added to Eq.(7.152) to close the system. Therefore, as for Option (3), the normal pressure gradient satisfies Eq.(7.159). Here also, the solution is free of spurious modes.

To close this section, we have to address the question of stability. There does not exist a complete stability analysis of the discrete problems represented by the various options considered above. For Option (1), Heinrichs (1998) checks numerically the stability of the method. However, he uses relatively small time-steps so that he does not verify if the stability is unconditional or not. Hugues and Randriamampianina (1998), and Raspo *et al.* (2002) have tested the incremental projection method [Options (3) and (4)] applied to the Navier-Stokes equations. They found that the critical



time-step is of the magnitude usually associated with the AB/BDI2 scheme. They did not consider, however, the possible unconditional stability of the discrete time-dependent Stokes equations.

### (e) Solution of the Darcy problem

In this section, we describe the algorithms for solving the algebraic system derived in the previous section. We remind ourselves that the calculation of the provisional velocity is the same for all four options considered above. The determination of  $(\tilde{u}^{n+1}, \tilde{v}^{n+1})$  reduces to the solution of the algebraic system resulting from the Chebyshev collocation approximation to the Dirichlet problem for the Helmholtz equation. This has been largely discussed in Chapter 3.

Therefore, the task here is to solve the algebraic system (7.143)-(7.146) supplemented with four boundary equations which differ according to the considered option. In a general way, the technique is to eliminate the velocity in order to obtain a system determining the pressure. With Options (1) and (2) this system is of Uzawa type while it is of Helmholtz type in the case of Options (3) and (4).

To simplify the notations we set

$$u_j^{n+1} = u_j, \quad v_j^{n+1} = v_j, \quad p_j^{n+1} = p_j, \quad \sigma = 3/(2 \Delta t),$$

such that Eqs. (7.143)-(7.145) simply become

$$\sigma u_j - k p_j = \varphi_{u,j} \quad (7.161)$$

$$\sigma v_j + D p_j = \varphi_{v,j} \quad (7.162)$$

$$k u_j + D v_j = 0 \quad (7.163)$$

$$v_0 = g_+, \quad v_N = g_-, \quad (7.164)$$

where

$$\varphi_{u,j} = \sigma \tilde{u}_j^{n+1} - k p_j^n, \quad \varphi_{v,j} = \sigma \tilde{v}_j^{n+1} + D p_j^n.$$

We define the following vectors containing the unknowns

$$U = \begin{pmatrix} u_0 \\ \vdots \\ u_N \end{pmatrix}, \quad V = \begin{pmatrix} v_0 \\ \vdots \\ v_N \end{pmatrix}, \quad P = \begin{pmatrix} p_0 \\ \vdots \\ p_N \end{pmatrix},$$

and

$$U_\star = \begin{pmatrix} u_1 \\ \vdots \\ u_{N-1} \end{pmatrix}, \quad V_\star = \begin{pmatrix} v_1 \\ \vdots \\ v_{N-1} \end{pmatrix}.$$

Note that  $U_\star = \mathcal{I}_1 U$  and  $V_\star = \mathcal{I}_1 V$  where  $\mathcal{I}_1$  is the  $(N-1) \times (N+1)$  matrix defined in Section 6.3.2.b [Eq.(6.33)].

Now, we consider successively the four options defined in the previous section.

*Option (1).* The system is constituted by Eq.(7.161) for  $j = 0, \dots, N$ , Eq.(7.162) for  $j = 1, \dots, N-1$ , Eq.(7.163) for  $j = 0, \dots, N$ , and Eq.(7.164). Therefore, the vectors of unknowns are  $U$ ,  $V_\star$ , and  $P$ , they are determined by

$$\sigma U - k P = S_u \quad (7.165)$$

$$\sigma V_\star + \mathcal{D}_5 P = S_v^\star \quad (7.166)$$

$$k U + \mathcal{D}_6 V_\star = S_Q. \quad (7.167)$$

The  $(N-1) \times (N+1)$  matrix  $\mathcal{D}_5$  and the  $(N+1) \times (N-1)$  matrix  $\mathcal{D}_6$  are defined, respectively, by Eqs.(7.51) and (7.52). The vectors  $S_u$ ,  $S_v^\star$ , and  $S_Q$  are, respectively, defined by

$$S_u = (\varphi_{u,0}, \dots, \varphi_{u,N})^T, \quad S_v^\star = (\varphi_{v,1}, \dots, \varphi_{v,N-1})^T,$$

$$S_Q = (\varphi_{Q,0}, \dots, \varphi_{Q,N})^T, \quad \varphi_{Q,j} = -\left(d_{j,0}^{(1)} g_+ + d_{j,N}^{(1)} g_-\right), \quad j = 0, \dots, N.$$

By eliminating  $U$  and  $V_\star$  from Eqs.(7.165)-(7.167) we obtain the system determining  $P$  :

$$(\mathcal{D}_6 \mathcal{D}_5 - k^2 \mathcal{I}_0) P = -\sigma S_Q + k S_u + \mathcal{D}_6 S_v, \quad (7.168)$$

where  $\mathcal{I}_0$  is the  $(N+1) \times (N+1)$  identity matrix. The operator  $\mathcal{U}_1 = \mathcal{D}_6 \mathcal{D}_5 - k^2 \mathcal{I}_0$  is an Uzawa operator where  $\mathcal{D}_6 \mathcal{D}_5$  could be qualified as “pseudo-discrete-Laplacian” since the  $j$ th element of the vector  $\mathcal{D}_6 \mathcal{D}_5 P$  is

$$\sum_{l=0}^N \left[ d_{j,l}^{(2)} - \left( d_{j,0}^{(1)} d_{0,l}^{(1)} + d_{j,N}^{(1)} d_{N,l}^{(1)} \right) \right] p_l,$$

making apparent the second-order derivative  $\sum_{l=0}^N d_{j,l}^{(2)} p_l$ .

After  $P$  has been calculated as the solution of (7.168),  $U$  and  $V_\star$  are determined by (7.165) and (7.166), respectively. For  $k \neq 0$ , the matrix  $\mathcal{U}_1$  is invertible. On the other hand, for  $k = 0$ , the matrix  $\mathcal{U}_1$  has two eigenvalues equal to zero. One of these corresponds to the fact that pressure is defined up to a constant. The second null eigenvalue corresponds to the spurious mode  $T_N(y)$  as discussed several times. In the present one-dimensional case, the solution of (7.161)-(7.164) for  $k = 0$  is easily calculated (see Section 7.3.2.b). In the two-dimensional problem approximated by Chebyshev collocation in both directions, the corresponding Uzawa matrix has four eigenvalues equal to zero. In this situation, it is possible to avoid these spurious modes by considering the so-called  $\mathcal{I}P_N - \mathcal{I}P_{N-2}$  approximation (Maday *et al.*, 1987 ; Heinrichs, 1993 ; Azaiez *et al.*, 1994b ; Botella, 1997) in which the pressure is approximated with a polynomial of degree  $N-2$

while the velocity is approximated with a polynomial of degree  $N$ . Such an approximation will be discussed in detail in Section 7.4.2.b.

*Option (2).* The system is made with Eqs.(7.161) and (7.162) for  $j = 1, \dots, N-1$ , Eq.(7.163) for  $j = 0, \dots, N$ , Eq.(7.164), and the boundary conditions  $u_0 = \gamma_+$ ,  $u_- = \gamma_-$ . The system determining the unknown vectors  $U_\star$ ,  $V_\star$ , and  $P$  is

$$\sigma U_\star - k \mathcal{I}_1 P = S_u^\star \quad (7.169)$$

$$\sigma V_\star + \mathcal{D}_5 P = S_v^\star \quad (7.170)$$

$$k \mathcal{I}_3 U_\star + \mathcal{D}_6 V_\star = S_Q', \quad (7.171)$$

where  $\mathcal{I}_3$  is the  $(N+1) \times (N-1)$  matrix defined by Eq.(7.53) and  $S_u^\star = \mathcal{I}_1 S_u$ . Finally, the vector  $S_Q'$  is defined by

$$S_Q' = (-k \gamma_+ + \varphi_{Q,0}, \varphi_{Q,1}, \dots, \varphi_{Q,N-1}, -k \gamma_- + \varphi_{Q,N})^T.$$

The elimination of  $U_\star$  and  $V_\star$  gives the system determining  $P$ , that is,

$$(\mathcal{D}_6 \mathcal{D}_5 - k^2 \mathcal{I}_3 \mathcal{I}_1) P = -\sigma S_Q' + k \mathcal{I}_3 S_u^\star + \mathcal{D}_6 S_v^\star. \quad (7.172)$$

The Uzawa operator  $\mathcal{U}_2 = \mathcal{D}_6 \mathcal{D}_5 - k^2 \mathcal{I}_3 \mathcal{I}_1$  has the same properties as  $\mathcal{U}_1$ . When  $P$  is calculated, Eq.(7.169) and (7.170) give, respectively,  $U_\star$  and  $V_\star$ .

*Option (3).* The system is made with Eqs.(7.161) and (7.162) for  $j = 0, \dots, N$ , Eq.(7.163) for  $j = 1, \dots, N-1$ , and Eq.(7.164). Therefore, the unknowns vectors are  $U$ ,  $V_\star$ , and  $P$ , that are determined by the equations

$$\sigma U - k P = S_u \quad (7.173)$$

$$\sigma \mathcal{I}_3 V_\star + \mathcal{D} P = S_v' \quad (7.174)$$

$$k \mathcal{I}_1 U + \mathcal{D}_1 V_\star = S_Q^\star, \quad (7.175)$$

where  $\mathcal{D}$  is the  $(N+1) \times (N+1)$  matrix defined by Eq.(3.51) and  $\mathcal{D}_1$  is the  $(N-1) \times (N-1)$  matrix defined by Eq.(4.66). The vector  $S_v'$  is defined by

$$S_v' = (-\sigma g_+ + \varphi_{v,0}, \varphi_{v,1}, \dots, \varphi_{v,N-1}, -\sigma g_- + \varphi_{v,N})^T$$

and  $S_Q^\star = \mathcal{I}_1 S_Q$ .

To derive the system determining  $P$  we have to make a combination of Eqs.(7.173) and (7.174) in order to exhibit the left-hand side of Eq.(7.175). This technique is similar to the one developed previously for the first two options, but it has to take into account the fact that the incompressibility equation (7.175) contains only  $N-1$  equations. Thus, Eq.(7.173) is multiplied by  $k \mathcal{I}_1$  and Eq.(7.174) is multiplied by  $\mathcal{D}_5$ , in order to extract

$(N - 1)$  equations from the  $(N + 1)$ -equation systems (7.173) and (7.174), respectively,

$$\sigma k \mathcal{I}_1 U - k \mathcal{I}_1 P = k \mathcal{I}_1 S_u = k S_u^*$$

$$\sigma \mathcal{D}_1 V_\star + \mathcal{D}_4 P = \mathcal{D}_5 S'_v,$$

where we have taken into account that  $\mathcal{D}_5 \mathcal{I}_3 = \mathcal{D}_1$  and  $\mathcal{D}_5 \mathcal{D} = \mathcal{D}_4$  with  $\mathcal{D}_4$  defined by Eq.(6.36). Now, by adding these two equations and using Eq.(7.175), we get the system

$$(\mathcal{D}_4 - k^2 \mathcal{I}_1) P = -\sigma S_Q^* + k S_u^* + \mathcal{D}_5 S'_v \equiv S_P. \quad (7.176)$$

This system contains  $(N - 1)$  equations. It is completed with the first and last equations of the system (7.174), that is, Eq.(7.162) for  $j = 0$  and  $j = N$ , so that

$$(\mathcal{D}_7 - k^2 \mathcal{I}_4) P = S'_P, \quad (7.177)$$

where the  $(N + 1) \times (N + 1)$  matrix  $\mathcal{D}_7$  has been defined in Eq.(7.59). The  $(N + 1)$ -element vector  $S'_P$  is such that

$$S'_P = (\varphi_{v,0}, S_P^T, \varphi_{v,N})^T$$

and  $\mathcal{I}_4$  is the  $(N + 1) \times (N + 1)$  matrix deduced from the identity matrix  $\mathcal{I}_0$  by setting to zero the first and last elements of the main diagonal. When  $P$  is known,  $U$  and  $V_\star$  are calculated from Eqs.(7.173) and (7.174), respectively.

It is easy to see that Eq.(7.177) represents the discrete approximation to a Helmholtz equation with Neumann boundary conditions. Obviously, this is nothing but the discrete counterpart of the projection method commonly presented in continuous form. To be more precise, let us consider the differential problem (7.134)-(7.138). The splitting scheme based on the BDI2 scheme for the predictor step leads to the following equations :

*Step 1*

$$\tilde{u}'' - (\sigma_1 + k^2) \tilde{u} = \tilde{f}_u, \quad -1 < y < 1$$

$$\tilde{v}'' - (\sigma_1 + k^2) \tilde{v} = \tilde{f}_v, \quad -1 < y < 1$$

$$\tilde{u}(\pm 1) = \gamma_\pm, \quad \tilde{v}(\pm 1) = g_\pm.$$

*Step 2*

$$\sigma u - k q = \sigma \tilde{u}, \quad -1 < y < 1$$

$$\sigma v + q' = \sigma \tilde{v}, \quad -1 < y < 1$$

$$k u + v' = 0, \quad -1 < y < 1$$

$$v(\pm 1) = g_\pm,$$

where  $\tilde{u} = \tilde{u}^{n+1}$ ,  $\tilde{v} = \tilde{v}^{n+1}$ ,  $u = u^{n+1}$ ,  $v = v^{n+1}$ ,  $q = p^{n+1} - p^n$ ,  $\sigma = 2/(3 \Delta t)$  and  $\sigma_1 = \sigma/\nu$ . Now, considering Step 2, we obtain the equation

for  $q$  by adding the first equation multiplied by  $k$  and the second one after differentiation, namely,

$$\sigma(ku + v') + (q'' - k^2 q) = \sigma(k\tilde{u} + \tilde{v}') .$$

Then, taking into account the incompressibility equation, we finally obtain

$$q'' - k^2 q = \sigma(k\tilde{u} + \tilde{v}') , \quad -1 < y < 1 , \quad (7.178)$$

to be solved with the boundary conditions deduced from the second equation evaluated at  $y = \pm 1$ , namely,

$$q'(\pm 1) = \sigma[\tilde{v}(\pm 1) - v(\pm 1)] = 0 . \quad (7.179)$$

When  $q$  is calculated, the two first equations of the above system give, respectively,  $u$  and  $v$ .

The Chebyshev collocation solution to the above Neumann problem for  $q$  coincides with the system (7.177) with only the (formal) difference of unknown :  $q$  instead of  $p^{n+1}$ . For  $k = 0$ , Eq.(7.178) becomes a Poisson equation and it is easy to verify that the compatibility condition is automatically satisfied since the Poisson equation and the Neumann condition come both from the same set of differential equations. It was obviously the same in the discrete presentation of the projection method described previously.

*Option (4).* The system is constituted by Eq.(7.161) for  $j = 1, \dots, N-1$ , Eq.(7.162) for  $j = 0, \dots, N$ , Eq.(7.163) for  $j = 1, \dots, N-1$ , Eq.(7.164), and the boundary conditions  $u_0 = \gamma_+$  and  $u_N = \gamma_-$ . The unknown vectors are  $U_*$ ,  $V_*$ , and  $P$ . They are determined by the system

$$\sigma U_* - k \mathcal{I}_1 P = S_u^* \quad (7.180)$$

$$\sigma \mathcal{I}_3 V_* + \mathcal{D} P = S_v' \quad (7.181)$$

$$k U_* + \mathcal{D}_1 V_* = S_Q^* . \quad (7.182)$$

Proceeding as in Option (3), we again get the system (7.177) for determining  $P$ . Then Eqs.(7.180) and (7.181) give  $U_*$  and  $V_*$ , respectively. The only difference with Option (3) lies in the determination of  $u_j^{n+1}$  at the boundaries.

### 7.3.3 Navier-Stokes equations

In this section, we consider the solution of the Navier-Stokes equations (7.1)-(7.2) in the domain  $\{0 \leq x \leq 2\pi, -1 \leq y \leq 1\}$ , assuming the solution to be  $2\pi$ -periodic in  $x$  and verifying the boundary conditions (7.18) and the initial condition (7.17). The difference with the time-dependent

Stokes equations considered in the previous section lies in the presence of the nonlinear convective term  $\mathbf{A}(\mathbf{V}, \mathbf{V})$ . Generally, this term is treated explicitly in order to avoid costly iterative procedures. However, it may happen, in some circumstances, that such an implicit treatment induces too strong a constraint on the size of the time-step. In this case, it may be preferable to consider an implicit treatment of the term  $\mathbf{A}(\mathbf{V}, \mathbf{V})$ . These two types of time-discretization will now be addressed, in a way similar to that developed in Chapter 6 for the vorticity-streamfunction equations.

**(a) Semi-implicit scheme**

Whatever the time-discretization scheme considered in Section 7.3.2, the term  $\mathbf{A}(\mathbf{V}, \mathbf{V})$  is evaluated in an explicit way through the Adams-Bashforth extrapolation. Thus, for the three-level BDI2 scheme defined by  $\varepsilon = 2$  and  $\theta = 1$ , the approximation to  $\mathbf{A}(\mathbf{V}, \mathbf{V})$  is

$$\mathbf{A}^{n+1} \cong 2\mathbf{A}^n - \mathbf{A}^{n-1}, \quad (7.183)$$

where  $\mathbf{A}^n = \mathbf{A}(\mathbf{V}^n, \mathbf{V}^n)$ . This term is included either in Eqs.(7.29)-(7.30) or in Eq.(7.128) according to the considered time-scheme. The evaluation of  $\mathbf{A}^n$  follows the development in Section 6.3.3 for the analogous term  $B^n$ , making use of the pseudospectral technique (Sections 2.8, 3.5.2, and 6.5.3) for the calculation of the Fourier coefficients of the various products at the collocation points  $y_j$ . The influence of the form of the nonlinear term  $\mathbf{A}(\mathbf{V}, \mathbf{V})$  on the stability will be analyzed in Section 7.4.5.

For some types of flows, the convective term  $\mathbf{A}$  can be treated partly implicitly, while avoiding the need for an iterative solution procedure, in a way similar to that mentioned for the Burgers equation in Section 3.5.2. More precisely, let us assume the existence of a constant velocity  $\mathbf{U}$  such that  $\mathbf{V} = \mathbf{U} + \mathbf{W}$  where  $\mathbf{W}$  is sufficiently “small” with respect to  $\mathbf{U}$ . The unknown vector in Eqs.(7.29)-(7.32) is then  $\mathbf{W}^{n+1}$  and the approximation (7.183) is replaced by

$$\begin{aligned} \mathbf{A}^{n+1} &= (\mathbf{V}^{n+1} \cdot \nabla) \mathbf{V}^{n+1} \\ &\cong (\mathbf{U} \cdot \nabla) \mathbf{W}^{n+1} + 2(\mathbf{W}^n \cdot \nabla) \mathbf{W}^n - (\mathbf{W}^{n-1} \cdot \nabla) \mathbf{W}^{n-1}. \end{aligned}$$

Of course, such a decomposition also applies to the splitting scheme (7.122). Therefore, for both time-discretization schemes, we are led to the solution, at each time-cycle, of advection-diffusion-type equations instead of Helmholtz equations. In the multi-dimensional case, the algebraic system resulting from the Chebyshev collocation approximation can be solved by the Schur decomposition technique described in Section 3.7. This semi-implicit time-scheme has been used by Forestier *et al.* (2000a,b) for the calculation of two- and three-dimensional wake flows.

**(b) Fully implicit scheme**

The fully implicit BDI2 scheme applied to the Navier-Stokes equations (7.1)-(7.2) leads to the following nonlinear problem, to be solved at each time-cycle

$$L(\mathbf{V}^{n+1}) + \mathbf{A}(\mathbf{V}^{n+1}, \mathbf{V}^{n+1}) + \nabla p^{n+1} = \mathbf{G}^{n+1} \quad \text{in } \Omega \quad (7.184)$$

$$\nabla \cdot \mathbf{V}^{n+1} = 0 \quad \text{in } \Omega \quad (7.185)$$

$$\mathbf{V}^{n+1}(x, \pm 1) = (\mathbf{V}_\Gamma^\pm)^{n+1}, \quad (7.186)$$

where  $L$  is the linear operator defined by

$$L(\mathbf{V}^{n+1}) = (\sigma I - \nu \nabla^2) \mathbf{V}^{n+1} \quad (7.187)$$

with  $\sigma = 3/(2\Delta t)$ . The forcing term  $\mathbf{G}^{n+1}$  is defined by

$$\mathbf{G}^{n+1} = \mathbf{f}^{n+1} - (4\mathbf{V}^n - \mathbf{V}^{n-1}) / (2\Delta t).$$

The nonlinear problem (7.184)-(7.186) is solved iteratively by using, for example, the preconditioned relaxation procedure (characterized by the superscript  $m$ ) considered by Fröhlich *et al.* (1991) :

$$L(\bar{\mathbf{V}}^{m+1}) + \nabla p^{n+1,m+1} = \mathbf{G}^{n+1} - \mathbf{A}(\mathbf{V}^{n+1,m}, \mathbf{V}^{n+1,m}) - L(\mathbf{V}^{n+1,m}) \quad \text{in } \Omega \quad (7.188)$$

$$\nabla \cdot \bar{\mathbf{V}}^{m+1} = 0 \quad \text{in } \Omega \quad (7.189)$$

$$\bar{\mathbf{V}}^{m+1}(x, \pm 1) = (\mathbf{V}_\Gamma^\pm)^{n+1} - \mathbf{V}^{n+1,m}(x, \pm 1), \quad (7.190)$$

and

$$\mathbf{V}^{n+1,m+1} = \mathbf{V}^{n+1,m} + \alpha \bar{\mathbf{V}}^{m+1}. \quad (7.191)$$

Therefore, at each iteration, we are faced with the solution of a Stokes problem. This is done by using one of the Chebyshev collocation (Uzawa or influence matrix) methods described in Sections 7.3.3.b and 7.3.3.c. The iterative procedure is started with

$$\mathbf{V}^{n+1,0} = \mathbf{V}^n.$$

The relaxation parameter  $\alpha$  was taken to be constant by Fröhlich *et al.* (1991), but improvement of the convergence could be obtained by using a dynamic procedure (at the price of the calculation of norms) as indicated in Section 3.8. The efficiency of the method has been estimated by Fröhlich *et al.* (1991) for the steady Rayleigh-Bénard problem considered in Section 6.5.1. The spatial approximation makes use of the Chebyshev collocation method based on a staggered mesh. The unknowns are the values of the velocity at the Gauss-Lobatto points. The momentum equation and incompressibility equation are enforced at, respectively, the Gauss-Lobatto points

and at the Gauss points. This type of approximation leads to an Uzawa equation determining the pressure in a way similar to the one described in Section 7.3.2.b. The Uzawa operator is inverted once and for all in a preprocessing stage. The conclusions concerning the properties of the iterative method, in comparison with the semi-implicit AB/BDI2 method, are quite similar to the ones drawn for the vorticity-streamfunction equations in Section 6.5.1.a.

In the case of the splitting method discussed in Section 7.3.3.d, the non-linear term  $\mathbf{A}(\tilde{\mathbf{V}}^{n+1}, \tilde{\mathbf{V}}^{n+1})$  is included in the first step of the scheme [Eq.(7.128) with  $\varepsilon = 2$ ,  $\theta = 1$ ]. The second step remains unchanged. Thus, the iterative procedure has to be applied to the calculation of the provisional velocity  $\tilde{\mathbf{V}}^{n+1}$  only, that is, to the solution of two-dimensional Burgers equations. This can be done by using a preconditioned relaxation procedure similar to the one described above.

## 7.4 Chebyshev-Chebyshev method

In this section we address the solution of the two-dimensional Navier-Stokes equations (7.1)-(7.2) in a square domain using a Chebyshev collocation approximation in both directions. The time-discretization schemes are similar to those discussed in the preceding section, except that higher-order accuracy is considered, thanks to the multistep AB/BDIk scheme.

First we consider time-schemes leading to the Stokes problem, and we discuss its solution using the influence matrix method. Then we consider the time-splitting schemes of projection-type leading to the solution of the Darcy problem. For this solution, we describe two methods, each of which corresponds to a special type of spatial approximation, namely,  $\mathcal{P}_N - \mathcal{P}_N$  and  $\mathcal{P}_N - \mathcal{P}_{N-2}$  approximations.

Every method discussed in this section has been applied to the computation of complex, unsteady, two- or three-dimensional flows. An application of the  $\mathcal{P}_N - \mathcal{P}_N$  projection method is described in Section 7.5. Examples of the application of the  $\mathcal{P}_N - \mathcal{P}_{N-2}$  projection method and the influence matrix method are given in Sections 8.4.2.b and 9.6.3, respectively.

The methods are described in the two-dimensional case only for the sake of simplicity. There is no fundamental obstacle to their three-dimensional extension. The use of the influence matrix method is, however, restricted to two Chebyshev directions (and, possibly, periodic in the third one) because of the huge size of the influence matrix whose order is equal to the number of collocation points belonging to the boundary.

Therefore, the Navier-Stokes equations (7.1)-(7.3) are solved in the domain  $\Omega = (-1, 1)^2$ , with the boundary condition

$$\mathbf{V} = \mathbf{V}_\Gamma \quad \text{on } \Gamma = \partial\Omega \quad (7.192)$$



and the initial condition

$$\mathbf{V} = \mathbf{V}_0 = (u_0, v_0) \quad \text{at } t = 0, \quad (7.193)$$

where  $\mathbf{V}_\Gamma$  and  $\mathbf{V}_0$  are assumed to satisfy the compatibility and regularity conditions mentioned in Section 5.1. In particular,  $\mathbf{V}_\Gamma$  is assumed to be regular enough to ensure the solution to be sufficiently differentiable.

#### 7.4.1 The influence matrix method

##### (a) Time-discretization

The time-discretization scheme makes use of the finite-difference multistep AB/BDIk scheme described in Section 4.5.1, namely

$$\frac{1}{\Delta t} \sum_{j=0}^k a_j \mathbf{V}^{n+1-j} + \sum_{j=0}^{k-1} b_j \mathbf{A}^{n-j} + \nabla p^{n+1} - \nu \nabla^2 \mathbf{V}^{n+1} = \mathbf{f}^{n+1} \quad (7.194)$$

$$\nabla \cdot \mathbf{V}^{n+1} = 0, \quad (7.195)$$

where  $\mathbf{A}^{n-j} = \mathbf{A}(\mathbf{V}^{n-j}, \mathbf{V}^{n-j})$ . The coefficients  $a_j$  and  $b_j$  are given in Table 4.4 for schemes of order  $2 \leq k \leq 4$ . Therefore, at each time-cycle, the following Stokes problem has to be solved

$$\nabla^2 \mathbf{V} - \sigma \mathbf{V} - \nabla p = \mathbf{F} \quad \text{in } \Omega \quad (7.196)$$

$$\nabla \cdot \mathbf{V} = 0 \quad \text{in } \Omega \quad (7.197)$$

$$\mathbf{V} = \mathbf{V}_\Gamma \quad \text{on } \Gamma, \quad (7.198)$$

where

$$\mathbf{V} = \mathbf{V}^{n+1}, \quad p = p^{n+1}/\nu, \quad \sigma = a_0/(\nu \Delta t),$$

$$\mathbf{F} = -\mathbf{f}^{n+1} + \frac{1}{\Delta t} \sum_{j=1}^k a_j \mathbf{V}^{n+1-j} + \sum_{j=0}^{k-1} b_j \mathbf{A}^{n-j}.$$

The question concerning the calculation of the first time-cycles with a multistep scheme like (7.194) is discussed in Section 4.5.1.c.

##### (b) Solution of the Stokes problem

The influence matrix method for solving the one-dimensional Stokes problem has largely been discussed in Section 7.3.2.c2. Therefore, we restrict ourselves here to presenting the method and to discussing the peculiar points characteristic of the two-dimensional configuration.

To begin with, the Poisson equation satisfied by the pressure is derived by applying the divergence operator to Eq.(7.196) and taking Eq.(7.197) into account, we obtain

$$\nabla^2 p = -\nabla \cdot \mathbf{F}.$$

This implies that  $Q = \nabla \cdot \mathbf{V}$  satisfies the Helmholtz equation

$$\nabla^2 Q - \sigma Q = 0, \quad (7.199)$$

so that  $Q = 0$  everywhere if  $Q = 0$  on the boundary  $\Gamma$ , which gives the missing boundary conditions. Therefore, the problem (7.196)-(7.198) is replaced by

$$\nabla^2 p = -\nabla \cdot \mathbf{F} \quad \text{in } \Omega \quad (7.200)$$

$$H \mathbf{V} - \nabla p = \mathbf{F} \quad \text{in } \Omega \quad (7.201)$$

$$\mathbf{V} = \mathbf{V}_\Gamma \quad \text{on } \Gamma \quad (7.202)$$

$$\nabla \cdot \mathbf{V} = 0 \quad \text{on } \Gamma, \quad (7.203)$$

where  $H$  is the Helmholtz operator

$$H = \nabla^2 - \sigma I. \quad (7.204)$$

The problem (7.200)-(7.203) is approximated by the Chebyshev collocation method based on the Gauss-Lobatto points

$$x_i = \cos \frac{\pi i}{N_x}, \quad i = 0, \dots, N_x, \quad y_j = \cos \frac{\pi j}{N_y}, \quad j = 0, \dots, N_y.$$

We denote by  $\Omega_N$  the set of collocation points in the interior of  $\Omega$ ,

$$\Omega_N = \{x_i, y_j; i = 1, \dots, N_x - 1, j = 1, \dots, N_y - 1\},$$

by  $\overline{\Omega}_N$ , the set of all collocation points,

$$\overline{\Omega}_N = \{x_i, y_j; i = 0, \dots, N_x, j = 0, \dots, N_y\},$$

and by  $\overline{\Omega}_N^I$  the set  $\overline{\Omega}_N$  minus the four corners. We denote by  $\Gamma_N$  the set of collocation points belonging to the boundary  $\Gamma$ , and by  $\Gamma_N^I$  this same set minus the four corners. Finally, we recall that  $\mathcal{P}_N$  represents the space of polynomials of degree at most equal to  $N_x$  in  $x$  and to  $N_y$  in  $y$ . Therefore, the solution  $[\mathbf{V} = (u, v), p]$  of problem (7.200)-(7.203) is approximated by  $[\mathbf{V}_N = (u_N, v_N), p_N]$ , where  $u_N$ ,  $v_N$ , and  $p_N$  belong to  $\mathcal{P}_N$ . Such an approximation, where the velocity components and the pressure are in the same polynomial space, is denoted by " $\mathcal{P}_N - \mathcal{P}_N$ ".

The Chebyshev collocation approximation to the problem (7.200)-(7.203) is

$$\nabla^2 p_N = -\nabla \cdot \mathbf{F}_N \quad \text{in } \Omega_N \quad (7.205)$$

$$H \mathbf{V}_N - \nabla p_N = \mathbf{F}_N \quad \text{in } \Omega_N \quad (7.206)$$

$$\mathbf{V}_N = \mathbf{V}_\Gamma \quad \text{on } \Gamma_N^I \quad (7.207)$$

$$\nabla \cdot \mathbf{V}_N = 0 \quad \text{on } \Gamma_N^I. \quad (7.208)$$

The various derivatives occurring in these equations are expressed by formulas of type (3.45). Thus, Eqs. (7.205)-(7.206) constitute a system of  $[3(N_x - 1)(N_y - 1) + 4(N_x + N_y - 2)]$  scalar equations for the same number of scalar unknowns, namely,  $(N_x - 1)(N_y - 1)$  for each velocity component  $u_N(x_i, y_j)$  and  $v_N(x_i, y_j)$ ,  $(x_i, y_j) \in \Omega_N$ , and  $[N_x N_y + (N_x + N_y - 3)]$  values of the pressure  $p_N(x_i, y_j)$ ,  $(x_i, y_j) \in \Omega_N^I$ . Note that the forcing term  $\mathbf{F}_N$  is evaluated either by matrix products or by the pseudospectral technique (Section 3.5.2).

As shown in Section 7.3.2.c, the velocity field  $\mathbf{V}_N$  defined by (7.205)-(7.208) is not divergence-free because the momentum equation normal to the boundary is not satisfied. Therefore, it is necessary to introduce a correction term  $\nabla \cdot \mathbf{B}$  into the pressure equation in order to recover an exactly divergence-free velocity field. The correction vector  $\mathbf{B}$  (whose components are polynomials belonging to  $\mathcal{P}_N$ ) is zero in  $\Omega_N$  and its normal component  $B_n$  at the boundary  $\Gamma_N^I$  is the residual of the momentum equation normal to the boundary.

Therefore, we have to solve the  $\mathcal{P}$ -Problem defined by the algebraic system

$$\nabla^2 p_N = -\nabla \cdot \mathbf{F}_N - \nabla \cdot \mathbf{B} \quad \text{in } \Omega_N \quad (7.209)$$

$$H \mathbf{V}_N - \nabla p_N = \mathbf{F}_N \quad \text{in } \Omega_N \quad (7.210)$$

$$\mathbf{V}_N = \mathbf{V}_\Gamma \quad \text{on } \Gamma_N^I \quad (7.211)$$

$$B_n = (H \mathbf{V}_N - \nabla p_N - \mathbf{F}_N) \cdot \mathbf{n} \quad \text{on } \Gamma_N^I, \quad (7.212)$$

where  $\mathbf{n}$  is the unit normal vector to  $\Gamma$  and  $B_n = \mathbf{B} \cdot \mathbf{n}$ . As noted above, the vector  $\mathbf{B}$  vanishes in  $\Omega_N$ , namely,

$$\mathbf{B} = \mathbf{0} \quad \text{in } \Omega_N. \quad (7.213)$$

Note that  $\mathbf{B}_t$ , the component of  $\mathbf{B}$  tangential to  $\Gamma$ , is not involved in the algorithm since  $\mathbf{B}$  appears in Eq.(7.209) only through its divergence.

Now the task is to solve the  $\mathcal{P}$ -problem in an efficient way. This is done through the influence matrix technique. As usual, we pose

$$p_N = \tilde{p}_N + \bar{p}_N, \quad \mathbf{V}_N = \tilde{\mathbf{V}}_N + \bar{\mathbf{V}}_N. \quad (7.214)$$

The grid values of the part  $(\tilde{p}_N, \tilde{\mathbf{V}}_N)$  are determined by the discrete  $\tilde{\mathcal{P}}$ -Problem constituted by

$$\begin{aligned} \nabla^2 \tilde{p}_N &= -\nabla \cdot \mathbf{F}_N & \text{in } \Omega_N \\ \tilde{p}_N &= 0 & \text{on } \Gamma_N^I, \end{aligned}$$

and by

$$\begin{aligned} H \tilde{\mathbf{V}}_N &= \mathbf{F}_N + \nabla \tilde{p}_N & \text{in } \Omega_N \\ \tilde{\mathbf{V}}_N &= \mathbf{V}_\Gamma & \text{on } \Gamma_N^I. \end{aligned}$$

The solution of these systems gives, successively,  $\tilde{p}_N$  and  $\tilde{\mathbf{V}}_N$ , from which we deduce

$$\tilde{B}_n = \left( H \tilde{\mathbf{V}}_N - \nabla \tilde{p}_N - \mathbf{F}_N \right) \cdot \mathbf{n} \quad \text{on } \Gamma_N^I,$$

where  $\tilde{B}_n$  is the normal component of vector  $\tilde{\mathbf{B}}$  such that

$$\tilde{\mathbf{B}} = \mathbf{0} \quad \text{in } \Omega_N$$

and  $\tilde{\mathbf{B}}_t = 0$  on  $\Gamma_N^I$ . Then the grid values of the part  $(\tilde{p}_N, \tilde{\mathbf{V}}_N)$  are a solution to the discrete  $\tilde{\mathcal{P}}$ -Problem :

$$\nabla^2 \tilde{p}_N = -\nabla \cdot \tilde{\mathbf{B}} \quad \text{in } \Omega_N \quad (7.215)$$

$$H \tilde{\mathbf{V}}_N - \nabla \tilde{p}_N = \mathbf{0} \quad \text{in } \Omega_N \quad (7.216)$$

$$\tilde{\mathbf{V}}_N = \mathbf{0} \quad \text{on } \Gamma_N^I \quad (7.217)$$

$$\nabla \cdot \tilde{\mathbf{V}}_N = -\nabla \cdot \tilde{\mathbf{V}}_N \quad \text{on } \Gamma_N^I, \quad (7.218)$$

and

$$B_n = \tilde{B}_n + \bar{B}_n \quad \text{on } \Gamma_N^I, \quad (7.219)$$

where

$$\bar{B}_n = \left( H \bar{\mathbf{V}}_N - \nabla \bar{p}_N \right) \cdot \mathbf{n} \quad \text{on } \Gamma_N^I.$$

As in the one-dimensional case, the  $\tilde{\mathcal{P}}$ -problem is solved through the superposition of elementary solutions according to

$$\phi = \sum_{l=1}^{2L} \xi_l \phi_l, \quad (7.220)$$

where  $\phi = \bar{p}_N, \bar{\mathbf{V}}_N, \bar{B}_n, \mathbf{B}$ . The integer  $L$  is the number of collocation points on  $\Gamma_N^I$ , namely  $L = 2(N_x + N_y - 2)$  and  $\xi_l, l = 1, \dots, 2L$ , are constants. The elementary solutions  $\phi_l$  are obtained from a sequence of  $\tilde{\mathcal{P}}_l$ -Problems,  $l = 1, \dots, 2L$ , defined as follows:

$\tilde{\mathcal{P}}_l$ -Problem,  $l = 1, \dots, L$  :

$$\nabla^2 \bar{p}_l = 0 \quad \text{in } \Omega_N$$

$$\bar{p}_l|_{\eta_m} = \delta_{m,l}, \quad \eta_m \in \Gamma_N^I,$$

and

$$H \bar{\mathbf{V}}_l = \nabla \bar{p}_l \quad \text{in } \Omega_N$$

$$\bar{\mathbf{V}}_l = \mathbf{0} \quad \text{on } \Gamma_N^I,$$

and

$$\mathbf{B}_l = \mathbf{0} \quad \text{in } \bar{\Omega}_N,$$

where  $\eta_m$ ,  $m = 1, \dots, L$ , refers to the collocation points on  $\Gamma_N^I$ . Again, when  $\bar{p}_l$  and  $\bar{\mathbf{V}}_l$  are determined, we calculate the residual  $\bar{B}_{ln}$  of the normal momentum equation at the boundary

$$\bar{B}_{ln} = (H \bar{\mathbf{V}}_l - \nabla \bar{p}_l) \cdot \mathbf{n} \quad \text{on } \Gamma_N^I. \quad (7.221)$$

$\bar{\mathcal{P}}_l$ -Problem,  $l = L + 1, \dots, 2L$ :

$$\begin{aligned} \nabla^2 \bar{p}_l &= -\nabla \cdot \mathbf{B}_l && \text{in } \Omega_N \\ \bar{p}_l &= 0 && \text{on } \Gamma_N^I \\ B_{ln}|_{\eta_m} &= \delta_{L+m,l}, && \eta_m \in \Gamma_N^I. \end{aligned}$$

Since

$$\mathbf{B}_l = \mathbf{0} \quad \text{in } \Omega_N$$

the above system completely determines  $\bar{p}_l$  in  $\bar{\Omega}_N^I$ . Then,

$$\begin{aligned} H \bar{\mathbf{V}}_l &= \nabla \bar{p}_l && \text{in } \Omega_N, \\ \bar{\mathbf{V}}_l &= \mathbf{0} && \text{on } \Gamma_N^I, \end{aligned}$$

and the residual  $\bar{B}_{ln}$  on  $\Gamma_N^I$  is again determined by Eq.(7.221).

It is not complicated to verify that the combination (7.220) satisfies Eqs.(7.215)-(7.217) of the  $\bar{\mathcal{P}}$ -problem, whatever the constants  $\xi_l$ ,  $l = 1, \dots, 2L$ . These constants are then determined by prescribing the combination (7.220) to also satisfy Eqs.(7.218) and (7.219), namely,

$$\begin{aligned} \sum_{l=1}^{2L} \xi_l (\nabla \cdot \bar{\mathbf{V}}_l)_{\eta_m} &= -(\nabla \cdot \tilde{\mathbf{V}}_l)_{\eta_m} && \eta_m \in \Gamma_N^I \\ \sum_{l=1}^{2L} \xi_l (B_{ln} - \bar{B}_{ln})_{\eta_m} &= \tilde{B}_n|_{\eta_m} && \eta_m \in \Gamma_N^I, \end{aligned}$$

or, in vector form,

$$\mathcal{M} \Xi = \tilde{E} \quad (7.222)$$

with  $\Xi = (\xi_1, \dots, \xi_{2L})^T$ . The matrix  $\mathcal{M}$  is the influence matrix. This matrix has four eigenvalues equal to zero. One of them has a physical meaning: it is associated to the property of the pressure field to be defined up to a constant. The other null eigenvalues are associated to the spurious pressure modes, which will be characterized later. Therefore, the solution  $\Xi$  of Eq.(7.222) is not determined in a unique way. It suffices, however, to have one solution  $(\xi_1, \dots, \xi_{2L})$  which defines a set of boundary values for  $p_N$  and  $B_n$  ensuring that Eqs.(7.215)-(7.219) are satisfied. The procedure proposed by Tuckerman (1989) consists of replacing the matrix  $\mathcal{M}$  in

Eq.(7.222) by a matrix  $\mathcal{M}_0$  defined as follows : first, calculate the eigenvalues  $\lambda_j$ ,  $j = 1, \dots, 2L$ , of the matrix  $\mathcal{M}$  and the associated eigenvectors, and let  $\mathcal{P}$  be the matrix made with these eigenvalues ; then construct the matrix

$$\mathcal{M}_0 = \mathcal{P} \Lambda_0 \mathcal{P}^{-1},$$

where  $\Lambda_0$  is the diagonal matrix made with the eigenvalues  $\lambda_j$  except that the null eigenvalues are replaced by any nonzero value  $\gamma$ , for example,  $\gamma = 1$ . Then  $\mathcal{M}_0$  can be inverted, so that

$$\Xi = \mathcal{M}_0^{-1} \tilde{E}. \quad (7.223)$$

Note that the matrix  $\mathcal{M}_0$  acts like  $\mathcal{M}$  except on its null space, on which it acts like the identity. It may be observed that this regularization procedure is similar to the one proposed in Section 3.7.1 for the solution of the Neumann problem for the Poisson equation using the matrix-diagonalization method. In this case there exists one null eigenvalue. The technique described, which consists of setting to zero the component of  $\tilde{V}$  corresponding to the null eigenvalue, would correspond to the special choice  $\gamma = \infty$ . Now,  $\Xi$  having been determined, it is easy to reconstruct the solution  $(p_N, \mathbf{V}_N)$  thanks to Eqs.(7.214) and (7.220). However, in order to avoid the storage of the elementary solutions as well as the summations (7.220), it is generally more efficient to calculate the boundary values of  $p_N$  and  $B_n$  and, then, to solve again an algebraic system. More precisely, from the expressions

$$p_N = \tilde{p}_N + \sum_{l=1}^{2L} \xi_l \bar{p}_l, \quad \mathbf{B} = \sum_{l=1}^{2L} \xi_l \mathbf{B}_l$$

and the definition of the boundary values of  $\bar{p}_l$  and  $B_{l_n}$  used in Problems  $\bar{\mathcal{P}}_l$ , we easily get

$$p_N|_{\eta_m} = \xi_m, \quad B_n|_{\eta_m} = \xi_{L+m}, \quad \eta_m \in \Gamma_N^I,$$

and we may calculate  $\nabla \cdot \mathbf{B}$  taking Eq.(7.213) into account. Then the solution  $(p_N, \mathbf{V}_N)$  is calculated by solving, successively, the two systems :

$$\nabla^2 p_N = -\nabla \cdot \mathbf{F}_N - \nabla \cdot \mathbf{B} \quad \text{in } \Omega_N \quad (7.224)$$

$$p_N|_{\eta_m} = \xi_m, \quad \eta_m \in \Gamma_N^I, \quad (7.225)$$

and

$$H \mathbf{V}_N = \mathbf{F}_N - \nabla p_N \quad \text{in } \Omega_N \quad (7.226)$$

$$\mathbf{V}_N = \mathbf{V}_\Gamma \quad \text{on } \Gamma_N^I. \quad (7.227)$$

The various algebraic systems are solved by means of the matrix-diagonalization technique described in Section 3.7.2. The  $\bar{\mathcal{P}}_l$ -problems are time-independent and can be solved in the preprocessing stage performed before

starting the time-integration. So, the matrix  $\mathcal{M}$  is calculated (using its symmetry properties) during this stage, and the matrix  $\mathcal{M}_0$  is calculated, inverted, and its inverse is stored. Generally, the elementary solutions are not stored. Therefore the algorithm is the following :

*A. Preprocessing stage*

A.1. Solve the  $\tilde{\mathcal{P}}_l$ -problems,  $l = 1, \dots, 2L$ .

A.2. Construct the matrix  $\mathcal{M}_0$ . Invert  $\mathcal{M}_0$  and store  $\mathcal{M}_0^{-1}$ .

*B. At each time-cycle*

B.1. Calculate the right-hand side  $\mathbf{F}_N$  in Eq.(7.200) by matrix products or by the pseudo-spectral technique (FFT).

B.2. Solve the  $\tilde{\mathcal{P}}$ -problem, calculate  $\tilde{E}$ .

B.3. Calculate  $\Xi$  by Eq.(7.223).

B.4. Calculate the final solution  $(p_N, \mathbf{V}_N)$  by solving Eqs.(7.224)-(7.227).

**Remarks**

To conclude the discussion, we make two remarks. The first one concerns the characterization and removal of the spurious pressure modes and the second remark is relative to the compatibility of the boundary conditions.

*1. Spurious pressure modes*

As mentioned in Section 7.3, the existence of spurious pressure modes is connected to the fact that some Chebyshev modes do not appear in the collocation at Gauss-Lobatto points. This is discussed now in more detail.

As for the one-dimensional case, the application of the influence matrix method with the correction term  $\mathbf{B}$  amounts to solving the discrete problem

$$\nabla^2 \mathbf{V}_N - \sigma \mathbf{V}_N + \nabla p_N = \mathbf{F}_N \quad \text{in } \Omega_N \quad (7.228)$$

$$\nabla \cdot \mathbf{V}_N = 0 \quad \text{in } \overline{\Omega}_N^I \quad (7.229)$$

$$\mathbf{V}_N = \mathbf{V}_\Gamma \quad \text{on } \Gamma_N^I. \quad (7.230)$$

Therefore the pressure appears only through its gradient evaluated at the inner collocation points. The polynomial  $p_N(x, y)$  can be expressed as

$$p_N(x, y) = \sum_{k=0}^{N_x} \sum_{l=0}^{N_y} \hat{p}_{k,l} T_k(x) T_l(y). \quad (7.231)$$

Now it is not difficult to detect the modes which are not involved in the gradient  $\nabla p_N$  evaluated at points belonging to  $\Omega_N$ , namely, those for which  $\nabla p_N(x_i, y_j) = 0$  for  $(x_i, y_j) \in \Omega_N$ . Thus, we find :

- (i) the constant mode  $T_0(x) T_0(y) = 1$ ,
- (ii) the line mode  $T_{N_x}(y) T_0(y) = T_{N_x}(x)$ ,
- (iii) the column mode  $T_0(x) T_{N_y}(y) = T_{N_y}(y)$ , and
- (iv) the checkboard mode  $T_{N_x}(x) T_{N_y}(y)$ .

Therefore, the Chebyshev coefficients  $\hat{p}_{0,0}$ ,  $\hat{p}_{N_x,0}$ ,  $\hat{p}_{0,N_y}$  and  $\hat{p}_{N_x,N_y}$  do not occur in the collocation approximation to the problem (7.228)-(7.230). This explains why some eigenvalues of  $\mathcal{M}$  are zero. The constant mode has a physical meaning, the others are spurious.

Moreover, since the algebraic system (7.228)-(7.230) does not involve the pressure values at the corners of the computational domain, there exists another set of special modes, the so-called “corner modes” defined by

$$(1 \pm x) (1 \pm y) T'_{N_x}(x) T'_{N_y}(y). \quad (7.232)$$

Each of these modes vanishes in  $\bar{\Omega}_N$  except at one corner. Moreover, the gradient of these modes is zero at any collocation point of  $\bar{\Omega}_N$  except on the two sides of the boundary adjacent to the associated corner. The values of the pressure at the corners cannot be determined by imposing the incompressibility equation at these points since the divergence of the velocity at a corner is completely defined (as zero, like here) by the knowledge of  $\mathbf{V}_\Gamma$ . The corner modes are generally qualified as “spurious,” although their nature and their effect are slightly different from the other ones. More precisely, they are not a possible source of null eigenvalues of the influence matrix or, more generally, of the algebraic system (7.228)-(7.230). The value of the pressure at the corners has no effect on the pressure values of the other collocation points, while the presence of the other spurious modes contaminates the pressure solution in the whole domain. Thus, the knowledge of the corner values of the pressure is unnecessary as long as the formal expression of the polynomial  $p_N(x, y)$  is not of interest. The existence of the spurious pressure modes was detected by Morchoisne (1983) and their theoretical analysis (*inf-sup* conditions) has been done by Bernardi *et al.* (1988).

An equivalent way to detect the spurious modes associated with the influence matrix method (Balachandar and Madabhushi, 1994) is to examine directly the system (7.209)-(7.213). Let us set  $\phi = \phi^I + \phi^{II}$ , where  $\phi = p_N, \mathbf{V}_N, \mathbf{B}$  and  $\phi^I$  represents the solution of (7.209)-(7.213) free of spurious modes. The spurious part  $\phi^{II}$  is such that  $\mathbf{V}_N^{II} = \mathbf{0}$  since these modes have no effect on the velocity. Therefore, the spurious part  $(p_N^{II}, \mathbf{B}^{II})$  is any nonzero solution to the homogeneous system

$$\nabla^2 p_N^{II} + \nabla \cdot \mathbf{B}^{II} = 0 \quad \text{in } \Omega_N$$

$$\nabla p_N^{II} = 0 \quad \text{in } \Omega_N$$

$$\mathbf{B}^{II} = \mathbf{0} \quad \text{in } \Omega_N$$

$$(\nabla p_N^{II} + \mathbf{B}^{II}) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_N^I.$$



A careful examination of the system shows that we have the following four nonzero solutions

$$\begin{aligned} p_N^{II} &= \text{constant}, & \mathbf{B}^{II} &= (0, 0), \\ p_N^{II} &= T_{N_x}(x), & \mathbf{B}^{II} &= (-T'_{N_x}(x), 0), \\ p_N^{II} &= T_{N_y}(y), & \mathbf{B}^{II} &= (0, -T'_{N_y}(y)), \\ p_N^{II} &= T_{N_x}(x) T_{N_y}(y), & \mathbf{B}^{II} &= (-T'_{N_x}(x) T_{N_y}(y), -T_{N_x}(x) T'_{N_y}(y)), \end{aligned}$$

exhibiting the spurious modes.

In the case where the correction term  $\mathbf{B}$  is not considered (the “simplified” influence matrix method), the above system reduces to

$$\nabla^2 p_N^{II} = 0, \quad \nabla p_N^{II} = 0 \quad \text{in } \Omega_N.$$

These equations do not admit a nonzero solution (except the constant) : there are no spurious modes. Of course, in both cases, the corner modes are always present.

It is important to remember that spurious modes do not affect the velocity field. Therefore, their presence is harmless if the pressure field is not of interest. However, in a number of physical problems the knowledge of the pressure is indispensable. In these conditions, it is necessary to recover an accurate pressure field. In the following, we propose two different ways to fulfill this requirement :

- (i) by filtering the spurious modes, or
- (ii) by determining the pressure as the solution of a Poisson equation with a Neumann boundary condition.

First, let us consider the filtering technique. This consists of the following operations :

1. From  $p_N(x_i, y_j)$ ,  $i = 0, \dots, N_x$ ,  $j = 0, \dots, N_y$ , calculate the Chebyshev coefficients  $\hat{p}_{k,l}$ ,  $k = 0, \dots, N_x$ ,  $l = 0, \dots, N_y$ .
2. Set to zero the coefficients  $\hat{p}_{N_x,0}$ ,  $\hat{p}_{0,N_y}$ , and  $\hat{p}_{N_x,N_y}$ .
3. From the filtered spectrum, calculate the new (filtered) pressure field  $\bar{p}_N(x_i, y_j)$ ,  $i = 0, \dots, N_x$ ,  $j = 0, \dots, N_y$ , defined to within a constant.

The implementation of this algorithm necessitates the knowledge of the values of the pressure  $p_N$  at the corners. These values  $p_l^c$ ,  $l = 1, \dots, 4$ , are not given by the collocation solution (corner modes) and, hence, their determination necessitates a special treatment. We describe, here, the technique proposed by Sabbah and Pasquetti (1998) which is based on a least-squares procedure. The pressure  $p_N(x, y)$  is expressed as

$$p_N = p_N^0 + \sum_{l=1}^4 p_l^c C_l, \quad (7.233)$$

where  $p_N^0$  is the calculated pressure field in  $\bar{\Omega}_N^I$  supplemented with the value zero at the corners and  $C_l$ ,  $l = 1, \dots, 4$ , are the polynomials constructed from (7.232) and normalized such that  $C_l = 1$  at the associated corner. Now the field (7.233) is filtered as explained above, to become

$$\bar{p}_N = \bar{p}_N^0 + \sum_{l=1}^4 p_l^c \bar{C}_l. \quad (7.234)$$

Then, with the objective of getting a smooth field  $\bar{p}_N$ , Sabbah and Pasquetti (1998) propose to determine the values  $p_l^c$ ,  $l = 1, \dots, 4$ , from the minimization of the functional

$$J(p_l^c) = \int_{\Omega} |\nabla \bar{p}_N|^2 w \, dx \, dy, \quad (7.235)$$

where  $w = (1 - x^2)^{-1/2} (1 - y^2)^{-1/2}$  is the Chebyshev weight. Thus, by taking Eq.(7.234) into account in Eq.(7.235) and by differentiating  $J$  with respect to  $p_m^c$ ,  $m = 1, \dots, 4$ , we get the least-squares equations

$$\int_{\Omega} \left[ \sum_{l=1}^4 p_l^c \nabla \bar{C}_l \cdot \nabla \bar{C}_m + \nabla \bar{p}_N^0 \cdot \nabla \bar{C}_m \right] w \, dx \, dy = 0, \quad m = 1, \dots, 4.$$

This integral is evaluated by means of the Gauss-Lobatto quadrature [see Eq.(3.14)], so that the system determining  $p_l^c$ ,  $l = 1, \dots, 4$ , is

$$\mathcal{A} P^c = S, \quad (7.236)$$

where  $P^c = (p_1^c, \dots, p_4^c)$ ,  $\mathcal{A}$  is the matrix, and  $S$  the vector whose elements are, respectively,

$$\begin{aligned} a_{m,l} &= \sum_{x_i, y_j} \frac{1}{\bar{c}_i \bar{c}_j} (\nabla \bar{C}_l \cdot \nabla \bar{C}_m)_{i,j}, & (x_i, y_j) \in \Gamma_N^I, \\ s_m &= - \sum_{x_i, y_j} \frac{1}{\bar{c}_i \bar{c}_j} (\nabla p_0^c \cdot \nabla \bar{C}_m)_{i,j}, & (x_i, y_j) \in \Gamma_N^I, \end{aligned}$$

with  $\bar{c}_k$  defined by Eq.(3.16). It must be noticed that the summations are taken on the boundary only, because the gradient  $\nabla \bar{C}_l$  vanishes in  $\Omega_N$ . This property can easily be proven. Let  $\bar{C}_l = C_l - C_l^*$  where  $C_l^*$  is a linear combination of the spurious modes. The result is that  $\nabla \bar{C}_l = \nabla C_l - \nabla C_l^*$  vanishes in  $\Omega_N$  because of the properties of the spurious and corner modes.

The second way to recover the correct pressure field is to define it as the solution of the Neumann problem (5.30)-(5.31) with  $\mathbf{V}$  replaced by the calculated velocity field  $\mathbf{V}_N$  which is exactly solenoidal owing to the application of the influence matrix method. This way is much more accurate than the filtering technique described above (see Section 7.4.4.a), whose

weakness is connected to the necessary determination of the pressure values at the corners. Moreover, the extra-cost in computing time induced by the solution of the Neumann problem remains unimportant since this latter is calculated (as post-processing) only when the pressure is really needed. Finally, it must be remarked that, if needed, the values of the pressure at the corners can be calculated as for any boundary value problem of Robin type (Section 3.7.2). More precisely, let us consider the side  $-1 \leq x \leq 1$ ,  $y = 1$ . The derivatives  $\partial_x p_N$  at the corners  $(-1, 1)$  and  $(1, 1)$  may be taken equal to the derivatives  $\partial_y p_N$  normal to the sides  $x = -1$  and  $x = 1$ , respectively, extended up to the corners. Then, from the first-order differentiation formula (3.45)-(3.46), associated to the known values of  $\partial_x p_N(\pm 1, 1)$  and  $p_N(x_i, 1)$ ,  $i = 1, \dots, N-1$ , we easily deduce the corner values  $p_N(\pm 1, 1)$ .

To close this remark, we mention the comparative study made by Schumack *et al.* (1991) on ways to avoid the spurious modes of pressure in the solution of the Stokes problem. Some of them have already been mentioned in Section 7.3. In the following sections we describe two projection methods exempt of spurious modes, one is based on the  $\mathcal{P}_N - \mathcal{P}_{N-2}$  approximation (Section 7.4.2) and the other makes use of a Neumann boundary condition for the pressure (Section 7.4.3).

## 2. Total flow rate condition

The influence matrix method provides an approximate velocity field whose divergence is the null polynomial. Therefore, consistency requires that the polynomial approximation  $\mathbf{V}_{\Gamma N} = (u_{\Gamma N}, v_{\Gamma N})$  to the boundary velocity  $\mathbf{V}_\Gamma$  satisfies exactly the flow rate condition

$$\int_{\Gamma} (\mathbf{V}_{\Gamma N} \cdot \mathbf{n}) d\Gamma = 0. \quad (7.237)$$

If the spatial resolution is sufficiently high, this condition is satisfied to machine accuracy. If not, the deviation from the zero total flow rate, even as small as  $10^{-6}$ , may induce long-time instabilities. The remedy is to slightly modify  $\mathbf{V}_{\Gamma N}$  on some part of the boundary in order to recover condition (7.237). This can be done in the following way.

Let  $\Gamma_1$  be the part of  $\Gamma$  where  $\mathbf{V}_{\Gamma N}$  is replaced by  $\mathbf{V}_{\Gamma N}^*$  such that

$$\mathbf{V}_{\Gamma N}^* \cdot \mathbf{n} = \begin{cases} \mathbf{V}_{\Gamma N} \cdot \mathbf{n} & \text{at the extremities of } \Gamma_1. \\ \mathbf{V}_{\Gamma N} \cdot \mathbf{n} + \varepsilon & \text{at the inner points of } \Gamma_1, \end{cases}$$

where  $\varepsilon$  is determined by requiring

$$\int_{\Gamma_1} \mathbf{V}_{\Gamma N}^* \cdot \mathbf{n} d\Gamma + \int_{\Gamma \setminus \Gamma_1} \mathbf{V}_{\Gamma N} \cdot \mathbf{n} d\Gamma = 0.$$

Another approach is the one used by Forestier *et al.* (2000b) in which  $\mathbf{V}_{\Gamma N}^*$  is represented on a polynomial basis satisfying the flux rate condition and is determined by the least-squares minimization of  $|\mathbf{V}_{\Gamma N}^* \cdot \mathbf{n} - \mathbf{V}_{\Gamma N} \cdot \mathbf{n}|^2$  on  $\Gamma_1$ .

### 7.4.2 The projection method

The projection method has been presented in Section 7.3.2, in the Fourier-Chebyshev case, associated with first- and second-order time-discretization schemes. Here, considering higher-order time schemes, we describe two commonly used projection methods based on a Chebyshev collocation approximation in both spatial directions (or in three, in the general case). The first one considers polynomial approximation of different degree for velocity and pressure but using the same grid of collocation points ( $\mathcal{P}_N - \mathcal{P}_{N-2}$  method). The second method makes use of the same polynomial space for velocity and pressure ( $\mathcal{P}_N - \mathcal{P}_N$  method) and is based, as is classical for projection methods, on a Neumann boundary condition for the pressure.

#### (a) The general semidiscrete projection method

As is known, the projection method consists of two steps. In the first step, a provisional value  $\tilde{\mathbf{V}}^{n+1}$  of the velocity is calculated without taking into account the incompressibility equation. Then, in the second step, this provisional value is corrected by determining the pressure ensuring that the final velocity field is solenoidal. The scheme is as follows:

*Step 1 :*

$$\frac{1}{\Delta t} \left( a_0 \tilde{\mathbf{V}}^{n+1} + \sum_{j=1}^k a_j \mathbf{V}^{n+1-j} \right) + \sum_{j=0}^{k-1} b_j \mathbf{A}^{n-j} + \nabla \bar{p}^{n+1} - \nu \nabla^2 \tilde{\mathbf{V}}^{n+1} = \mathbf{f}^{n+1} \quad \text{in } \Omega \quad (7.238)$$

$$\tilde{\mathbf{V}}^{n+1} = \mathbf{V}_{\Gamma}^{n+1} \quad \text{on } \Gamma, \quad (7.239)$$

with  $\mathbf{A}^{n-j} = \mathbf{A}(\mathbf{V}^{n-j}, \mathbf{V}^{n-j})$ .

*Step 2 :*

$$\frac{a_0}{\Delta t} (\mathbf{V}^{n+1} - \tilde{\mathbf{V}}^{n+1}) + \nabla (p^{n+1} - \bar{p}^{n+1}) = \mathbf{0} \quad \text{in } \Omega \quad (7.240)$$

$$\nabla \cdot \mathbf{V}^{n+1} = 0 \quad \text{in } \Omega \quad (7.241)$$

$$\mathbf{V}^{n+1} \cdot \mathbf{n} = \mathbf{V}_{\Gamma}^{n+1} \cdot \mathbf{n} \quad \text{on } \Gamma. \quad (7.242)$$

The coefficients  $a_j$  and  $b_j$  for  $2 \leq k \leq 4$  are given in Table 4.4 of Section 4.5.1.a. They ensure, respectively, the material derivative of  $\mathbf{V}$  and the nonlinear term  $\mathbf{A}$  to be approximated at time  $(n+1)\Delta t$  with an error  $O(\Delta t^k)$ . Moreover, the global accuracy of the scheme depends on the definition of the pressure  $\bar{p}^{n+1}$  as discussed now.

By eliminating the provisional velocity  $\tilde{\mathbf{V}}^{n+1}$  from Eqs.(7.238) and (7.240), we obtain the semidiscrete momentum equation actually solved,

namely,

$$\frac{1}{\Delta t} \sum_{j=0}^k a_j \mathbf{V}^{n+1-j} + \sum_{j=0}^{k-1} b_j \mathbf{A}^{n-j} + \nabla p^{n+1} - \nu \nabla^2 \mathbf{V}^{n+1} + \Delta t \mathbf{E}^{n+1} = \mathbf{f}^{n+1} \quad (7.243)$$

with

$$\mathbf{E}^{n+1} = -\frac{\nu}{a_0} \nabla^2 [\nabla (p^{n+1} - \bar{p}^{n+1})] . \quad (7.244)$$

Obviously, when the problem (7.238)-(7.242) is approximated by a Chebyshev collocation, the discrete form of  $\mathbf{E}^{n+1}$  depends on the type of the polynomial approximation ( $\mathcal{P}_N - \mathcal{P}_N$  or  $\mathcal{P}_N - \mathcal{P}_{N-2}$ ) and on the way in which the algebraic system is closed. In any case, however, the discrete operator in  $\mathbf{E}^{n+1}$  acts on the polynomial approximation of the quantity  $q^{n+1} = p^{n+1} - \bar{p}^{n+1}$ , so that the order of the error in time of Eq.(7.243) is unchanged.

The examination of Eqs.(7.243) and (7.244) shows that the truncation error of this equation is  $O(\Delta t^k)$  if  $p^{n+1} - \bar{p}^{n+1} = O(\Delta t^{k-1})$ . Therefore, the choice of  $\bar{p}^{n+1}$  determines the accuracy of the method :

1. The original method (Chorin, 1968 ; Temam, 1969) simply uses

$$\bar{p}^{n+1} = 0$$

and the truncation error is  $O(\Delta t)$ .

2. The incremental method (Goda, 1979) makes use of

$$\bar{p}^{n+1} = p^n$$

so that the truncation error is  $O(\Delta t^2)$ .

3. A general high-order approximation is obtained (Karniadakis *et al.*, 1991 ; Botella, 1997) from the extrapolation formula

$$\bar{p}^{n+1} = \sum_{j=0}^{k-2} c_j p^{n-j} , \quad k \geq 2, \quad (7.245)$$

with  $c_0 = 1$  for  $k = 2$ ,  $c_0 = 2$ ,  $c_1 = -1$  for  $k = 3$  and, more generally,  $c_j(k) = b_j(k-1)$ . Formula (7.245) leads to a truncation error of  $O(\Delta t^k)$  for Eq.(7.243).

4. Hugues and Randriamampianina (1998) determine  $\bar{p}^{n+1}$  as the solution of a Neumann problem for a Poisson equation, such that  $p^{n+1} - \bar{p}^{n+1} = O(\Delta t^2)$ . This technique is described in Section 7.4.2.b.

In *Step 1*, Eqs.(7.238) and (7.239) define a Dirichlet problem for the Helmholtz equation satisfied by each component of  $\tilde{\mathbf{V}}^{n+1}$ . These Helmholtz problems are easily solved by means of the Chebyshev collocation method

associated with the matrix-diagonalization technique (Section 3.7.2) for the solution of the associated algebraic system. Note that the nonlinear convective term in Eq.(7.238) is evaluated through matrix products or using the pseudospectral technique (Section 3.5.2).

*Step 2* corresponds to a Darcy problem (or div-grad problem) whose solution is the subject of the next two sections.

The Darcy problem is written as

$$\sigma \mathbf{V}^{n+1} + \nabla q^{n+1} = \sigma \tilde{\mathbf{V}}^{n+1} \quad \text{in } \Omega \quad (7.246)$$

$$\nabla \cdot \mathbf{V}^{n+1} = 0 \quad \text{in } \Omega \quad (7.247)$$

$$\mathbf{V}^{n+1} \cdot \mathbf{n} = \mathbf{V}_\Gamma^{n+1} \cdot \mathbf{n} \quad \text{on } \Gamma, \quad (7.248)$$

and

$$p^{n+1} = q^{n+1} + \bar{p}^{n+1}, \quad (7.249)$$

where  $\sigma = a_0/\Delta t$ .

Now let us assume that the velocity components  $u^{n+1}$  and  $v^{n+1}$ , as well as the pressure  $p^{n+1}$  (and consequently  $q^{n+1}$ ) are approximated with polynomials of the same degree ( $\mathcal{P}_N - \mathcal{P}_N$  method). Then suppose Eq.(7.246) to be enforced on  $\Omega_N$  and Eq.(7.247) on  $\bar{\Omega}_N^I$  (see Section 7.4.1.b for the definition of  $\Omega_N$  and  $\bar{\Omega}_N^I$ ). The resulting pressure solution is contaminated by spurious modes, since  $q^{n+1}$  appears only through its gradient at inner collocation points. Spatial approximations avoiding spurious modes may be of two types, which are now described.

1. The first type of method consists of approximating the pressure with a polynomial whose degree is lower than the one used for the velocity components, so that the pressure gradient does not vanish at points where the momentum equation is enforced. This can be done in either of the following approximation techniques :

(i) Use a staggered grid of MAC type where the velocity components and the pressure are computed on different grids (see, e.g., Peyret and Taylor, 1983). This is relatively simple to implement in the one-dimensional case (Malik *et al.*, 1985 ; Zang and Hussaini, 1985 ; Canuto *et al.*, 1988 ; Fröhlich and Peyret, 1990). On the other hand, in the two-dimensional case (Bernardi and Maday, 1988 ; Le Quéré, 1989) the implementation is appreciably more complicated since it involves three different grids and appropriate interpolation. For these reasons, the staggered grid method is not of common use.

(ii) Use a unique collocation grid (Gauss-Lobatto points) but approximate the velocity components and polynomials of different degree. This  $\mathcal{P}_N - \mathcal{P}_{N-2}$  method, introduced by Bernardi *et al.* (1990) and developed in several works (e.g., Heinrichs, 1993 ; Azaiez *et al.*, 1994b; Botella, 1997), is described in detail in Section 7.4.2.b.

2. The second type of method consists of using the normal momentum equation at the boundary to close the discrete system rather than the incompressibility equation. In this case, the velocity components as well as the pressure are approximated with polynomials of the same degree ( $\mathbb{P}_N - \mathbb{P}_N$  method). There are no spurious pressure modes since the normal gradient of the pressure at the boundary involves the derivative of the high-degree Chebyshev polynomial which does not vanish at  $\pm 1$  while it does at the inner collocation points. The  $\mathbb{P}_N - \mathbb{P}_N$  method is described in detail in Section 7.4.2.c.

**(b) The  $\mathbb{P}_N - \mathbb{P}_{N-2}$  projection method**

The pressure  $\bar{p}^{n+1}$  in Eq.(7.238) is evaluated through the extrapolation formula (7.245). Let us define  $\mathbb{P}_N$  as the space of polynomials of degree at most  $N_x$  in  $x$  and  $N_y$  in  $y$ . The velocity components are approximated with polynomials belonging to the  $\mathbb{P}_N$ -space. Thus, the provisional velocity  $\tilde{\mathbf{V}}^{n+1}$  is approximated by  $\tilde{\mathbf{V}}_N^{n+1} = (\tilde{u}_N^{n+1}, \tilde{v}_N^{n+1})$  and the velocity  $\mathbf{V}^{n+1}$  by  $\mathbf{V}_N^{n+1} = (u_N^{n+1}, v_N^{n+1})$ . The pressure is approximated with a polynomial of degree less than two in both directions, namely,  $p^{n+1} \cong p_{N-2}^{n+1} \in \mathbb{P}_{N-2}$  and, consequently,  $q^{n+1} \cong q_{N-2}^{n+1} \in \mathbb{P}_{N-2}$ . Equations (7.246) and (7.247) are enforced at the Gauss-Lobatto points  $(x_i, y_j) \in \Omega_N$ . This necessitates the evaluation of the pressure gradient at the same points. Hence, one must define the Lagrange interpolation polynomial of degree  $N_x - 2$  (or  $N_y - 2$ ) based on the points  $x_i = \cos(\pi i/N_x)$ ,  $i = 1, \dots, N_x - 1$ , [or  $y_j = \cos(\pi j/N_y)$ ,  $j = 1, \dots, N_y - 1$ ].

Let us consider the function  $\phi(x)$  defined in  $[-1, 1]$  and the inner Gauss-Lobatto points  $x_i = \cos \pi i/N$ ,  $i = 1, \dots, N - 1$ . The Lagrange interpolation polynomial based on this set of points is defined by

$$\phi_{N-2}(x) = \sum_{j=1}^{N-1} \phi(x_j) \hat{h}_j(x), \quad (7.250)$$

where  $\hat{h}_j(x)$  is the polynomial of degree  $N - 2$  such that

$$\hat{h}_j(x_i) = \delta_{i,j}, \quad i, j = 1, \dots, N - 1.$$

The polynomial  $\hat{h}_j(x)$  has the expression

$$\hat{h}_j(x) = \frac{(-1)^{j+1} (1 - x_j^2) T'_N(x)}{N^2 (x - x_j)}, \quad j = 1, \dots, N - 1. \quad (7.251)$$

This expression is obtained by using the fact that the Gauss-Lobatto points  $x_i$ ,  $i = 1, \dots, N - 1$ , are the zeros of  $T'_N(x)$  and by taking into account that

$$\frac{T'_N(x)}{x - x_j} \rightarrow (-1)^{j+1} \frac{N^2}{1 - x_j^2} \quad \text{for } x \rightarrow x_j, \quad j = 1, \dots, N - 1.$$

We observe that

$$\hat{h}_j(x) = \frac{1 - x_j^2}{1 - x^2} h_j(x),$$

where  $h_j(x)$  is the Lagrange interpolation coefficient associated with the  $P_N$  approximation and given by Eq.(3.44). The differentiation formula associated with the approximation (7.250) is written

$$\phi'_{N-2}(x_i) = \sum_{j=1}^{N-1} \hat{d}_{i,j}^{(1)} \phi(x_j), \quad i = 1, \dots, N-1 \quad (7.252)$$

with

$$\hat{d}_{i,j}^{(1)} = \hat{h}'_j(x_i) = \frac{1 - x_j^2}{1 - x_i^2} h'_j(x_i) + \frac{2x_i}{1 - x_i^2} \delta_{i,j},$$

or

$$\begin{aligned} \hat{d}_{i,j}^{(1)} &= \frac{(-1)^{j+1} (1 - x_j^2)}{(1 - x_i^2) (x_i - x_j)}, \quad i, j = 1, \dots, N-1, \quad i \neq j \\ \hat{d}_{i,i}^{(1)} &= \frac{3x_i}{2(1 - x_i^2)}, \quad i = 1, \dots, N-1. \end{aligned} \quad (7.253)$$

Now the collocation method applied to the solution of problem (7.238)-(7.239) determines the value of the approximate provisional velocity  $\tilde{\mathbf{V}}_N^{n+1}$  at the Gauss-Lobatto points, namely,

$$\sigma \tilde{\mathbf{V}}_N^{n+1} - \nu \nabla^2 \tilde{\mathbf{V}}_N^{n+1} = \mathbf{F}_N^{n+1} \quad \text{in } \Omega_N \quad (7.254)$$

$$\tilde{\mathbf{V}}_N^{n+1} = \mathbf{V}_\Gamma^{n+1} \quad \text{on } \Gamma_N^I, \quad (7.255)$$

where

$$\mathbf{F}_N^{n+1} = \mathbf{f}_N^{n+1} - \frac{1}{\Delta t} \sum_{j=1}^k a_j \mathbf{V}_N^{n+1-j} - \sum_{j=0}^{k-1} b_j \mathbf{A}_N^{n-j} - \sum_{j=0}^{k-2} c_j \nabla p_{N-2}^{n-j}$$

and  $\sigma = a_0/\Delta t$ . The gradient  $\nabla p_{N-2}^{n-j}$  is evaluated at the Gauss-Lobatto points by means of formulas of type (7.252)-(7.253). The nonlinear term in  $\mathbf{F}_N^{n+1}$  is evaluated by matrix products or by the pseudospectral technique (Section 3.5.2). Equations (7.254) and (7.255) define an algebraic system for each component  $\tilde{u}_N^{n+1}$  and  $\tilde{v}_N^{n+1}$ , that is easily solved by using the matrix-diagonalization technique (Section 3.7.2).

Then the collocation approximation of the Darcy problem (7.246)-(7.249) yields the algebraic system

$$\sigma \mathbf{V}_N^{n+1} + \nabla q_{N-2}^{n+1} = \sigma \tilde{\mathbf{V}}_N^{n+1} \quad \text{in } \Omega_N \quad (7.256)$$

$$\nabla \cdot \mathbf{V}_N^{n+1} = 0 \quad \text{in } \Omega_N \quad (7.257)$$



$$\mathbf{V}_N^{n+1} \cdot \mathbf{n} = \mathbf{V}_\Gamma^{n+1} \cdot \mathbf{n} \quad \text{on } \Gamma_N^I, \quad (7.258)$$

and

$$p_{N-2}^{n+1} = q_{N-2}^{n+1} + \bar{p}_{N-2}^{n+1} \quad \text{in } \Omega_N, \quad (7.259)$$

where the divergence operator acting on  $\mathbf{V}_N^{n+1}$  is defined by the usual differentiation formulas (3.45)-(3.46), while the gradient  $\nabla q_{N-2}^{n+1}$  is expressed by formulas like (7.252)-(7.253).

It is observed that the system (7.256)-(7.259) contains sufficient algebraic equations to completely determine the velocity and pressure fields in  $\Omega_N$ . Therefore, one of the main advantages of this method is that it does not require the prescription of boundary conditions for the pressure. On the boundary  $\Gamma_N^I$ , the normal component of the velocity is defined by (7.258). On the other hand, the tangential component is not involved in the above system. Therefore, this component is not deduced from the solution of the Darcy problem (as it should be in the continuous case) but is simply defined by the boundary condition (7.192) of the Navier-Stokes problem.

We can estimate the error in time of the method by analyzing the truncation error of the discrete equations obtained after the elimination of the provisional velocity  $\tilde{\mathbf{V}}_N^{n+1}(x_i, y_j)$ . Thus, the discrete form of Eq.(7.243) involves the term

$$\mathbf{E}_{N-2}^{n+1}(x_i, y_j) = -\frac{\nu}{a_0} \left[ \sum_{l=1}^{N_x-1} d_{i,l}^{(2)} \mathbf{G}_{N-2}^{n+1}(x_l, y_j) + \sum_{l=1}^{N_y-1} d_{j,l}^{(2)} \mathbf{G}_{N-2}^{n+1}(x_i, y_l) \right]$$

where  $\mathbf{G}_{N-2}^{n+1}$  is the discrete gradient of  $q_{N-2}^{n+1}$  defined by differentiation formulas of type (7.252). The discrete equations resulting from Eq.(7.243) are applied to the inner points  $(x_i, y_j) \in \Omega_N$ , supplemented by the incompressibility equation  $\nabla \cdot \mathbf{V}_N = 0$  enforced on  $\Omega_N$  and the boundary conditions  $\mathbf{V}_N^{n+1} = \mathbf{V}_\Gamma^{n+1}$  on  $\Gamma_N^I$ . Therefore it is clear that the truncation error of the overall discrete problem is  $O(\Delta t^k)$  if  $\bar{p}^{n+1}$  is defined by Eq.(7.245). Moreover, the error-splitting term  $\Delta t \mathbf{E}_{N-2}^{n+1}$  vanishes at steady state.

Equations (7.256)-(7.259) are written in matrix form by introducing the  $(N_x - 1) \times (N_y - 1)$  matrices containing the grid values of the unknowns

$$\mathcal{U} = [u_N^{n+1}(x_i, y_j)], \quad i = 1, \dots, N_x - 1, \quad j = 1, \dots, N_y - 1$$

$$\mathcal{V} = [v_N^{n+1}(x_i, y_j)], \quad i = 1, \dots, N_x - 1, \quad j = 1, \dots, N_y - 1$$

$$\mathcal{Q} = [q_{N-2}^{n+1}(x_i, y_j)], \quad i = 1, \dots, N_x - 1, \quad j = 1, \dots, N_y - 1.$$

The system is written

$$\sigma \mathcal{U} + \hat{\mathcal{D}}_x \mathcal{Q} = \sigma \tilde{\mathcal{U}} \quad (7.260)$$

$$\sigma \mathcal{V} + \mathcal{Q} \hat{\mathcal{D}}_y^T = \sigma \tilde{\mathcal{V}} \quad (7.261)$$

$$\mathcal{D}_x \mathcal{U} + \mathcal{V} \mathcal{D}_y^T = \mathcal{S}, \quad (7.262)$$

where  $\mathcal{D}_x$  and  $\mathcal{D}_y$  are, respectively, the  $(N_x - 1) \times (N_x - 1)$  and  $(N_y - 1) \times (N_y - 1)$  matrices analogous to the matrix  $\mathcal{D}_1$  defined in Eq.(4.66) ;  $\hat{\mathcal{D}}_x$  and  $\hat{\mathcal{D}}_y$  are similar matrices made with the coefficients  $\hat{d}_{i,j}^{(1)}$ . The matrices  $\tilde{\mathcal{U}}$  and  $\tilde{\mathcal{V}}$  are defined by

$$\begin{aligned}\tilde{\mathcal{U}} &= [\tilde{u}_N^{n+1}(x_i, y_j)] , \quad i = 1, \dots, N_x - 1, \quad j = 1, \dots, N_y - 1 \\ \tilde{\mathcal{V}} &= [\tilde{v}_N^{n+1}(x_i, y_j)] , \quad i = 1, \dots, N_x - 1, \quad j = 1, \dots, N_y - 1\end{aligned}$$

and  $\mathcal{S}$  is the  $(N_x - 1) \times (N_y - 1)$  matrix containing the boundary values of the velocity, namely,

$$\mathcal{S} = - \left( \overline{\mathcal{D}}_x \mathcal{U}_\Gamma + \mathcal{V}_\Gamma \overline{\mathcal{D}}_y^T \right) ,$$

where

$$\begin{aligned}\mathcal{U}_\Gamma &= [u_\Gamma^{n+1}(x_i, y_j)] , \quad i = 0, N_x, \quad j = 1, \dots, N_y - 1 \\ \mathcal{V}_\Gamma &= [v_\Gamma^{n+1}(x_i, y_j)] , \quad i = 1, \dots, N_x - 1, \quad j = 0, N_y\end{aligned}$$

and

$$\begin{aligned}\overline{\mathcal{D}}_x &= [d_{i,j}^{(1)}] , \quad i = 1, \dots, N_x - 1, \quad j = 0, N_x \\ \overline{\mathcal{D}}_y &= [d_{i,j}^{(1)}] , \quad i = 1, \dots, N_y - 1, \quad j = 0, N_y.\end{aligned}$$

The solution of the large system (7.260)-(7.262) is obtained through the elimination of  $\mathcal{U}$  and  $\mathcal{V}$  to get the system determining  $\mathcal{Q}$ , let

$$\mathcal{D}_x \hat{\mathcal{D}}_x \mathcal{Q} + \mathcal{Q} \left( \mathcal{D}_y \hat{\mathcal{D}}_y \right)^T = -\sigma \left( \mathcal{S} - \mathcal{D}_x \tilde{\mathcal{U}} - \tilde{\mathcal{V}} \mathcal{D}_y^T \right) . \quad (7.263)$$

The operator acting on  $\mathcal{Q}$  is of Uzawa type. The eigenvalues of the matrix  $\mathcal{D}_x \hat{\mathcal{D}}_x$  (or  $\mathcal{D}_y \hat{\mathcal{D}}_y$ ) are real, distinct, and negative except one and only one which is zero, corresponding to the constant mode (Heinrichs, 1993). Moreover, when  $N_x \rightarrow \infty$ , the smaller eigenvalue  $|\lambda|_{min} \rightarrow \pi^2/4$  (discarding the null eigenvalue) and the larger one  $|\lambda|_{max} \cong 0.047 N_x^4$ . This is typically the behaviour of the eigenvalues of the second-order derivative operator with Dirichlet conditions (see Table 3.1). Therefore, the Uzawa operator acting on  $\mathcal{Q}$  in (7.263) may be characterized as a “pseudo-discrete-Laplacian”.

System (7.263) is solved by means of the matrix-diagonalization procedure (Section 3.7.2) using, for the null eigenvalue, the technique proposed in Section 3.7.1 for solving the Neumann problem for the Poisson equation. Then, when  $\mathcal{Q}$  is known, Eqs.(7.260) and (7.261) give  $\mathcal{U}$  and  $\mathcal{V}$ , respectively. Note that in the three-dimensional case, the system determining  $\mathcal{Q}$  is put in the tensor product form and the solution procedure follows the one described in Section 3.7.3.

Note that the value of the pressure at the boundary is not known from the discrete system (7.260)-(7.262). If this value is needed, it is rather simple to

deduce an approximation of it from the Lagrange interpolation polynomial of type (7.250) evaluated at the extremities  $\pm 1$ .

Moreover, we recall that the incompressibility equation is enforced at the inner collocation points but not on the boundary. The result is that the approximate divergence velocity  $Q_N = \nabla \cdot \mathbf{V}_N$  is not the null polynomial as it is in the case of the influence matrix method. This drawback may be seen as an advantage in some situations where the solution exhibits a singularity at the boundary. Moreover, the method is less sensitive to the way in which the total flow rate condition is satisfied (see Remark 2 of Section 7.4.1.b).

Returning to the time-discretization of the Navier-Stokes equations, we are faced with the usual problem associated with multistep schemes, namely the calculation of the solution of the first time-cycles. More precisely, the calculation of  $(\mathbf{V}^1, p^1)$  for the second-order scheme ( $k = 2$ ),  $(\mathbf{V}^1, p^1)$  and  $(\mathbf{V}^2, p^2)$  for the third-order scheme ( $k = 3$ ), and so on.

As mentioned in Section 4.5.1.d, the starting scheme may be one order less accurate than the current scheme without effect on the accuracy of the approximate solution. Therefore, for the second-order scheme one can set  $\mathbf{V}^{-1} = \mathbf{V}^0 = \mathbf{V}_0$  and  $p^0 = 0$ , that leads to a first-order scheme provided the time-step is taken equal to  $3\Delta t/2$ .

For the third-order scheme, Botella and Peyret (2001) suggest the calculation of  $(\mathbf{V}^1, p^1)$  and  $(\mathbf{V}^2, p^2)$  by means of a second-order scheme derived from the RK2/CN scheme (4.85):

*Step 1a : Calculation of  $\mathbf{V}_1$  :*

$$\begin{aligned} \frac{\mathbf{V}_1 - \mathbf{V}^n}{\Delta t} + \mathbf{A}(\mathbf{V}^n, \mathbf{V}^n) + \nabla p^n - \nu \nabla^2 \mathbf{V}^n &= \mathbf{f}^n & \text{in } \Omega \\ \mathbf{V}_1 &= \mathbf{V}_\Gamma^{n+1} & \text{on } \Gamma. \end{aligned}$$

*Step 1b : Calculation of  $\tilde{\mathbf{V}}^{n+1}$  :*

$$\begin{aligned} \frac{2\tilde{\mathbf{V}}^{n+1} - \mathbf{V}_1 - \mathbf{V}^n}{\Delta t} + \mathbf{A}(\mathbf{V}_1, \mathbf{V}_1) + \nabla p^n - \nu \nabla^2 \tilde{\mathbf{V}}^{n+1} &= \mathbf{f}^{n+1} & \text{in } \Omega \\ \tilde{\mathbf{V}}^{n+1} &= \mathbf{V}_\Gamma^{n+1} & \text{on } \Gamma. \end{aligned}$$

*Step 2 : Projection :*

$$\begin{aligned} \frac{\mathbf{V}^{n+1} - \tilde{\mathbf{V}}^{n+1}}{2\Delta t} + \frac{1}{2} \nabla (p^{n+1} - p^n) &= \mathbf{0} & \text{in } \Omega \\ \nabla \cdot \mathbf{V}^{n+1} &= 0 & \text{in } \Omega \\ \mathbf{V}^{n+1} \cdot \mathbf{n} &= \mathbf{V}_\Gamma^{n+1} \cdot \mathbf{n} & \text{on } \Gamma. \end{aligned}$$

It is possible to show (Botella, 1998) that this scheme is equivalent at  $O(\Delta t^2)$  to the classical Crank-Nicolson scheme. The scheme is applied to  $n = 0$  and  $n = 1$ . Therefore, the knowledge of an initial pressure field  $p^0$  is required. As shown by Heywood and Rannacher (1982) and by Temam (1982), the initial pressure field (which is not data of the exact continuous problem) can be determined from the Neumann problem (5.30)-(5.31) with  $\mathbf{V} = \mathbf{V}_0$ , taking into account that the initial velocity field  $\mathbf{V}_0$  satisfies the compatibility condition (5.11). We refer to Marion and Temam (1998) for a general discussion on the initial conditions. Examples of the application of the  $\mathcal{I}P_N - \mathcal{I}P_{N-2}$  method are presented in Sections 7.6.4 and 8.4.2.b.

**(c) The predicted  $\mathcal{I}P_N - \mathcal{I}P_N$  projection method**

The  $\mathcal{I}P_N - \mathcal{I}P_N$  projection method was defined as Option (3) in Section 7.3.2.d, and the weakness of the method concerning the Neumann condition for the pressure, in the framework of the strong collocation approximation, was discussed. This drawback, however, can be removed by using, for the pressure  $\bar{p}^{n+1}$  occurring in Eq.(7.235), the solution of a Poisson equation with Neumann boundary conditions. This method, proposed by Hugues and Randriamampianina (1998), is now presented associated with the second-order time-scheme [ $k = 2$  in Eqs.(7.238)-(7.242)], but higher-order schemes could be used as long as stability is ensured.

The pressure  $\bar{p}^{n+1}$  is calculated, in a predicted step, from Eqs.(5.30)-(5.31) using an explicit formulation, namely,

$$\nabla^2 \bar{p}^{n+1} = \nabla \cdot [\mathbf{f}^{n+1} - (2\mathbf{A}^n - \mathbf{A}^{n-1})] \quad \text{in } \Omega \quad (7.264)$$

$$\begin{aligned} \partial_n \bar{p}^{n+1} = & [\mathbf{f}^{n+1} - (2\mathbf{A}^n - \mathbf{A}^{n-1}) + \nu(2\mathbf{D}^n - \mathbf{D}^{n-1}) \\ & - \frac{1}{2\Delta t} (3\mathbf{V}_\Gamma^{n+1} - 4\mathbf{V}_\Gamma^n + \mathbf{V}_\Gamma^{n-1})] \cdot \mathbf{n} \quad \text{on } \Gamma. \end{aligned} \quad (7.265)$$

The use of the classical form of the diffusive term  $\mathbf{D} = \nabla^2 \mathbf{V}$  leads to instability. On the other hand, the form

$$\mathbf{D} = \nabla(\nabla \cdot \mathbf{V}) - \nabla \times (\nabla \times \mathbf{V}) = -\nabla \times (\nabla \times \mathbf{V}) \quad (7.266)$$

considered by Orszag *et al.* (1986) and by Karniadakis *et al.* (1991) is found to be stable. As a matter of fact, because the constraint  $\nabla \cdot \mathbf{V} = 0$  is not enforced at the boundary, the compatibility condition of the Neumann problem (7.264)-(7.265) would not be satisfied if the term  $\nabla(\nabla \cdot \mathbf{V})$  was retained in  $\mathbf{D}$ , as it is when using  $\mathbf{D} = \nabla^2 \mathbf{V}$ . Formally (and numerically checked) the value  $\bar{p}^{n+1}$  is an approximation to  $p^{n+1}$  with an error  $O(\Delta t^2)$ . Therefore, the term  $\Delta t E^{n+1}$  in Eq.(7.243) is  $O(\Delta t^3)$ .

It should be recalled that, in the  $\mathcal{I}P_N - \mathcal{I}P_N$  approximation method, the velocity components and the pressure are approximated with polynomials of the same degree, namely,  $u_N^{n+1}$ ,  $v_N^{n+1}$ ,  $p_N^{n+1}$ ,  $\bar{p}_N^{n+1}$  (and  $q_N^{n+1} = p_N^{n+1} - \bar{p}_N^{n+1}$ ) are polynomials of degree at most  $N_x$  in  $x$  and  $N_y$  in  $y$ .

The Chebyshev-collocation solution of the Neumann problem (7.264)-(7.265) gives  $\bar{p}_N^{n+1}$ . Then, when  $\bar{p}_N^{n+1}$  has been calculated, the polynomial approximation  $\tilde{\mathbf{V}}_N^{n+1} = (\tilde{u}_N^{n+1}, \tilde{v}_N^{n+1})$  to the polynomial velocity  $\tilde{\mathbf{V}}^{n+1} = (\tilde{u}^{n+1}, \tilde{v}^{n+1})$  is determined from the solution of Eqs.(7.238)-(7.239) with  $k = 2$ , namely,

$$\sigma \tilde{\mathbf{V}}^{n+1} - \nu \nabla^2 \tilde{\mathbf{V}}^{n+1} = \mathbf{F}^{n+1} \quad \text{in } \Omega \quad (7.267)$$

$$\nabla \cdot \tilde{\mathbf{V}}^{n+1} = 0 \quad \text{in } \Omega \quad (7.268)$$

$$\tilde{\mathbf{V}}^{n+1} = \mathbf{V}_\Gamma^{n+1} \quad \text{on } \Gamma, \quad (7.269)$$

with  $\sigma = 3/(2 \Delta t)$  and

$$\mathbf{F}^{n+1} = \mathbf{f}^{n+1} - \frac{4 \mathbf{V}^n - \mathbf{V}^{n-1}}{2 \Delta t} - (2 \mathbf{A}^n - \mathbf{A}^{n-1}) - \nabla \bar{p}^{n+1}. \quad (7.270)$$

The collocation approximation to Eqs.(7.267)-(7.269) gives an algebraic system for the grid values of each component of  $\tilde{\mathbf{V}}^{n+1}$ .

Now it remains to solve the Darcy problem (7.246)-(7.249). As explained in Section 7.3.2.e this is done by deriving a Poisson equation for  $q^{n+1}$  supplemented with a Neumann boundary condition. This can be done either directly from the discrete system resulting from the Chebyshev collocation approximation to problem (7.246)-(7.249) or from the continuous equations themselves. This latter derivation is simpler in the multidimensional case. Therefore, the Poisson equation is obtained by applying the divergence operator to Eq.(7.246) and taking into account the incompressibility equation (7.247). The associated Neumann condition is derived by projecting Eq.(7.246) normal to the boundary. We obtain the Neumann problem

$$\nabla^2 q^{n+1} = \sigma \nabla \cdot \tilde{\mathbf{V}}^{n+1} \quad \text{in } \Omega \quad (7.271)$$

$$\partial_n q^{n+1} = \sigma (\tilde{\mathbf{V}}^{n+1} - \mathbf{V}^{n+1}) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma, \quad (7.272)$$

which automatically satisfies the compatibility condition. Equation (7.272) shows that  $p^{n+1}$  satisfies the Neumann condition

$$\partial_n p^{n+1} = \partial_n \bar{p}^{n+1} \quad \text{on } \Gamma,$$

that is consistent with the Navier-Stokes problem and accurate to second-order. Problem (7.271)-(7.272) is solved by means of the Chebyshev collocation method. Once  $q_N^{n+1}$  is known, Eq.(7.246) applied to  $\bar{\Omega}_N^I$  determines the approximate velocity field  $\mathbf{V}_N^{n+1}$  whose divergence is zero at the inner collocation points.

In summary, the algorithm is the following :

1. Solve Eqs.(7.264)-(7.265) for determining  $\bar{p}_N^{n+1}$  in  $\bar{\Omega}_N^I$ .
2. Solve Eqs.(7.267)-(7.269) for determining  $\tilde{\mathbf{V}}_N^{n+1}$  in  $\bar{\Omega}_N^I$ .

3. Solve Eqs.(7.271)-(7.272) for determining  $q_N^{n+1}$  in  $\bar{\Omega}_N^I$ .
4. Calculate  $p_N^{n+1}$  in  $\bar{\Omega}_N^I$  from Eq.(7.249), namely,  $p_N^{n+1} = q_N^{n+1} + \bar{p}_N^{n+1}$ .
5. Calculate  $\mathbf{V}_N^{n+1}$  in  $\bar{\Omega}_N^I$  from Eq.(7.246), namely,

$$\mathbf{V}_N^{n+1} = \tilde{\mathbf{V}}_N^{n+1} - \frac{1}{\sigma} \nabla q_N^{n+1}.$$

The calculation of  $(\mathbf{V}^1, p^1)$  at the first time-cycle ( $n = 0$ ) with the three-level scheme ( $k = 2$ ) is performed by setting  $\mathbf{V}^{-1} = \mathbf{V}^0 = \mathbf{V}_0$  and the time-step equal to  $3\Delta t/2$ ,  $\bar{p}^1$  being determined by Eqs.(7.264)-(7.265).

Note that the boundary condition on the tangential velocity component is not exactly satisfied since this velocity is obtained from Eq.(7.246), namely,

$$\mathbf{V}_N^{n+1} \cdot \mathbf{t} = \mathbf{V}_\Gamma^{n+1} \cdot \mathbf{t} - \frac{\Delta t}{a_0} [\nabla (p_N^{n+1} - \bar{p}_N^{n+1})] \cdot \mathbf{t} \quad \text{on } \Gamma_N^I. \quad (7.273)$$

The evaluation of the tangential gradient of  $q_N^{n+1} = p_N^{n+1} - \bar{p}_N^{n+1}$  on  $\Gamma_N^I$  necessitates the knowledge of the values of  $q_N^{n+1}$  at the corners. These values are calculated by using the Neumann condition  $\partial_n q_N^{n+1} = 0$  according to the technique described in Section 3.7.2 in the general case of Robin conditions and applied to the calculation of the pressure at the corners in Remark 1 of Section 7.4.1.

The error term (“slip velocity”) in the right-hand side of Eq.(7.273) is  $O(\Delta t^3)$  since  $p_N^{n+1} - \bar{p}_N^{n+1} = O(\Delta t^2)$  as numerically checked (Hugues and Randriamampianina, 1998; Raspo *et al.*, 2002). Therefore, the effect of this slip velocity on the global accuracy  $O(\Delta t^2)$  is negligible. However, the slip velocity does not vanish at steady state because the difference  $p_N^{n+1} - \bar{p}_N^{n+1}$  is not zero at steady state. In order to avoid the slip velocity, one might be tempted to define the tangential velocity from the exact boundary condition (7.192) rather than from Eq.(7.273). This is the Option (4) defined in Section 7.3.2.d. Unfortunately, such a procedure appeared unstable in some applications, while it is stable in the case where  $\bar{p}^{n+1} = p^n$ .

Lastly, one could wonder whether it is quite necessary to calculate the predicted pressure  $\bar{p}^{n+1}$ . One would then restrict oneself to set  $\bar{p}^{n+1} = p^n$ , namely,  $q^{n+1} = p^{n+1} - p^n$ , and to solve the Poisson equation (7.271) with a Neumann boundary condition similar to Eq.(7.265). However, the compatibility condition associated with this Neumann problem is satisfied at  $O(\Delta t)$  only, whatever the form of  $\mathbf{D}$ , and the method appears to be unstable. On the other hand, a stable projection method involving such a Neumann condition can be constructed (Karniadakis *et al.*, 1991) by inverting the two steps of the procedure. In the first step the pressure field and a solenoidal velocity field are calculated by discarding the diffusive term. The pressure field  $p^{n+1}$  is a solution of the Neumann problem (7.264)-(7.265) which satisfies the compatibility condition. In the second step, the final velocity is calculated by taking the diffusion into account. Although providing accurate results, this method presents the drawback of not controlling the

divergence  $Q_N^{n+1}$  of the approximation of the final velocity field  $\mathbf{V}^{n+1}$ . The divergence  $Q_N^{n+1}$  satisfies a Helmholtz equation, as in the influence matrix method, but without the boundary condition ensuring that  $Q_N^{n+1}$  is effectively zero. However, it must be pointed out that the polynomial  $Q_N^{n+1}$  tends toward zero when the spatial resolution is increased. A similar splitting scheme has been proposed by Batoul *et al.* (1994), associated with the  $\mathcal{P}_N - \mathcal{P}_{N-2}$  approximation, avoiding the need for a boundary condition for the pressure. This method presents the same behaviour concerning the divergence of the velocity.

#### 7.4.3 Shear-stress-free boundary

When a part of the boundary  $\Gamma$  is stress-free, the no-slip condition is replaced by

$$V_n = 0, \quad \partial_n (V_t) = 0, \quad (7.274)$$

where  $V_n$  and  $V_t$  are, respectively, the normal and tangential components of the velocity vector  $\mathbf{V}$  with respect to the boundary  $\Gamma$ . Of course, it is assumed here that the boundary velocity  $\mathbf{V}_\Gamma$  on the complementary part of the stress-free boundary is compatible with condition (7.274). In the following, and this to fix the ideas, the side  $\Gamma_1 = \{y = 1, -1 < x < 1\}$  will be supposed to be stress-free, that is, Eq.(7.274) becomes

$$v = 0, \quad \partial_y u = 0 \quad \text{on } \Gamma_1. \quad (7.275)$$

##### (a) Influence matrix method

The implementation of the boundary conditions (7.275) in the influence matrix method associated with the Fourier-Chebyshev approximation has been discussed at the end of Section 7.3.2.c. The extension to the Chebyshev-Chebyshev case is straightforward.

##### (b) $\mathcal{P}_N - \mathcal{P}_{N-2}$ projection method

In *step 1* of the projection method, the calculation of the provisional velocity  $\tilde{\mathbf{V}}^{n+1}$  is done by simply replacing the no-slip condition on  $\Gamma_1$  by

$$\tilde{v}^{n+1} = 0, \quad \partial_y \tilde{u}^{n+1} = 0 \quad \text{on } \Gamma_1. \quad (7.276)$$

The solution of the Darcy problem in *step 2* requires a little more work. For the normal component on  $\Gamma_1$ , there is no difficulty in prescribing

$$v^{n+1} = 0 \quad \text{on } \Gamma_1. \quad (7.277)$$

The difficulty appears in determining the tangential velocity  $u^{n+1}$  on  $\Gamma_1$  such that it satisfies the second condition (7.275). This can be done by differentiating the tangential part of Eq.(7.246) and, taking the second equation (7.275) into account, we get the discrete equation

$$\partial_y u_N^{n+1} = -\frac{1}{\sigma} \partial_{yx} q_N^{n+1} \quad \text{on } \Gamma_{1N}^I \quad (7.278)$$

where  $\Gamma_{1N}^I$  refers to the inner collocation points belonging to  $\Gamma_1$ . The term  $\partial_{yx} q_N^{n+1}(x_i, 1)$ ,  $i = 1, \dots, N_x - 1$ , is evaluated from the values  $q_N^{n+1}(x_i, y_j)$ ,  $i = 1, \dots, N_x - 1$ ,  $j = 1, \dots, N_y - 1$ , that is,

$$\partial_x q_N^{n+1}(x_i, y_j) = \sum_{k=1}^{N_x-1} \hat{d}_{i,k}^{(1)} q_N^{n+1}(x_k, y_j)$$

then

$$\partial_x q_N^{n+1}(x_i, y_j) = \sum_{l=1}^{N_y-1} [\partial_x q_N^{n+1}(x_i, y_l)] \hat{h}_l(y)$$

and

$$\partial_{yx} q_N^{n+1}(x_i, 1) = \sum_{l=1}^{N_y-1} [\partial_x q_N^{n+1}(x_i, y_l)] \hat{h}'_l(1),$$

where  $\hat{h}'_l(1)$  is calculated from Eq.(7.251).

Now the derivative  $\partial_y u_N^{n+1}(x_i, 1)$  is evaluated through the usual differentiation formula using the known values  $u_N^{n+1}(x_i, y_j)$ ,  $j = 1, \dots, N_y$ . Thus, Eq.(7.278) determines the boundary value  $u_N^{n+1}(x_i, 1)$ .

Note that Eq.(7.278) exhibits the error term  $(\Delta t/a_0) \partial_{yx} q_N^{n+1}$  which is formally  $O(\Delta t^2)$ . This is the analog of the slip velocity occurring in the  $\mathcal{P}_N - \mathcal{P}_N$  projection method [Eq.(7.273)].

### (c) $\mathcal{P}_N - \mathcal{P}_N$ projection method

The implementation of the stress-free conditions is similar to that described for the  $\mathcal{P}_N - \mathcal{P}_{N-2}$  method. It is, however, simpler since the quantity  $q_N^{n+1}$  is known on  $\Gamma_{1N}^I$  from the solution of Eqs.(7.271)-(7.272) and is calculated at the corners as explained above. It may be useful to observe that the pressure satisfies the homogeneous Neumann condition  $\partial_n p = 0$  on a stress-free boundary provided that  $\mathbf{f} \cdot \mathbf{n} = 0$ .

#### 7.4.4 Assessment of methods

The purpose of this section is to assess and compare the various methods by considering simple examples : analytical solutions and regularized cavity flow. The considered methods are based on the AB/BDIk scheme with  $k = 2$  or  $k = 3$  and the nonlinear term  $\mathbf{A}(\mathbf{V}, \mathbf{V})$  is considered in the convective form (see Section 7.4.5 for a discussion on the various forms of this term). These methods are :

1. The AB/BDI2,  $\mathcal{P}_N - \mathcal{P}_N$ , influence matrix method [Section 7.4.1] : IM.
2. The AB/BDI2,  $\mathcal{P}_N - \mathcal{P}_N$ , original projection method [Section 7.4.2,  $\bar{p}^{n+1} = 0$ ] : OP.
3. The AB/BDI2,  $\mathcal{P}_N - \mathcal{P}_N$ , incremental projection method [Section



7.4.2,  $\bar{p}^{n+1} = p^n$ ] : IP.

4. The AB/BDI2,  $\mathcal{I}P_N - \mathcal{I}P_N$ , predicted projection method [Section 7.4.2,  $\bar{p}^{n+1}$  solution of Eqs.(7.264)-(7.265)] : PP.

5. The AB/BDI2,  $\mathcal{I}P_N - \mathcal{I}P_{N-2}$ , projection method [Section 7.4.2,  $\bar{p}^{n+1} = p^n$ ] : P2.

6. The AB/BDI3,  $\mathcal{I}P_N - \mathcal{I}P_{N-2}$ , projection method [Section 7.4.2,  $\bar{p}^{n+1} = 2p^n - p^{n-1}$ ] : P3.

The following points will be discussed :

(a) Comparison of the accuracy of the pressure calculated by the IM and P2 methods. In the case of the IM method, the pressure is recovered either by filtering or by the solution of a Poisson equation.

(b) Evaluation of the effect of  $\bar{p}^{n+1}$  on the accuracy of the  $\mathcal{I}P_N - \mathcal{I}P_N$  projection methods (OP, IP and PP).

(c) Comparison of the accuracy in time of the IM, PP, P2 and P3 methods.

(d) Comparison of the stability of the IM, PP, P2 and P3 methods.

(e) Comparison of the spatial accuracy of the IM, PP and P2 methods.

(f) Comparison of the cost of the IM, PP and P2 methods.

These questions are now successively addressed.

#### (a) Accuracy of the pressure in IM and P2 methods

The application of the influence matrix method with the correction term  $\mathbf{B}$  produces spurious pressure modes which have to be removed to get an accurate pressure field. As discussed in Section 7.4.1.b, this can be done either by a filtering process or by the solution of a Poisson equation with a Neumann condition. These two techniques are evaluated and their results are compared to those given by the P2 method. This is done by considering the calculation of the analytical steady solution in  $\Omega = \{-1 < x, y < 1\}$  :

$$\begin{aligned} u &= \sin(\pi x/2) \cos(\pi y/2), \quad v = -\cos(\pi x/2) \sin(\pi y/2), \\ p &= (\cos \pi x + \cos \pi y) / 4 + 10(x + y). \end{aligned} \quad (7.279)$$

This defines the forcing term  $\mathbf{f}$  in Eq.(7.1) and the boundary value  $\mathbf{V}_\Gamma$ . The initial condition is the exact solution (7.279), and the time-integration (with  $\nu = 10^{-1}$ ) is pursued until the residual of the velocity is stabilized (at less than  $2 \times 10^{-12}$ ). The arbitrary constant in the pressure field is determined by matching the calculated and exact pressures at the center point of the domain  $\Omega_N$ . The calculations (Botella, 1998; Sabbah, 2000) are done with  $N_x = N_y = N$ . The accuracy of a quantity  $\phi$  is evaluated through the discrete  $L^2$ -norm based on  $\Omega_N^I$  :

$$\bar{E}_\phi = \left\{ \frac{1}{(N-1)^2} \sum_{i,j=1}^{N-1} [\phi_N^{n+1}(x_i, y_j) - \phi(x_i, y_j, t_{n+1})]^2 \right\}^{1/2} \quad (7.280)$$

where  $\phi_N$  is the computed solution and  $\phi$  is the exact one. Table 7.2 gives the errors on the  $u$ -component of the velocity (error  $\overline{E}_u$ ) and on the pressure (error  $\overline{E}_p$ ). The error  $\overline{E}_p^I$  refers to the pressure field obtained by the filtering process and the error  $\overline{E}_p^{II}$  to the one computed from the Poisson equation. The second technique is much more accurate and its accuracy is similar to the one associated with the P2 method (note that different computers have been used for the IM and P2 methods). The last column in Table 7.2 gives the maximal value  $Q_{max}$  of  $\nabla \cdot \mathbf{V}_N$  on the boundary given by the P2 method.

$N$	IM method		
	$\overline{E}_u$	$\overline{E}_p^I$	$\overline{E}_p^{II}$
12	$5.26 \times 10^{-11}$	$1.20 \times 10^{-5}$	$2.80 \times 10^{-8}$
16	$1.08 \times 10^{-14}$	$2.00 \times 10^{-6}$	$2.27 \times 10^{-12}$

$N$	P2 method		
	$\overline{E}_u$	$\overline{E}_p$	$Q_{max}$
12	$1.99 \times 10^{-11}$	$5.16 \times 10^{-7}$	$2.39 \times 10^{-9}$
16	$3.75 \times 10^{-13}$	$7.46 \times 10^{-11}$	$1.29 \times 10^{-11}$

Table 7.2. Errors given by the IM and P2 methods for the steady solution (7.279).

**(b) Influence of  $\overline{p}^{n+1}$  in OP, IP and PP methods**

This example is intended to evaluate the effect of the choice of  $\overline{p}^{n+1}$  on the accuracy in time of the  $\mathbb{P}_N - \mathbb{P}_N$  projection methods. This is done by considering a three-dimensional unsteady solution (Raspo *et al.*, 2002) in a cylindrical coordinate system  $(r, \theta, z)$  with  $\mathbf{V} = (u, v, w)$ . The annular domain  $\Omega$  is defined by  $1 \leq r \leq 2$ ,  $0 \leq \theta \leq 2\pi$ ,  $-1 \leq z \leq 1$ . The coordinate  $r$  is changed in  $\rho = r - 2$  so that  $-1 \leq \rho \leq 1$ . The numerical tests are done with the analytical time-periodic solution

$$\begin{aligned}
u &= (1/2\pi) (1 + \cos^2 4\pi t) \sin^2 \pi \rho \cdot \sin 2\pi z \cdot \cos \theta \\
v &= -(1/2\pi) (1 + \cos^2 4\pi t) \sin^2 \pi \rho \cdot \sin 2\pi z \cdot \sin \theta \\
w &= -(1/2\pi) (1 + \cos^2 4\pi t) \sin 2\pi \rho \cdot \sin^2 \pi z \cdot \cos \theta \\
p &= (\cos \pi \rho + \cos \pi z) \cos \theta + 10(x + y) \cos \theta \cdot \cos 4\pi t,
\end{aligned} \tag{7.281}$$

which defines the forcing term  $\mathbf{f}$  in Eq.(7.1), the boundary value  $\mathbf{V}_\Gamma$ , and the initial condition  $\mathbf{V}_0$ .

The spatial approximation makes use of the Fourier Galerkin method in the  $\theta$ -direction and Chebyshev collocation in the  $\rho$ - and  $z$ -directions (see Section 7.5). We denote by  $K$  the cut-off frequency of the truncated Fourier

series, and by  $N_\rho$  and  $N_z$  the degree of the polynomial approximation in  $\rho$ - and  $z$ -directions, respectively. The calculations are done with  $\nu = 2 \times 10^{-3}$  and the resolution  $2K = N_\rho = N_z = 40$ . This resolution is expected to be sufficient to ensure that the measured error is due only to the error in time for the considered values of the time-step  $\Delta t$ . However, as will be observed, the presence of a rather large spatial error associated with the normal pressure gradient at the boundary may require higher resolutions.

The interest in using a time-periodic solution like (7.281) is that the error itself is periodic after a transient stage. Thus, it is recommended integrating over a large number of periods in order to check the stability of the method, characterized by a constant maximal error defined by

$$E_\phi = \max_t \overline{E}_\phi(t) \quad (7.282)$$

where  $\overline{E}_\phi$  is defined by an equation similar to (7.280) based on the collocation points  $(\rho_i, \theta_j, z_k)$ . The error  $E_\phi$  characterizes the accuracy in time of the method. Figures 7.1 and 7.2 show, respectively, the errors  $E_u$  and  $E_p$  calculated on the inner collocation points. Figure 7.3 shows the error  $E_p^\Gamma$  calculated on the points belonging to the boundary of the computational domain. It may be observed that the spatial error of the IP method [curve (2)] becomes preponderant as soon as  $\Delta t < 10^{-3}$ . This threshold is lowered by increasing the resolution to  $2K = N_\rho = N_z = 54$  [curve (3)]. On the other hand, the temporal error of the OP method [curve (1)] is too large to make apparent the spatial error for the values of  $\Delta t$  considered. Lastly, the error associated with the PP method [curve (4)] shows a perfect decrease in  $\Delta t^2$ . Results concerning the rate of decrease of the errors are reported in Table 7.3. This table also gives the behaviour of the slip velocity  $V_s$  exhibited by Eq.(7.273).

Method	$\overline{E}_u$	$V_s$	$\overline{E}_p$	$\overline{E}_p^\Gamma$
OP	$\Delta t$	$\Delta t$	$\Delta t^{0.88}$	$\Delta t^{0.74}$
IP	$\Delta t^2$	$\Delta t^2$	$\Delta t^{1.65}$	$\Delta t$
PP	$\Delta t^2$	$\Delta t^3$	$\Delta t^2$	$\Delta t^2$

Table 7.3. Rate of decrease of the errors given by the OP, IP, and PP methods for the time-periodic solution (7.281).

These various numerical results clearly show the weaknesses of the OP and IP methods compared to the PP method. This latter constitutes an efficient way to apply the concept of projection associated with the  $\mathcal{P}_N - \mathcal{P}_N$  approximation in the strong collocation framework.

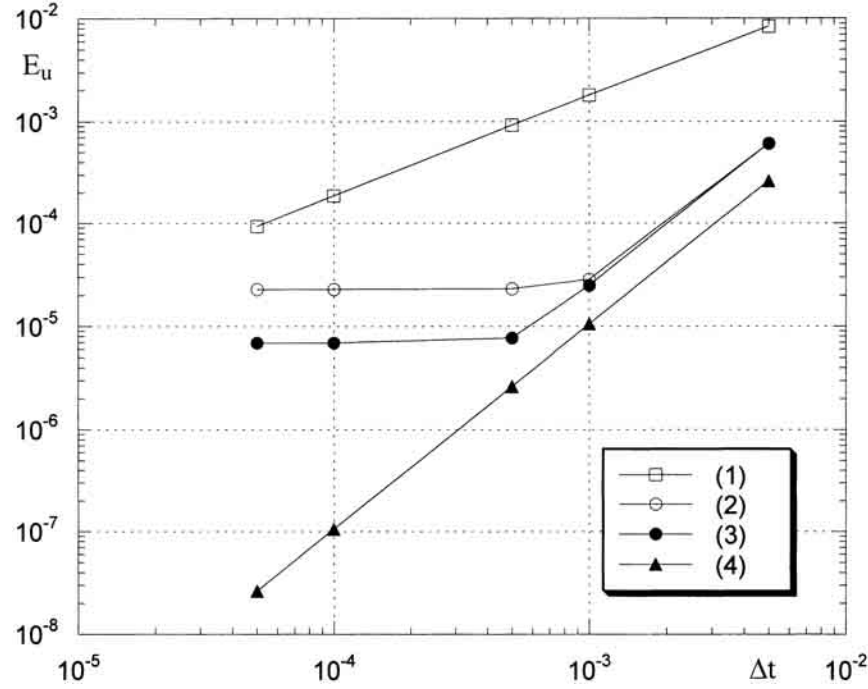


FIGURE 7.1. Error  $E_u$  for the solution (7.281) given by the projection methods : (1) OP method ( $40 \times 40 \times 40$  resolution), (2) IP method ( $40 \times 40 \times 40$  resolution), (3) IP method ( $54 \times 54 \times 54$  resolution), (4) PP method ( $40 \times 40 \times 40$  resolution).

### (c) Accuracy in time of the IM, PP, P2, and P3 methods

The purpose of this test case is to compare the accuracy in time of the IM, PP, P2, and P3 methods. The comparison is based on the calculation of the two-dimensional time-periodic solution similar to the steady solution (7.279), namely,

$$\begin{aligned} u &= \cos(5t) \sin(\pi x/2) \cos(\pi y/2), \quad v = -\cos(5t) \cos(\pi x/2) \sin(\pi y/2), \\ p &= \cos^2(5t) (\cos \pi x + \cos \pi y) / 4 + 10(x + y) \cos(5t), \end{aligned} \quad (7.283)$$

which defines the forcing term  $\mathbf{f}$  in Eq.(7.1), the boundary value  $\mathbf{V}_\Gamma$ , and the initial condition  $\mathbf{V}_0$ .

Numerical tests (Hugues and Randriamampianina, 1998 ; Botella, 1997; Sabbah, 2000) are performed for  $\nu = 10^{-2}$ ,  $N_x = N_y = 32$ , and  $\Delta t$  varying between  $10^{-4}$  and  $10^{-2}$ . The three second-order IM, PP and P2 methods give essentially the same errors  $E_u$  and  $E_p$  [defined by Eq.(7.282)]: they behave like  $\Delta t^2$  and their magnitude is nearly identical, namely,  $E_u \simeq 2.1 \times 10^{-4}$  and  $E_p \simeq 3.6 \times 10^{-3}$  for  $\Delta t = 10^{-2}$ . For the P3 method, we

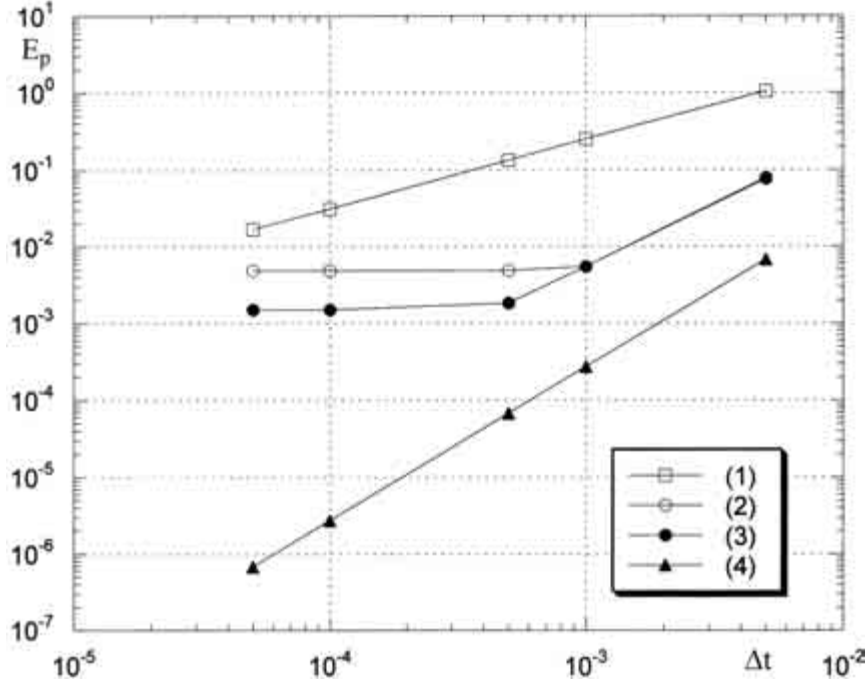


FIGURE 7.2. Error  $E_p$ , at inner points, for the solution (7.281) given by the projection methods : (1) OP method ( $40 \times 40 \times 40$  resolution), (2) IP method ( $40 \times 40 \times 40$  resolution), (3) IP method ( $54 \times 54 \times 54$  resolution), (4) PP method ( $40 \times 40 \times 40$  resolution).

obtain the  $\Delta t^3$ -behaviour with the typical values  $E_u \simeq 5.2 \times 10^{-5}$  and  $E_p \simeq 3.1 \times 10^{-4}$  for  $\Delta t = 10^{-2}$ .

#### (d) Stability of the methods

The stability of the various methods is compared by evaluating the maximum allowable time-steps in the calculation of the regularized driven cavity flow. This flow, which takes place in the square  $0 \leq X, Y \leq 1$ , is defined by the boundary conditions

$$u(X, 1) = -16X^2(1 - X)^2, \quad v(X, 1) = 0,$$

on the side  $Y = 1$  and  $u = 0, v = 0$  on the other three sides. The forcing term  $\mathbf{f}$  in Eq.(7.1) is zero. The initial condition is  $\mathbf{V}_0 = \mathbf{0}$ . The Reynolds number  $Re = \nu^{-1}$ , based on the side of unit length, is  $Re = 400$ . The application of the Chebyshev method necessitates the change of variable in order to transform the square  $[0, 1] \times [0, 1]$  into the square  $[-1, 1] \times [-1, 1]$ .

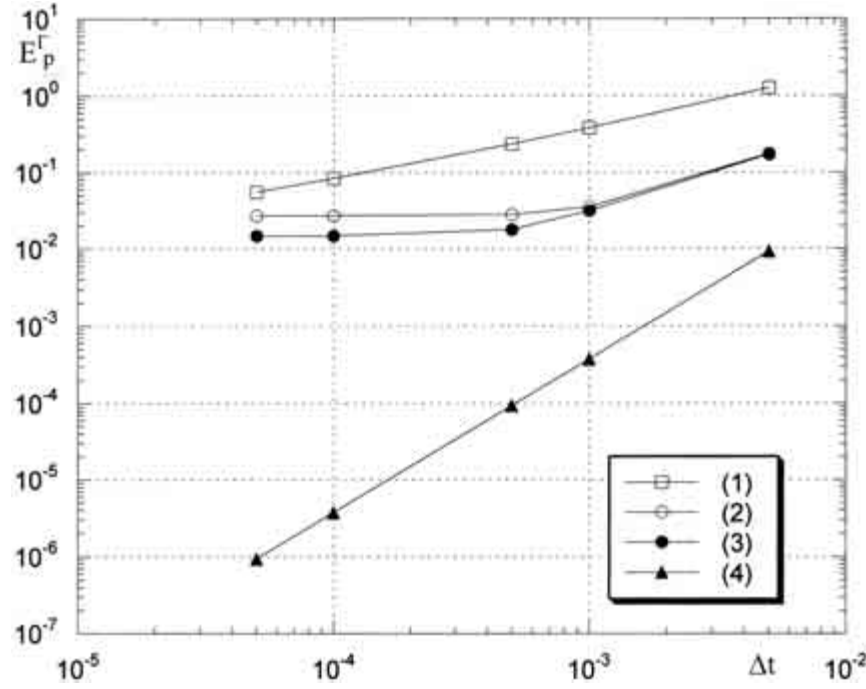


FIGURE 7.3. Error  $E_p^\Gamma$ , at the boundary, for the solution (7.281) given by the projection methods : (1) OP method ( $40 \times 40 \times 40$  resolution), (2) IP method ( $40 \times 40 \times 40$  resolution), (3) IP method ( $54 \times 54 \times 54$  resolution), (4) PP method ( $40 \times 40 \times 40$  resolution).

$N$	IM	PP	P2	P3	$\omega-\psi$
16	0.082	0.055	0.081	0.049	0.067
24	0.049	0.033	0.047	0.032	0.047
32	0.032	0.027	0.030	0.023	0.033

Table 7.4. Critical time-step  $\Delta t_\star$  for the regularized cavity flow at  $Re = 400$ .

Table 7.4. gives the critical time-step  $\Delta t_\star$  (estimated within an error of  $\pm 10^{-3}$ ) in the case of the IM, PP, P2, P3 methods and the AB/BDI2 influence matrix method applied to the vorticity-streamfunction equations (Section 6.4). The calculations are done by using the same polynomial degree in both directions  $N_x = N_y = N$ . The results show that the PP method requires time-steps slightly smaller than the other methods, the difference diminishing when the resolution is increased. Also, we may observe the decrease of the critical time-step when the order of accuracy increases (P2 and P3 methods).

$N$	IM	PP	P2	$\omega$ - $\psi$
16	24.6799 (0.60)	25.0387 (0.60)	24.7799 (0.60)	25.2329 (0.60)
24	24.9144 (0.63)	24.9180 (0.63)	24.9157 (0.63)	24.9344 (0.63)
32	24.7845 (0.65)	24.7844 (0.65)	24.7845 (0.65)	24.7845 (0.65)

Table 7.5. Maximal value  $M_1$  of the vorticity on  $Y = 1$  for the regularized cavity flow at  $Re = 400$ . The abscissa  $X_M$  of the point where the maximum is reached is given in parentheses.

$N$	IM	PP	P2	$\omega$ - $\psi$
16	25.0476 (0.63)	25.3854 (0.62)	25.1604 (0.625)	25.4675 (0.62)
24	24.9135 (0.63)	24.9170 (0.63)	24.9148 (0.63)	24.9333 (0.63)
32	24.9109 (0.63)	24.9107 (0.63)	24.9110 (0.63)	24.9110 (0.63)
40	24.9109 (0.63)	-	24.9109 (0.63)	-

Table 7.6. Maximal value  $M_2$  of the vorticity on  $Y = 1$  for the regularized cavity flow at  $Re = 400$ . The abscissa  $X_M$  of the point where the maximum is reached is given in parentheses.

#### (e) Spatial accuracy of the methods

The convergence of the solution with respect to the polynomial degree  $N$  is assessed by considering the steady regularized cavity flow at  $Re = 400$ , already considered in the previous section. The convergence is measured by examining a sensitive quantity such as the maximal value of the vorticity  $\omega = \partial_X v - \partial_Y u$  on the side  $Y = 1$ . Thus, we may define  $M_1$  as the maximum of  $\omega_N$  on the collocation points belonging to  $Y = 1$ . Such a quantity, however, is not really significant due to the unequal distribution of the Gauss-Lobatto points and the large variations of the boundary vorticity. So, it is much more significant to consider the maximal value of the polynomial  $\omega_N(x, 1)$  on  $x \in [-1, 1]$ . From the knowledge of the grid values  $\omega_N(x_i, 1)$ ,  $i = 0, \dots, N$ , the polynomial  $\omega_N(x, 1)$  is reconstructed either through the Lagrange interpolation polynomial (3.43) or through the Chebyshev expansion (3.17) after having calculated the Chebyshev coefficients with Eq.(3.36). Therefore, we denote by  $M_2$  the maximal value of  $\omega_N(x, 1)$  taken on a large number of equispaced points, say 201. Tables 7.5 and 7.6, respectively, display the values of  $M_1$  and  $M_2$  obtained with the IM, PP, P2 methods and with the influence matrix method applied to the  $\omega$ - $\psi$  equation (Section 6.4).

A comparison between the behaviour of  $M_1$  and  $M_2$  is very instructive. The evolution of  $M_1$ , when  $N$  increases, does not give any indication on the convergence of the solution, while the results for  $M_2$  exhibit a perfectly monotonic convergence. In particular, the relative difference between  $M_2$  obtained for  $N = 32$  and the converged value ( $M_2 = 24.9109$ ) is less than  $1.6 \times 10^{-4}$  for any method. This illustrates not only the high accuracy of the

Chebyshev method but also its superiority over local methods like finite-differences concerning the information which can be extracted from the numerical results. Indeed, the finite-difference method gives the solution only at some points of the computational domain. The spectral method gives much more, since it gives the approximate solution in the form of a polynomial valid for any location in the computational domain.

**(f) Cost of the IM, PP and P2 methods**

For every method the CPU time needed by the calculations in the preprocessing stage is negligible compared to the time spent during the integration in time, even for the IM method which necessitates important precalculations. For each of the IM, PP, and P2 methods, the computations performed at each time-cycle are essentially : (1) the evaluation of the right-hand sides, and (2) the solution of the Helmholtz problems. We assume that the right-hand sides are evaluated by matrix-matrix products and that the algebraic systems associated with the Helmholtz problems are solved by the matrix-diagonalization procedure. Therefore, the computational effort amounts to matrix-matrix products.

The cost of a method, in terms of CPU time, depends on :

- (1) the number of time-cycles needed to reach some prescribed time, with a prescribed accuracy in time,
- (2) the number of matrix-matrix products to be done at each time-cycle, and
- (3) the spatial resolution needed to reach some prescribed accuracy, since the size of the matrices has an effect on the CPU time needed to perform these products.

From the results discussed in previous Sections, it appears that the temporal and spatial accuracy is comparable for the three IM, PP, and P2 methods, whereas the size of the time-step required for stability is slightly smaller for the PP method (1.2-1.5 times). Moreover, the number of matrix-matrix products done at each time-cycle is 34 for the IM method, 28 for the PP method, and 20 for the P2 method. In conclusion, the P2 method is less costly whereas the PP method is generally more costly.

#### 7.4.5 *The form of the nonlinear term*

In Section 7.2.2, the influence of the form of the nonlinear term of the Navier-Stokes equations on the stability of large Reynolds number time-dependent solutions has been discussed for the Fourier method. It was pointed out that, in the absence of the forcing term  $\mathbf{f}$ , the semiconservation properties of the Navier-Stokes equations (i.e., conservation with  $\nu = 0$ ) were preserved for the Fourier Galerkin approximation. Therefore, Fourier Galerkin or, equivalently, dealiased Fourier collocation, approximation of the nonlinear term does not introduce instability. On the other hand, the same properties of semi-conservation do not hold for the Chebyshev approx-



imation (Canuto *et al.*, 1988). Since a theoretical analysis for the Chebyshev method is not available, we must have recourse to numerical experiments in order to understand the influence of the form of the nonlinear term.

Wilhelm and Kleiser (2000) have studied the case of a spectral-element method based on Legendre polynomials and staggered mesh. They consider the four forms of the nonlinear terms, namely, the convective, conservative, skew-symmetric, and rotational form (Section 5.1) and also the alternative rotational form where  $\mathbf{A}(\mathbf{V}, \mathbf{V}) = \boldsymbol{\omega} \times \mathbf{V} + \nabla(|\mathbf{V}|^2/2)$  while retaining  $\nabla p$ . Their numerical results, for channel flow at a Reynolds number equal to 7500, show that only the convective form and the above alternative rotational forms are stable, whereas the conservative, the skew-symmetric, and the usual rotational forms are unstable, whatever the time-discretization scheme, time-step, and spatial resolution. The argument put forward by Wilhelm and Kleiser (2000) to explain the different behaviour between the various forms is the lack of verification of the constraint  $\nabla \cdot \mathbf{V} = 0$  at the Gauss-Lobatto points where the nonlinear term is evaluated. However, it is likely that aliasing also has an influence.

From the numerical experiments (Botella *et al.*, 2001) reported below it seems that, in Chebyshev collocation methods, two effects are responsible for instability : aliasing and defective verification of the incompressibility condition. These experiments have been performed on the problem of the regularized cavity flow (Sections 7.4.4.d and 7.4.4.e) by means of the influence matrix method (IM-method, Section 7.4.1) and the  $\mathcal{P}_N - \mathcal{P}_{N-2}$  projection method (P2-method, Section 7.4.2). In both methods, the same second-order AB/BDI2 time-scheme is used and the nonlinear term is evaluated without aliasing removal. This term is considered in the four classical forms : convective, conservative, skew-symmetric, and rotational defined in Section 5.1.

In the IM method, the divergence of the approximate velocity  $\mathbf{V}_N$  is exactly zero everywhere, namely,  $Q_N = \nabla \cdot \mathbf{V}_N$  is the null polynomial, while in the P2 method  $Q_N$  is zero only at the inner collocation points. At the collocation points belonging to the boundary  $\Gamma_N^I$ ,  $Q_N$  is not zero but tends toward zero when the resolution is increased. This situation is different from the one considered by Wilhelm and Kleiser since a single collocation mesh is used here to enforce both momentum and incompressibility equations. For a given form of the nonlinear term, the aliasing error at inner collocation points is the same in both the IM and P2 methods. The only difference between both methods, concerning the nonlinear term, lies in the value of  $Q_N$  at the boundary.

For small values of  $Re$  (say  $Re = 400$ ) all the forms are stable, whatever the method, because viscous effects are preponderant. Different behaviours appear for larger values of  $Re$ .

For example, Table 7.7 displays the results obtained for  $Re = 2000$ . For the considered time-steps and spatial resolutions ( $N_x = N_y = N$ ) it is observed that :

Form	Method	$N = 32$		$N = 50$		$N = 80$	
		$10^2 \Delta t = 1$	0.5	1	0.05	0.5	0.25
CONV	IM	S	S	S	S	S	S
	P2	S	S	S	S	S	S
CONS	IM	S	S	S	S	S	S
	P2	U	U	U	U	U	U
S-S	IM	S	S	S	S	S	S
	P2	U	U	S	S	S	S
ROT	IM	S	S	S	S	S	S
	P2	S	S	S	S	S	S

Table 7.7. Stability properties, in the case  $Re = 2000$ , associated with the various forms of the nonlinear term (CONV = convective, CONS = conservative, S-S = skew-symmetric, ROT = rotational, for the influence matrix method (IM) and the  $\mathcal{P}_N - \mathcal{P}_{N-2}$  projection method (P2), S = stable, U = unstable).

Form	Method	
	IM	P2
CONV	S	S
CONS	U	U
S-S	S	U
ROT	S	S

Table 7.8. Stability properties, in the case  $Re = 7500$ ,  $N = 50$ , and  $\Delta t = 5 \times 10^{-3}$ , associated with the various form of the nonlinear term (CONV = convective, CONS = conservative, S-S = skew-symmetric, ROT = rotational, for the influence matrix method (IM), and the  $\mathcal{P}_N - \mathcal{P}_{N-2}$  projection method (P2), S = stable, U = unstable).

1. The IM method is stable whatever the form of the nonlinear term.
2. For the P2 method, the conservative form is unstable.
3. For the P2 method, the skew-symmetric form is unstable for low resolution ( $N = 32$ ).

Further computations performed with the P2 method and with the conservative form show that stability is obtained with  $N = 160$  and  $\Delta t = 2.5 \times 10^{-3}$ . The fact that the calculations are stable for these values, while they are unstable for the same time-step but with  $N = 80$ , makes evident that instability is not a result of the choice of too large time-steps. The nature of the instability is connected to the spatial approximation itself as was observed in the Chebyshev approximation to the advection-diffusion (Section 4.2.3) but the reasons are obviously different. From these results, it seems that, for  $Re = 2000$ , the observed instability is most likely due

more to the defective verification of the incompressibility condition at the boundary than the inner aliasing effects. By the way, it is interesting to point out that the same cavity flow at  $Re = 2000$  has been calculated using the vorticity-streamfunction equations and the collocation influence matrix method given in Section 6.4. The computations done for  $N = 32$  and  $\Delta t = 2 \times 10^{-2}$  are stable whatever the form of the nonlinear term [Eqs.(5.16) and (5.17)]. This is in agreement with the above argument since the computed velocity field is automatically solenoidal.

Other computations done with a higher Reynolds number, namely,  $Re = 7500$  with  $N = 50$  and  $\Delta t = 5 \times 10^{-3}$ , reported in Table 7.8, show that the conservative form is unstable even with the IM method. A possible explanation is that viscous effects are no longer sufficient to prevent the influence of the aliasing error. This is strengthened by the fact that stability is recovered by increasing the resolution to  $N = 80$  having, for effect, to reduce the aliasing terms.

In conclusion, it is clear that, for the considered Chebyshev collocation methods, the convective and rotational forms are the more stable. However, it must be pointed out that, at the present state of knowledge, the explanations put forward above must be taken with caution. Definitive conclusions could only be drawn after justification of the numerical results by theoretical analyses.

## 7.5 Example of application: three-dimensional flow in a rotating annulus

In this section we present an application of the  $\mathcal{P}_N - \mathcal{P}_N$  predicted projection method (PP method) presented in Section 7.4.2. The application (Serre *et al.*, 2001) concerns three-dimensional flow in a rotating annulus with forced flux. This configuration has already been considered in Section 6.5.2 in the axisymmetric case characterized by a relatively low value of the dimensionless flow rate :  $C_w \leq 120$ . Here we consider the three-dimensional flow which appears for larger values of  $C_w$ . The problem is made dimensionless by means of the following characteristic quantities :  $h$  for length,  $\Omega^{-1}$  for time,  $\Omega^{-1}R_1$  for velocity, and  $(R_m + 1)L^2\Omega^2h^2$  for pressure. We refer to Section 6.5.2 for the definition of these various quantities. The dimensionless equations, considered in the rotating frame of reference  $(\rho, \theta, z)$  with  $\mathbf{V} = (u, v, w)$ , are

$$\begin{aligned} & \partial_t u + A_u + \partial_\rho p \\ & -E \left[ \nabla^2 u - \frac{u}{L^2(R_m + \rho)^2} - \frac{2}{L^2(R_m + \rho)^2} \partial_\theta v \right] - 2v = 0 \end{aligned} \quad (7.284)$$

$$\partial_t v + A_v + \frac{1}{R_m + \rho} \partial_\theta p - E \left[ \nabla^2 v - \frac{v}{L^2 (R_m + \rho)^2} + \frac{2}{L^2 (R_m + \rho)^2} \partial_\theta u \right] + 2u = 0 \quad (7.285)$$

$$\partial_t w + A_w + L \partial_z p - E \nabla^2 w = 0 \quad (7.286)$$

$$\partial_\rho u + \frac{u}{R_m + \rho} + \frac{1}{R_m + \rho} \partial_\theta v + L \partial_z w = 0, \quad (7.287)$$

where  $(A_u, A_v, A_w)$  is the convective term,  $\nabla^2$  is the Laplacian operator,

$$\nabla^2 = \frac{1}{L^2} \partial_{\rho\rho} + \frac{1}{L^2 (R_m + \rho)} \partial_\rho + \frac{1}{L^2 (R_m + \rho)^2} \partial_{\theta\theta} + \partial_{zz}, \quad (7.288)$$

and  $E$  is the Ekman number defined by Eq.(6.149).

The above equations are solved in the domain  $(-1 \leq \rho \leq 1, 0 \leq \theta \leq 2\pi, -1 \leq z \leq 1)$ . The boundary conditions prescribed at  $\rho = \pm 1$  are deduced from the Ekman boundary layer solution, namely,

$$\begin{aligned} u &= V_g [e^{Z_+} \sin Z_+ - e^{Z_-} \sin Z_-], \\ v &= -V_g [e^{Z_+} \cos Z_+ - e^{Z_-} \cos Z_-], \\ w &= 0, \end{aligned}$$

where  $V_g = C_w E^{1/2} / [2\pi L (R_m + \rho)]$  and  $Z_\pm = -E^{-1/2} (1 \pm z)$ . The no-slip conditions  $u = 0, v = 0, w = 0$  are prescribed at  $z = \pm 1$ . The initial condition will be specified later. The problem being periodic in the azimuthal direction, the spatial approximation makes use of truncated Fourier series in  $\theta$ . Thus, each dependent variable  $\phi = u, v, w, p$  is approximated according to

$$\phi_K(\rho, \theta, z, t) = \sum_{k=-K+1}^K \hat{\phi}_k(\rho, z, t) e^{ik\theta}. \quad (7.289)$$

Then the Galerkin equations (see Section 6.2.1) determining the Fourier coefficients  $\hat{u}_k, \hat{v}_k, \hat{w}_k$ , and  $\hat{p}_k$  are

$$\begin{aligned} \partial_t \hat{u}_k + \hat{A}_{u,k} + \partial_\rho \hat{p}_k - E \left[ \nabla^2 \hat{u}_k - \frac{k^2 + 1}{L^2 (R_m + \rho)^2} \hat{u}_k - \frac{ik}{L^2 (R_m + \rho)^2} \hat{v}_k \right] - 2\hat{v}_k &= 0 \end{aligned} \quad (7.290)$$

$$\begin{aligned} \partial_t \hat{v}_k + \hat{A}_{v,k} + \frac{ik}{R_m + \rho} \hat{p}_k - E \left[ \nabla^2 \hat{v}_k - \frac{k^2 + 1}{L^2 (R_m + \rho)^2} \hat{v}_k + \frac{i}{L^2 (R_m + \rho)^2} \hat{u}_k \right] + 2\hat{u}_k &= 0 \end{aligned} \quad (7.291)$$

$$\partial_t \hat{w}_k + \hat{A}_{w,k} + L \partial_z \hat{p}_k - E \left[ \bar{\nabla}^2 \hat{w}_k - \frac{k^2}{L^2 (R_m + \rho)^2} \hat{w}_k \right] = 0 \quad (7.292)$$

$$\partial_\rho \hat{u}_k + \frac{\hat{u}_k}{R_m + \rho} + i \frac{k \hat{v}_k}{R_m + \rho} + L \partial_z \hat{w}_k = 0, \quad (7.293)$$

with associated boundary conditions at  $\rho = \pm 1$ ,  $z = \pm 1$ . The operator  $\bar{\nabla}^2$  is the Laplacian operator (7.288) without the  $\theta$ -term. The problem is solved by using the second-order version of the splitting scheme (7.238)-(7.242) and the  $\mathcal{P}_N - \mathcal{P}_N$  projection method described in Section 7.4.3.e. Thus, the solution method is as follows :

1. The predicted pressure  $\bar{p}^{n+1}$  is calculated from equations similar to Eqs.(7.264)-(7.265).

2. The provisional velocity  $\tilde{\mathbf{V}}_k^{n+1} = (\tilde{u}_k^{n+1}, \tilde{v}_k^{n+1}, \tilde{w}_k^{n+1})$  is calculated from equations similar to Eqs.(7.238)-(7.239). The nonlinear term  $(\hat{A}_{u,k}, \hat{A}_{v,k}, \hat{A}_{w,k})$  is calculated through the pseudospectral technique (Section 6.3.3) based on even collocation (Section 2.4) defined by the collocation points  $\theta_j = 2\pi j/N_\theta$ ,  $j = 0, \dots, N_\theta$ , with  $N_\theta = 2K$  and  $\rho_i = \cos(\pi i/N_\rho)$ ,  $i = 0, \dots, N_\rho$ ,  $z_k = \cos(\pi k/N_z)$ ,  $k = 0, \dots, N_z$ . To avoid the coupling of Eqs.(7.290) and (7.291) through the terms  $i k \hat{v}_k$  and  $i k \hat{u}_k$ , Serre *et al.* (2001) evaluated these terms in an explicit way using the Adams-Bashforth extrapolation. Note that, if the equations were considered in a fixed frame of reference [the Coriolis term  $(-2 \hat{v}_k, \hat{u}_k)$  being absent], the coupling could be avoided, while considering an implicit discretization, by introducing (Orszag and Patera, 1983) the new dependent variables

$$\tilde{\lambda}_k^{n+1} = \tilde{u}_k^{n+1} + i \tilde{v}_k^{n+1}, \quad \tilde{\mu}_k^{n+1} = \tilde{u}_k^{n+1} - i \tilde{v}_k^{n+1}.$$

Then uncoupled equations for  $\tilde{\lambda}_k^{n+1}$  and  $\tilde{\mu}_k^{n+1}$  are obtained by addition and subtraction of the two momentum equations.

3. The solution  $(\hat{u}_k^{n+1}, \hat{v}_k^{n+1}, \hat{p}_k^{n+1})$  is obtained from the projection step (7.240)-(7.242) solved through a Neumann problem analogous to (7.271)-(7.272).

Serre *et al.* (2001) did a detailed study of the problem by varying the flow rate  $C_w$  and the curvature parameter  $R_m$ . Here we want to illustrate the application of the PP projection method by considering a typical case, namely,  $C_w = 530$ ,  $E = 2.24 \times 10^{-3}$ ,  $L = 3.37$ , and  $R_m = 5$ . The calculations are done with  $N_\rho = N_z = 48$ ,  $K = 32$ , and  $\Delta t = 4 \times 10^{-3}$ . This flow is computed by using, as an initial condition, the axisymmetric flow at  $C_w = 460$ . The resulting flow is time-periodic and again axisymmetric but it becomes three-dimensional after an instantaneous small disturbance was applied to the azimuthal velocity near the entrance. This three-dimensional



FIGURE 7.4. Instantaneous iso-surface of the axial velocity component ( $w = -3.5 \times 10^{-4}$ ) of the time-periodic three-dimensional flow for  $C_w = 530$ ,  $E = 2.24 \times 10^{-3}$ ,  $L = 3.37$  and  $R_m = 5$ .

flow is illustrated in Figure 7.4 which displays the instantaneous iso-surface of the axial velocity component ( $w = -3.5 \times 10^{-4}$ ). The iso-surface exhibits a spiral pattern whose number of arms (twelve) depends on the wavelength of the imposed disturbance. Figure 7.5 displays the representation of the same iso-surface pattern in a Cartesian frame.

### Remark

#### *The axis singularity*

In the example considered above, the computational domain does not contain the axis  $r = 0$ , that is, the coordinate singularity associated with the axis is avoided. In the general cylindrical case  $(r, \theta, z)$  with  $0 \leq r \leq R$ , the behaviour of the solution as  $r \rightarrow 0$  must be taken into account. The requirement for analyticity of the physical (scalar or vector) quantities  $\phi(r, \theta, z, t)$  implies some conditions in the form of the approximation in the  $r$ -direction (Orszag and Patera, 1983 ; Tuckerman, 1989 ; Pasquetti and Bwemba, 1994; Priymak, 1995). In practice, only some of these conditions are actually prescribed, for example the Laplacian of  $\phi$  is required to be nonsingular. The weaker form of the regularity condition is the requirement that  $\phi$  is single-valued at  $r = 0$ . This implies the vanishing of the azimuthal derivative

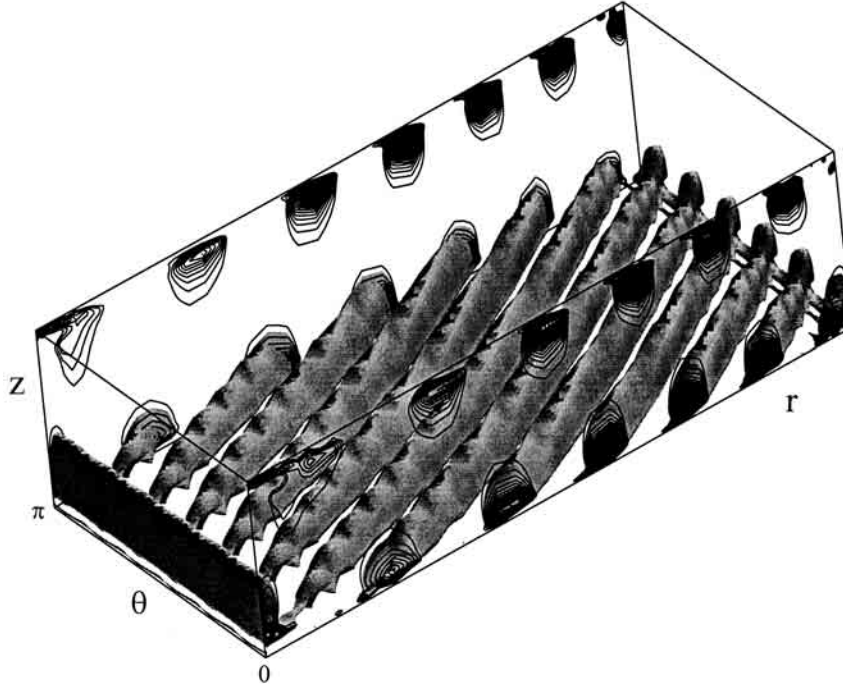


FIGURE 7.5. Representation in a cartesian frame of the instantaneous iso-surface of the axial velocity component ( $w = -3.5 \times 10^{-4}$ ).

$\partial_\theta \phi$  at  $r = 0$ . The result is that the Fourier coefficients  $\hat{\phi}_k(r, z, t)$  are such that  $\hat{\phi}_k(0, z, t) = 0$  for  $k \neq 0$ , but the value  $\hat{\phi}_0(0, z, t)$  remains undetermined.

Various ways are possible to avoid having to prescribe boundary conditions on the axis :

(1) Consider the interval  $-R \leq r \leq R$  and the Gauss-Lobatto points  $\rho_i = r_i/R$ ,  $i = 0, \dots, N_r$ , where  $N_r$  is even, so that  $r = 0$  is not a collocation point (Pulicani and Ouazzani, 1991 ; Pasquetti and Bwemba, 1994).

(2) Consider the interval  $0 \leq r \leq R$  and the Gauss-Radau points (see Appendix A), so that the origin  $r = 0$  does not belong to the set of collocation points (Le Marec *et al.*, 1996 ; Manna and Vacca, 1999).

(3) Consider the interval  $0 \leq r \leq R$  and the Gauss-Lobatto points, but introduce the change of dependent variable  $\tilde{\phi} = r\phi$  so that  $\tilde{\phi}$  satisfies the boundary condition  $\tilde{\phi}(0, \theta, z, t) = 0$  (Pulicani and Ouazzani, 1991 ; Serre and Pulicani, 2001). The value of  $\phi$  on the axis  $r = 0$  is computed from the value of  $\partial_r \phi$  at  $r = R$ . Because this later value depends on  $\theta$ , the continuity of  $\phi$  at  $r = 0$  is obtained only approximately and needs the resolution to be large enough.

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