To Albert and Antoni

Preface

The basic theory of plane birational maps was first stated by L. Cremona in his two memoirs [14] (1863), [15] (1865), and henceforth plane birational maps are known as plane Cremona maps. Other geometers soon brought substantial additions. Historical résumés can be found in [34] XVII and [12].

To start with, let us explain the connection between plane Cremona maps, linear systems and clusters of points. To a given plane Cremona map $\Phi : \mathbb{P}_1^2 \dashrightarrow \mathbb{P}_2^2$ we associate the linear system (net) $\mathcal{H}$ in $\mathbb{P}_1^2$ without fixed part which is the inverse image by Φ of the net of lines of $\mathbb{P}_2^2$. The net $\mathcal{H}$ determines the map Φ up to a projectivity of $\mathbb{P}_2^2$: there is a projectivity $u : \mathbb{P}_2^2 \longrightarrow \mathcal{H}^*$ so that $u \circ \Phi$ is equal to the map $\mathbb{P}_2^2 \dashrightarrow \mathcal{H}^*$, with $\mathcal{H}^*$ the projective space dual to $\mathcal{H}$, which sends $x \in \mathbb{P}_1^2$ to the hyperplane in $\mathcal{H}$ consisting of the divisors passing through x. Observe that a point $x \in \mathbb{P}_1^2$ is fundamental for Φ (i.e., x belongs to the closed subset where Φ cannot be defined as a morphism) if and only if x is a base point of the net $\mathcal{H}$. Consider the set of base points of $\mathcal{H}$: not only the fundamental points of Φ, which are proper points in $\mathbb{P}_1^2$, but also the infinitely near base points, which are proper points in a suitable surface obtained from $\mathbb{P}_1^2$ by successive blowing-ups. They form a weighted cluster $\mathcal{K} = (K, \mu)$, where K are the base points of $\mathcal{H}$ and μ assigns to each $p \in K$ the multiplicity at p of generic curves in $\mathcal{H}$. The pair $\mathcal{K}$ will be called the weighted cluster of base points of Φ. This book studies the plane Cremona maps from the viewpoint of the geometry of their weighted clusters of base points.

Accounts of the classical development of the theory of plane Cremona maps are given by Hudson [34], Godeaux [28], [29], Coble [11], Enriques-Chisini [26], Semple-Roth [47], Coolidge [13]. All of them deal systematically only with plane Cremona maps whose base points are all proper; infinitely near base points do appear just in examples. By contrast, infinitely near points already appear in the easiest examples of plane Cremona maps, as for instance the quadratic ones. Besides the well-known ordinary quadratic transformation there are two other types (special quadratic transformations) which have, respectively, one and two infinitely near base points (see section 2.8).

The structure of the group of plane Cremona transformations (plane Cremona group) has received a good deal of attention by modern literature:

modern proofs of Noether's factorization theorem (to which chapter 8 is devoted) can be found in [41] and [48], the relations between the generators of the plane Cremona group are given in [27] and [36], the classification of birational plane involutions, studied in the classic works of [4], [34] V or [28] 39, has been treated recently in [2], the result of Enriques [25] of the determination of the maximal connected algebraic subgroups of the plane Cremona group, also collected in the classic treatises of [28], [29], [34], has modern versions and extensions to dimension $n > 2$ in [16], [32], [50], [49]. Also the classic work of Coble [11] about the relationship between plane Cremona transformations, ordered finite point sets in the projective plane, and automorphisms of rational surfaces obtained by blowing up certain point sets in the plane and their extension to dimension $n > 2$ has a modern treatment in [17], [18], [19], [33] and [40].

On the other hand, in modern times little attention seems to have been paid to the examination of plane Cremona maps for their own sake. The purpose of this book is to contribute to filling this gap: to recover the classical results in updated versions, to extend them for arbitrary plane Cremona maps, dropping the hypothesis of proper base points by allowing any configuration of singularities for the base points, and to develop further properties. In order to present these matters adequately it has been necessary to include an exposition of the whole theory, embracing the classic results. Thus this book presents an exposition, in a reasonably self-contained way, of the theory of plane Cremona maps, studying the configurations of singularities of the base points, without the restrictive classical hypothesis that all base points be proper points. It is not the purpose of this book to study the above quoted aspects concerning the structure of the plane Cremona group, beyond Noether's factorization theorem, and their relation to the ordered finite point sets in the plane, as for them modern versions are available.

Throughout this book the base field is the complex one and we shall mean by surface a smooth projective irreducible surface over the field of the complex numbers $\mathbb{C}$.

Before giving an outline of the different chapters, let us introduce some definitions and set the framework where the study of plane Cremona maps is developed. There exists a triple (S, Π_K, Π_L), with S a surface, $\Pi_K : S \longrightarrow \mathbb{P}^2_1$ the blowing-up of the points of K and $\Pi_L : S \longrightarrow \mathbb{P}^2_2$ the blowing-up of the cluster of base points L of the inverse map Φ^{-1}, which is determined by the universal property of being final object of the category whose objects are the triples (S', g_1, g_2), with $g_i : S' \longrightarrow \mathbb{P}^2_i$ for $i = 1, 2$ morphisms of surfaces that commute with the map Φ. For each base point $p \in K$ let $\overline{E}_p$ be its inverse image on S and denote by E_p the component of $\overline{E}_p$ which is the strict transform on S of the exceptional divisor of blowing up p. Analogously, for each base point $q \in L$ of the inverse we use the notation $\overline{F}_q$ and F_q. For any fixed $p \in K$, consider the divisor $\Pi_{L*}(E_p)$ on $\mathbb{P}^2_2$. It is zero if and only if there is a point $q \in L$ so that $E_p = F_q$. Then p is called non-expansive and we say that (p,q) is a pair of non-expansive corresponding points. Otherwise $\Pi_{L*}(E_p)$ is an irreducible, rational curve Ω_p, which is then called the principal curve of Φ relative to p. We say in this case that p is expansive. By contrast, if we consider the inverse image $\overline{E}_p$, then the divisor $\Pi_{L*}(\overline{E}_p)$ is always a curve Θ_p that is called total principal curve.

A map all of whose base points are proper points in the plane is called simple. A simple map whose inverse is also simple is called bisimple (which is the case studied by the classics). In the simple case for each base point $p \in K$ we have a curve in $\mathbb{P}^2_2$ which equals $\Omega_p = \Theta_p$ and is the materialization in the plane of the tangent directions at p. Thus, in the bisimple case we handle all the information when working on the plane, and nice results hold as, e.g., the fact that for any $p \in K$ and $q \in L$ the multiplicity of Ω_p at q equals that of Ω_q at p. But when non-expansive base points appear, something is lost by blowing down from S to the plane, and for instance the above quoted symmetry between principal curves and base points ceases to hold in the general case. Hence most of the classic proofs, which limit to reasonings of plane projective geometry, are not valid.

The net $\mathcal{H}$ associated to Φ has the property of being homaloidal: the pencil of curves of $\mathcal{H}$ that go through a generic point x in the plane has no other (proper or infinitely near) base points further than $\{x\} \cup K$. The curves in $\mathcal{H}$ are called homaloidal curves and $\mathcal{H}$ is called homaloidal net of Φ. This property is equivalent to saying that there is a dense Zariski-open subset U of $\mathbb{P}^2_1$ so that for all $x \in U$ the fibre $\Phi^{-1}(\Phi(x))$ equals $\{x\}$ as a scheme, that is, Φ induces an isomorphism of U onto $\Phi(U)$, which means that Φ is birational.

Now we briefly summarize the main contents of each chapter.

Chapter 1 is of preliminary nature: concepts and well-known results about birational maps of surfaces, blowing-ups and weighted clusters are reviewed and some consequences, which are needed in subsequent chapters, are derived.

Chapters 2 and 3 are devoted to a series of classical results. Chapter 2 introduces the basic concepts related to plane Cremona maps such as simplicity and bisimplicity, principal and total principal curve, expansive and non-expansive base point, and studies their properties.

The first part of chapter 3 extends to an arbitrary plane Cremona map a pair of classic theorems proved by Clebsch [9]. The degree n of the homaloidal curves (which is called degree of Φ) and the multiplicities $\mu_1, \dots, \mu_\sigma$ of all the base points of Φ make up a vector $(n; \mu_1, \dots, \mu_\sigma)$ that is called characteristic of the map. The characteristics of the direct and inverse maps have a high degree of symmetry. As an illustration, we shall reproduce here the characteristics $(6; 4, 2, 2, 2, 2, 1, 1, 1)$ of the sextic plane Cremona map appearing in 2.1.14 and $(6; 3, 3, 3, 2, 1, 1, 1, 1)$ of its inverse. If we group equal multiplicities together, there is a bijection between groups of the direct and inverse maps so that corresponding groups have the same cardinal. This phenomenon is formalized in the following result (that is called Clebsch's theorem, because it generalizes to an arbitrary map the same thesis proved by Clebsch for bisimple maps): there exists a bijection $\beta : \mathbb{N} \longrightarrow \mathbb{N}$ so that for each $m \in \mathbb{N}$ the number of base points of Φ with multiplicity m equals the number of base points of Φ^{-1} with multiplicity $\beta(m)$. The second part of chapter 3 describes the components of the jacobian of the homaloidal net. Given a net, its jacobian is defined as the locus of the singular points of the curves of the net. For a homaloidal net we prove that its jacobian is the sum of the total principal curves relative to all the base points of the inverse map (each one counted once). This extends the expression already known to the classics: the jacobian of the homaloidal net of a bisimple map is the reduced curve composed of the principal curves.

The goal of chapter 4 is to obtain information about the composite map from the component maps. Given a plane Cremona map Φ, we define its characteristic matrix as the $(\sigma + 1)$-square matrix whose first row is the characteristic of Φ, whose first column is the characteristic of Φ^{-1} (save a change of sign of the multiplicities), and the rest of entries are the intersection numbers $-\overline{E}_p \cdot \overline{F}_q$ on S for each $p \in K$ and $q \in L$. Once we are given the characteristic matrices and the clusters of base points of the direct and inverse component maps and also the relative position of their base points and principal curves in the intermediate plane, we give the characteristic matrix of the composition map, its cluster of base points and that of its inverse. Classical results about composite maps are partial and always refer to bisimple maps.

Chapter 5 is devoted to studying the characteristic matrix of an arbitrary plane Cremona map Φ. First we focus on its first row, $(n; \mu_1, \ldots, \mu_\sigma)$, which is the characteristic of Φ. The following two relations (already known to the classics for bisimple maps) do hold: $\mu_1^2 + \cdots + \mu_\sigma^2 = n^2 - 1$ and $\mu_1 + \cdots + \mu_\sigma = 3n - 3$. According to the classical nomenclature we will refer to them as the first and second equations of condition. The first equation of condition expresses that two generic homaloidal curves cut transversally at a unique point other than those in K. The second relation says that the homaloidal curves are rational. This leads to define a homaloidal type as a solution to the equations of condition; it is called proper if it is essentially (by dropping the zero entries) the characteristic of some plane Cremona map; otherwise it is called improper. The problem that we tackle is to know which solutions to the equations of condition are in fact characteristics of some plane Cremona map, that is, to characterize the proper homaloidal types. Next we study the same question for a row (not the first one) $-(\nu; \varepsilon_1, \ldots, \varepsilon_\sigma)$ of the characteristic matrix: its entries satisfy the equations $\varepsilon_1^2 + \cdots + \varepsilon_\sigma^2 = \nu^2 + 1$ and $\varepsilon_1 + \cdots + \varepsilon_\sigma = 3\nu - 1$. An exceptional type is defined as a solution to these equations; it is called proper if it comes from a row of the characteristic matrix of some plane Cremona map; otherwise it is called improper. Here the goal is to characterize the proper exceptional types. Lastly, the analogous problem for the whole characteristic matrix is raised in the following terms: the characteristic matrix of a plane Cremona map satisfies two arithmetical properties, which include, in particular, the equations of condition and those of the exceptional types. The invertible σ-square matrices with integral entries fulfilling these arithmetical properties form a group Γ_σ; denote by W_σ its subgroup of the characteristic matrices of all plane Cremona maps with at most σ base points. The question in this context is to know which elements of Γ_σ belong to W_σ. We prove that the above three problems are deeply related: an element $T \in \Gamma_\sigma$ belongs to W_σ if and only if the first row of T is a proper homaloidal type, and this is equivalent to saying that each row (but for the first) of T is a proper exceptional type. To characterize the proper homaloidal types Hudson [34] in 1927 outlined a test without proof, which seemed to fall into oblivion afterwards (cf. later works of Semple-Roth [47] in 1949 and Coble [11] in 1961, where this question is left open). In this book we prove three characterizations of proper homaloidal types; one of them is Hudson's algorithmic test. Its proof requires a result that is worth mentioning: there exist simple plane Cremona maps with fixed characteristic and whose base points can be chosen generically in the plane. Hudson's test is adapted to characterize proper exceptional types as well. We come to the main property of the characteristic matrix: its entries depend only on its first row, which is the characteristic of the map. A method to compute the whole characteristic matrix and, in particular, the characteristic of the inverse map from the characteristic of the direct map is given. Hence the proximity relations among the base points do not affect the characteristic matrix, neither do their particular projective positions. Coolidge [13] in 1931, Du Val [21] in 1936 and Coble [11] in 1961 dealt with the characterization of the elements of Γ_σ belonging to W_σ: Coolidge wrongly affirms the equality $W_\sigma = \Gamma_\sigma$, Du Val sees that the equality $W_\sigma = \Gamma_\sigma$ holds if and only if $\sigma \leq 9$ and Coble considers the problem as open. We provide a complete and updated proof of Du Val's result, which is based on the fact already noticed by the classics that all the homaloidal (exceptional) types are proper if and only if $\sigma \leq 8$ ($\sigma \leq 9$). Still following Du Val's line we identify W_σ as a Weyl group, we reprove the known fact that the order of W_σ is finite if and only if $\sigma \leq 8$, we establish the connection between proper exceptional types and exceptional curves of the first kind on surfaces obtained from the plane by successive blowing-ups, and we infer the two well-known classical results about the number of lines on a del Pezzo surface and the existence of rational surfaces carrying infinitely many exceptional curves of the first kind. Lastly, as a consequence of the techniques developed in this chapter (and those of chapter 4), it is inferred that, if the characteristics of the component maps and the multiplicities at the coincident base points in the intermediate plane are given, then the characteristic of the composition is completely determined.

Chapter 6 contains two parts dealing with different topics, both essentially novel. The first one deals with total principal curves of a plane Cremona map Φ, studies their effective behaviour at the base points K of the map and compares it to two virtual behaviours determined from the characteristic of Φ.

More precisely, fixed a base point q of the inverse, we define two other systems of multiplicities for the same underlying cluster K, besides the effective multiplicities of Θ_q at K. The first, $\mathcal{K}_q$, is formed from the entries $\{\overline{E}_p \cdot \overline{F}_q\}_{p \in K}$ of the q-th row of the characteristic matrix of Φ; it is called virtual behaviour of Θ_q, because Θ_q goes virtually through $\mathcal{K}_q$ (furthermore Θ_q is the unique curve going through $\mathcal{K}_q$ of degree ν_q, which is the multiplicity of Φ^{-1} at q). Secondly, we take the system of effective multiplicities of generic curves of a suitably high degree going through $\mathcal{K}_q$, and we call it generic behaviour. Equalities between effective, virtual and generic behaviours are characterized. Notice that for a bisimple map these three behaviours coincide, which explains why this question is not tackled in the classical literature. The second part of chapter 6 studies the homaloidal curves whose effective multiplicities at the base points are different from those of generic homaloidal curves. By Bertini's theorem on linear systems, a generic homaloidal curve is irreducible and goes through $\mathcal{K}$, the weighted cluster of base points of the map, with effective multiplicities equal to the virtual ones. We characterize homaloidal curves failing to go through $\mathcal{K}$ with effective multiplicities equal to the virtual ones, we prove that they are reducible and we determine their effective multiplicities at K.

In chapter 5 we have calculated the characteristic of the inverse map from the characteristic of the direct map. We go further away in chapter 7 and we ask for the relative position of the base points L of the inverse map, that is, we want to know whether they are proper or infinitely near and, in this case, to find out which points they are proximate to (a point p is said to be proximate to another point q if p lies on the exceptional divisor of blowing up q or on one of its strict transforms). This information is encoded in a matrix $\mathbf{P}_L$ called proximity matrix of the cluster L. Two different approaches to describe $\mathbf{P}_L$ from the weighted cluster $\mathcal{K}$ and the relative position in the plane of the points of K (which appear to be new) are discussed. They involve not only the proximity relations between the points of K but also some projective information about K, as it can be expected from the existence of simple plane Cremona maps that are not bisimple (phenomenon already observed by the classics, e.g. [28]). The projective information is collected in the first procedure from the detection in $\mathbb{P}^2_1$ of the principal curves of Φ^{-1} and in the second method by checking some inclusion relations in $\mathbb{P}^2_1$ between the total principal curves of Φ^{-1}.

In chapter 8 we use the tools and results developed in this book to give a new proof of the well-known Noether factorization theorem [43], [44] which says that every plane Cremona map Φ is composed of ordinary quadratic transformations. Most of the classic proofs are based on Noether's inequality: the sum of the three highest multiplicities at base points exceeds the degree n of Φ. If we had a quadratic map τ whose three base points were coincident with these of Φ having the three highest multiplicities, then the degree of $\Phi \circ \tau^{-1}$ would be strictly lower than n, and by induction the result would follow. But such a τ does not always exist, e.g. if two of the base points are proximate to the third. We follow Castelnuovo's line (which had not received a rigorous treatment until now), and we factorize any plane Cremona map into de Jonquières maps and any de Jonquières map into ordinary quadratic transformations. We use two criteria to know whether the linear system $\mathcal{L}$ defined by a given cluster in the plane and a given proper homaloidal type is a homaloidal net: a result due to Enriques that asserts that $\mathcal{L}$ is a homaloidal net if and only if generic curves in $\mathcal{L}$ are irreducible is recovered (in chapter 5) and it is adapted to achieve an arithmetical characterization.

This book is intended to be accessible to any mathematician who is interested in the topic of plane Cremona maps. For this sake, we chose the notation and nomenclature mostly inspired by the classical ones and those of [3] and [6], which suit very well when studying plane Cremona transformations.

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