

$$\lim_{n \rightarrow \infty} \|w_n(s_j \otimes 1) - (s_j \otimes 1)w_n\| = 0,$$

and

$$\begin{aligned} \text{dist}(w_n^*(1 \otimes s_j)w_n, \mathcal{O}_2 \otimes 1) &\leq \|w_n^*(1 \otimes s_j)w_n - \rho_n(s_j) \otimes 1\| \\ &= \|\psi_n(w^*(1 \otimes s_j)w - s_j \otimes 1)\| < \varepsilon, \end{aligned}$$

for $j = 1, 2$. We can therefore take v to be w_n for some large enough n . \square

Theorem 5.2.1 clearly implies that

$$\bigotimes_{n=1}^m \mathcal{O}_2 \cong \mathcal{O}_2$$

for all natural numbers m . Applying this and Corollary 5.1.5 to the sequence

$$\mathcal{O}_2 \longrightarrow \mathcal{O}_2 \otimes \mathcal{O}_2 \longrightarrow \mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \mathcal{O}_2 \longrightarrow \cdots \longrightarrow \bigotimes_{n=1}^{\infty} \mathcal{O}_2,$$

where the connecting maps are given by $a \mapsto a \otimes 1$, yields:

Corollary 5.2.4. *The infinite tensor product $\bigotimes_{n=1}^{\infty} \mathcal{O}_2$ is isomorphic to \mathcal{O}_2 .*

Chapter 6

Nuclear and Exact C^* -algebras

6.1 Nuclear and Exact C^* -algebras

We list some more results about nuclear C^* -algebras (already encountered in Section 2.1), nuclear maps, and exact C^* -algebras. The reader can find a more detailed treatment of these subjects in [4] as well as in several textbook such as Paulsen [106] and Wassermann [145].

A closed linear subspace E of a C^* -algebra A is called an *operator space*. If, moreover, A is unital, E is self-adjoint, and E contains the unit of A , then E is called an *operator system*. An operator space E is endowed with a canonical norm on each matrix algebra $M_n(E)$ over E (viewing $M_n(E)$ as a subspace of $M_n(A)$). Operator spaces have been characterized axiomatically by Ruan in [126].

A linear map $\rho: E \rightarrow F$ between operator spaces is called *completely bounded* if

$$\|\rho\|_{\text{cb}} \stackrel{\text{def}}{=} \sup_{n \in \mathbb{N}} \|\rho_n\| < \infty, \quad (6.1.1)$$

where $\rho_n: M_n(E) \rightarrow M_n(F)$ is the linear mapping obtained by applying ρ entry-wise and $\|\rho_n\|$ is the operator norm of this linear mapping. A linear map $\rho: E \rightarrow F$ is called a *contraction* if $\|\rho\| \leq 1$, and ρ is called a *complete contraction* if $\|\rho\|_{cb} \leq 1$.

A linear map $\rho: A \rightarrow B$ between C^* -algebras A and B is called *completely positive* if each of the linear mappings $\rho_n: M_n(A) \rightarrow M_n(B)$ is positive (i.e., ρ_n maps the positive elements in the C^* -algebra $M_n(A)$ into positive elements in $M_n(B)$). If A and B are unital and $\rho: A \rightarrow B$ is completely positive, then $\|\rho\|_{cb} = \|\rho\| = \|\rho(1)\|$ (see [106, Proposition 3.5]) and ρ is completely bounded.

Completely positive maps are characterized in Stinespring's theorem from [131]:

Theorem 6.1.1 (Stinespring). *Let ρ be a completely positive map from a C^* -algebra A into $\mathcal{L}(\mathcal{H})$, the bounded operators on some Hilbert space \mathcal{H} . Then there is a representation π of A on some Hilbert space \mathcal{H}' and a bounded operator $V: \mathcal{H}' \rightarrow \mathcal{H}$ such that*

$$\rho(a) = V^* \pi(a) V, \quad a \in A.$$

If A has a unit and ρ is unital, then V is necessarily an isometry.

One can apply Stinespring's theorem to a completely positive map between two arbitrary C^* -algebras by representing the target C^* -algebra on some Hilbert space.

Definition 6.1.2 (Nuclear Mappings). A completely positive contraction $\rho: A \rightarrow B$ between C^* -algebras A and B is called *nuclear* if for every finite subset F of A and for every $\varepsilon > 0$ there is a natural number n and completely positive contractions σ and η making the diagram

$$\begin{array}{ccc} A & \xrightarrow{\rho} & B \\ \sigma \searrow & & \nearrow \eta \\ & M_n(\mathbb{C}) & \end{array}$$

commutative on F within ε , i.e., $\|\rho(a) - (\eta \circ \sigma)(a)\| \leq \varepsilon$ for all a in F .

If A and B are unital C^* -algebras and $\rho: A \rightarrow B$ is a nuclear, unital, completely positive map, then $\sigma: A \rightarrow M_n(\mathbb{C})$ and $\eta: M_n(\mathbb{C}) \rightarrow B$ in Definition 6.1.2 can be chosen to be unital, completely positive maps. The next result was proved in [27, Theorem 3.1].

Theorem 6.1.3 (Choi–Effros). *The following conditions are equivalent for each separable C^* -algebra A*

- (i) A is nuclear;
- (ii) the identity map $\text{id}_A: A \rightarrow A$ is nuclear;
- (iii) the identity map $\text{id}_A: A \rightarrow A$ is the point–norm limit of finite rank, completely positive contractions.

Condition (iii) says that for every finite subset F of A and for every $\varepsilon > 0$ there is a completely positive contraction $\rho: A \rightarrow A$ with finite-dimensional range such that $\|\rho(a) - a\| \leq \varepsilon$ for every a in F .

Theorem 6.1.4 (Choi–Effros’s Lifting Theorem). *Let A be a unital C^* -algebra, let I be an ideal in A , let $\pi: A \rightarrow A/I$ be the quotient mapping, and let E be a separable operator system. For each nuclear, unital, completely positive map $\rho: E \rightarrow A/I$ there is a unital, completely positive map $\lambda: E \rightarrow A$ such that $\pi \circ \lambda = \rho$; i.e., ρ has a unital, completely positive lift:*

$$\begin{array}{ccc} & & A \\ & \nearrow \lambda & \downarrow \pi \\ E & \xrightarrow{\rho} & A/I \end{array}$$

It follows from Theorem 6.1.4 that if A is separable and A/I is nuclear, then there is a unital, completely positive lift $\lambda: A/I \rightarrow A$ of the quotient mapping $\pi: A \rightarrow A/I$ (because nuclearity of π is automatic when A/I is nuclear).

The original reference is [26]. See also [106] and [145]. We mention some important extension results for completely positive and completely bounded maps.

Theorem 6.1.5 (Arveson’s Extension Theorem). *Let E be an operator system in a unital C^* -algebra A . Each unital, completely positive map ρ from E into $\mathcal{L}(\mathcal{H})$, the bounded operators on some Hilbert space \mathcal{H} , extends to a unital, completely positive map $\bar{\rho}: A \rightarrow \mathcal{L}(\mathcal{H})$.*

Consult for example [106, Theorem 6.5] for a proof. The reader can find a proof of the next result in [106, Theorem 7.2].

Theorem 6.1.6 (Wittstock’s Extension Theorem). *Let E be an operator space in a unital C^* -algebra A . Every unital, completely bounded map $\rho: E \rightarrow \mathcal{L}(\mathcal{H})$ extends to a unital, completely bounded map $\bar{\rho}: A \rightarrow \mathcal{L}(\mathcal{H})$ with $\|\bar{\rho}\|_{\text{cb}} = \|\rho\|_{\text{cb}}$.*

We shall use the following version (from [84, Lemma 1.9]) of Wittstock’s extension theorem.

Lemma 6.1.7. *Let E be an operator system in a unital C^* -algebra A . For every self-adjoint, unital, completely bounded map $\rho: E \rightarrow \mathcal{L}(\mathcal{H})$ there is a unital, completely positive map $\bar{\rho}: A \rightarrow \mathcal{L}(\mathcal{H})$ with $\|\bar{\rho}|_E - \rho\|_{\text{cb}} \leq \|\rho\|_{\text{cb}} - 1$.*

A linear unital map ρ from an operator system into a unital C^* -algebra is completely positive if and only if $\|\rho\|_{\text{cb}} = 1$ (see [106, Proposition 2.11]).

Exact C^* -algebras are defined in terms of properties of the minimal (or spatial) tensor product \otimes_{\min} (see [102]). The minimal tensor product has the following functorial property:

Proposition 6.1.8. *Let A_1, A_2, B_1 , and B_2 be C^* -algebras, and let $\varphi_1: A_1 \rightarrow B_1$ and $\varphi_2: A_2 \rightarrow B_2$ be $*$ -homomorphisms. Then there is one and only one $*$ -homomorphism*

$$\varphi_1 \otimes \varphi_2: A_1 \otimes_{\min} A_2 \rightarrow B_1 \otimes_{\min} B_2$$

that satisfies $(\varphi_1 \otimes \varphi_2)(a_1 \otimes a_2) = \varphi_1(a_1) \otimes \varphi_2(a_2)$ for all $a_1 \in A_1$ and all $a_2 \in A_2$.

If φ_1 and φ_2 are injective, respectively surjective, then so is $\varphi_1 \otimes \varphi_2$.

Proposition 6.1.8 is proved in [102, Theorem 6.5.1]. It shows that $A \otimes_{\min} -$ is a functor from the category of C^* -algebras into itself for each fixed C^* -algebra A . In contrast to the maximal tensor product, $A \otimes_{\max} -$, this functor is in general not exact. More specifically, if $0 \rightarrow I \rightarrow B \rightarrow B/I \rightarrow 0$ is a short exact sequence of C^* -algebras, then the image of $A \otimes_{\min} I \rightarrow A \otimes_{\min} B$ can be a proper subset of the kernel of the map $A \otimes_{\min} B \rightarrow A \otimes_{\min} B/I$ (see Wassermann, [145, Corollary 3.7]).

Definition 6.1.9 (Exact C^* -algebras). A C^* -algebra A is said to be *exact* if the functor $A \otimes_{\min} -$ is exact; i.e., if every short-exact sequence of C^* -algebras

$$0 \longrightarrow I \xrightarrow{\iota} B \xrightarrow{\pi} B/I \longrightarrow 0$$

induces an exact sequence

$$0 \longrightarrow A \otimes_{\min} I \xrightarrow{\text{id}_A \otimes \iota} A \otimes_{\min} B \xrightarrow{\text{id}_A \otimes \pi} A \otimes_{\min} B/I \longrightarrow 0.$$

Every nuclear C^* -algebra is exact (see eg. [102, Theorem 6.5.2]).

The next result is taken from Kirchberg ([81, Proposition 7.1]), and it shows that the class of exact C^* -algebras is closed under some natural operations somewhat similar to the situation for nuclear C^* -algebras (see Proposition 2.1.2). Note however the differences between the two statements Proposition 2.1.2 and Proposition 6.1.10. For instance, a sub- C^* -algebra of a nuclear C^* -algebra need not be nuclear, and extensions of exact C^* -algebras are in general not exact. The deepest result in Proposition 6.1.10 is part (ii).

Proposition 6.1.10 (Permanence).

- (i) *Every sub- C^* -algebra of an exact C^* -algebra is again exact.*
- (ii) *Every quotient of an exact C^* -algebra is again exact.*
- (iii) *If A and B are exact C^* -algebras, then so is $A \otimes_{\min} B$.*
- (iv) *Any inductive limit of exact C^* -algebras is exact.*
- (v) *If A is an exact C^* -algebra and if G is an amenable, locally compact group acting on A , then the crossed product $A \rtimes G$ is exact.*

Theorem 6.1.11 (Kirchberg). *A C^* -algebra is exact if and only if it admits a nuclear embedding into $\mathcal{L}(\mathcal{H})$ for some Hilbert space \mathcal{H} .*

The reader can find a proof in Wassermann's book, [145, Theorem 9.1]. As a corollary to this theorem we obtain that the identity map on a unital, separable, exact C^* -algebra admits a local factorization through matrix algebra via unital, completely bounded maps with cb-norm just above 1:

Corollary 6.1.12. *Let A be an exact, unital, separable C^* -algebra, let E be a finite dimensional operator system in A , and let $\varepsilon > 0$. Then there is a natural number n , a unital, completely positive map $\sigma : E \rightarrow M_n(\mathbb{C})$, and a unital, completely bounded map $\eta : \sigma(E) \rightarrow E$ such that $\eta \circ \sigma = \text{id}_E$ and $\|\eta\|_{\text{cb}} \leq 1 + \varepsilon$.*

Proof: Let $\{e_1 = 1, e_2, \dots, e_m\}$ be a basis for the finite dimensional vector space E and let $\varepsilon > 0$. Since A is nuclearly embeddable (by Theorem 6.1.11) we may assume that A is a sub- C^* -algebra of $\mathcal{L}(\mathcal{H})$ for some Hilbert space \mathcal{H} and that the inclusion map $\iota : A \rightarrow \mathcal{L}(\mathcal{H})$ is nuclear. Upon replacing \mathcal{H} with a subspace if necessary we may also assume that ι is unital. Take an approximate factorization

$$\begin{array}{ccc} A & \xrightarrow{\iota} & \mathcal{L}(\mathcal{H}) \\ & \searrow \sigma & \nearrow \eta' \\ & M_n(\mathbb{C}) & \end{array} \quad (6.1.2)$$

for a suitable n , where σ and η' are unital, completely positive mappings, and where the triangle commutes to within a certain small tolerance (to be determined later) on e_1, e_2, \dots, e_m , so that the elements $f_j = (\eta' \circ \sigma)(e_j)$ are close to e_j . As explained in [84, Lemma 1.8] if $\|f_j - e_j\|$ are sufficiently small, then $1 = f_1, f_2, \dots, f_m$ are linearly independent, and the linear map π from the span of f_1, f_2, \dots, f_m to E given by $\pi(f_j) = e_j$ is unital, self-adjoint, and satisfies $\|\pi\|_{\text{cb}} \leq 1 + \varepsilon$. Set $\eta = \pi \circ \eta' : \sigma(A) \rightarrow E$. Then $\|\eta\|_{\text{cb}} \leq 1 + \varepsilon$ and $\eta \circ \sigma|_E = \text{id}_E$. \square

6.2 Limit Algebras

Limit algebras are used as a tool to transform a map with an approximate property into a map with an exact version of that property — for example the property of being multiplicative.

Definition 6.2.1 (The C^* -algebra $(A)_\infty$). For each C^* -algebra A let $\ell^\infty(A)$ denote the C^* -algebra of all bounded functions from \mathbb{N} into A with entry-wise defined algebraic operations, and let $c_0(A)$ be the closed two-sided ideal in $\ell^\infty(A)$ consisting of those sequences $\{a_n\}_{n=1}^\infty$ for which $\|a_n\| \rightarrow 0$. Put

$$(A)_\infty = \ell^\infty(A)/c_0(A),$$

and let $\pi_\infty : \ell^\infty(A) \rightarrow (A)_\infty$ be the quotient mapping.

Let $\delta_A : A \rightarrow \ell^\infty(A)$ denote the diagonal embedding $\delta_A(a) = (a, a, a, \dots)$, and set $\iota_A = \pi_\infty \circ \delta_A : A \rightarrow (A)_\infty$.

We turn now to a more general class of limit algebras. Recall that a *filter* ω on \mathbb{N} is an upward directed family of subsets of \mathbb{N} that is closed under finite intersections

and does not contain the empty set. That ω is upward directed means that if X and Y are subsets of \mathbb{N} such that $X \in \omega$ and $X \subseteq Y$, then $Y \in \omega$. A sequence $\{x_n\}_{n=1}^\infty$ (for example of complex numbers) is said to converge to x_0 along ω , in symbols,

$$\lim_{\omega} x_n = x_0,$$

if for each $\varepsilon > 0$ there exists X in ω such that $|x_n| < \varepsilon$ for all n in X . A maximal filter is called an *ultrafilter*. Equivalently, a filter ω is an ultrafilter if for each $X \subseteq \mathbb{N}$ either X or $\mathbb{N} \setminus X$ belong to ω . The family ω_n of all subsets of \mathbb{N} that contain n is an ultrafilter. A filter ω is said to be *free* if $\bigcap_{X \in \omega} X = \emptyset$. An ultrafilter is free precisely when it is not equal to ω_n for any n in \mathbb{N} . An application of Zorn's Lemma shows that every filter is contained in an ultrafilter, and — consequently — that free ultrafilters do exist. The salient feature of ultrafilters is that every sequence of points in a compact set converges along any ultrafilter. (The space of ultrafilters on \mathbb{N} is equal to the β -compactification, $\beta\mathbb{N}$, of \mathbb{N} .)

Definition 6.2.2 (The Ultrapower C^* -algebra A_ω). Let ω be a filter on \mathbb{N} , let $c_\omega(A)$ be the ideal in $\ell^\infty(A)$ consisting of those sequences $\{a_n\}_{n=1}^\infty$ in $\ell^\infty(A)$ for which $\lim_{\omega} \|a_n\| = 0$, and put

$$A_\omega = \ell^\infty(A)/c_\omega(A).$$

We call A_ω the *ultrapower* of A with respect to the filter ω . Let $\pi_\omega: \ell^\infty(A) \rightarrow A_\omega$ be the quotient mapping, and let $\delta_A: A \rightarrow \ell^\infty(A)$ be the diagonal embedding $\delta_A(a) = (a, a, a, \dots)$. Define an embedding $\iota_A: A \rightarrow A_\omega$ by $\iota_A(a) = \pi_\omega(\delta_A(a))$.

We shall often suppress ι_A in Definitions 6.2.1 and 6.2.2 and view A as a sub- C^* -algebra of $(A)_\infty$ and of A_ω . For each filter ω and for each sequence $\{x_n\}$ of real numbers, define

$$\limsup_{\omega} x_n = \inf_{X \in \omega} \sup_{n \in X} x_n. \quad (6.2.1)$$

If ω is free, then

$$\limsup_{\omega} x_n \leq \limsup_{n \rightarrow \infty} x_n, \quad (6.2.2)$$

because for each k in \mathbb{N} there is X in ω such that $X \subseteq \{k, k+1, k+2, \dots\}$.

If ω is the free filter of all cofinite subsets of \mathbb{N} , then $\lim_{\omega} x_n = \lim_{n \rightarrow \infty} x_n$ for all convergent sequences $\{x_n\}_{n=1}^\infty$ (of complex numbers), and hence $c_\omega(A) = c_0(A)$ and $A_\omega = (A)_\infty$.

Lemma 6.2.3. *Let A be a C^* -algebra and let ω be a filter on \mathbb{N} . For each $a = \{a_n\}_{n=1}^\infty$ in $\ell^\infty(A)$ one has $\|\pi_\omega(a)\| = \limsup_{\omega} \|a_n\|$; and if ω is an ultrafilter, then $\|\pi_\omega(a)\| = \lim_{\omega} \|a_n\|$.*

Let $a^{(1)}, \dots, a^{(k)}$ be a finite set of elements in $\ell^\infty(A)$, write $a^{(j)} = \{a_n^{(j)}\}_{n=1}^\infty$, let $\varepsilon > 0$, and assume that $\|\pi_\omega(a^{(j)})\| < \varepsilon$ for all j . Then there is a subset X in ω such that $\|a_n^{(j)}\| < \varepsilon$ for all $n \in X$ and for all $j = 1, \dots, k$.

Proof: Put $v(\pi_\omega(a)) = \limsup_\omega \|a_n\|$. Then v is a well-defined C^* -norm on A_ω , and by uniqueness of the C^* -norm it must coincide with the given C^* -norm on A_ω . The bounded sequence $\{\|a_n\|\}_{n=1}^\infty$ converges along any ultrafilter, and so $\lim_\omega \|a_n\| = \|\pi_\omega(a)\|$ when ω is an ultrafilter.

For the second claim, let $\varepsilon > 0$ be given. It follows from the first part of the lemma that $\limsup_\omega \|a_n^{(j)}\| < \varepsilon$. Accordingly, we can find X_1, \dots, X_k in ω such that $\sup_{n \in X_j} \|a_n^{(j)}\| < \varepsilon$. Put $X = X_1 \cap X_2 \cap \dots \cap X_k$. Then X belongs to ω and $\|a_n^{(j)}\| < \varepsilon$ for all n in X and for $j = 1, \dots, k$. \square

Lemma 6.2.4. *Let A be a C^* -algebra and let ω be a filter on \mathbb{N} . Then each projection in A_ω lifts to a projection in $\ell^\infty(A)$. If A is unital, then each isometry and each unitary element in A_ω lifts to an isometry, respectively, a unitary element in $\ell^\infty(A)$.*

Proof: We only prove the statement about unitary elements — the proofs for the projection and the isometry cases are similar. Take a unitary element u in A_ω and lift it to a sequence $a = (a_1, a_2, a_3, \dots)$ in $\ell^\infty(A)$. Then

$$0 = \limsup_\omega \|a_n^* a_n - 1\| = \limsup_\omega \|a_n a_n^* - 1\|$$

by Lemma 6.2.3. Choose X in ω such that $\|a_n^* a_n - 1\| < 1$ and $\|a_n a_n^* - 1\| < 1$ for all n in X . Then a_n is invertible for all n in X , and we can therefore write $a_n = v_n |a_n|$ where v_n is a unitary element in A and $|a_n| = (a_n^* a_n)^{1/2}$. The unitary v_n is close to a_n if $\|a_n^* a_n - 1\|$ and $\|a_n a_n^* - 1\|$ are small. Set $v_n = 1$ if n does not belong to X , and put $v = (v_1, v_2, v_3, \dots)$. Then v is a unitary element in $\ell^\infty(A)$, $\lim_\omega \|v_n - a_n\| = 0$, and hence $\pi_\omega(v) = u$. \square

The next lemma says that any two approximately unitarily equivalent $*$ -homomorphisms $A \rightarrow B$ are exactly unitarily equivalent in B_ω ; and if they are approximately equivalent in B_ω then they are also approximately unitarily equivalent in B . More is true: any two approximately unitarily equivalent $*$ -homomorphisms $A \rightarrow B_\omega$ are exactly unitarily equivalent when A is separable (see eg. [120, Proposition 3.3]).

Lemma 6.2.5. *Let A, B be C^* -algebras, and let $\varphi, \psi : A \rightarrow B$ be $*$ -homomorphisms. Let ω be a free filter on \mathbb{N} and let $\iota : B \rightarrow B_\omega$ be the inclusion mapping.*

- (i) *If $\iota \circ \varphi \approx_u \iota \circ \psi$ in B_ω , then $\varphi \approx_u \psi$ in B .*
- (ii) *If A is separable and if $\iota \circ \varphi \approx_u \iota \circ \psi$, then $\iota \circ \varphi \sim_u \iota \circ \psi$.*

Proof: (i). Let F be a finite subset of A and let $\varepsilon > 0$ be given. By assumption there is a unitary u in A_ω such that $\|u(\iota \circ \varphi)(a)u^* - (\iota \circ \psi)(a)\| < \varepsilon$ for all a in F . Use Lemma 6.2.4 to find a unitary $v = (v_1, v_2, v_3, \dots)$ in $\ell^\infty(A)$ that lifts u . Then

$$\limsup_\omega \|v_n \varphi(a) v_n^* - \psi(a)\| = \|u(\iota \circ \varphi)(a)u^* - (\iota \circ \psi)(a)\| < \varepsilon, \quad a \in F,$$

by Lemma 6.2.3. The second part of Lemma 6.2.3 implies that there are infinitely many natural numbers n such that $\|v_n \varphi(a) v_n^* - \psi(a)\| < \varepsilon$ for all a in F .

(ii). If A is separable and if $\iota \circ \varphi \approx_u \iota \circ \psi$, then $\varphi \approx_u \psi$ by (i), and hence $\text{Ad } v_n \circ \varphi \rightarrow \psi$ for some sequence $\{v_n\}_{n=1}^\infty$ of unitaries in A . Put $v = (v_1, v_2, v_3, \dots)$ in $\ell^\infty(A)$ and put $u = \pi_\omega(v)$. By Lemma 6.2.3 and equation (6.2.2) we get

$$\begin{aligned} \|u(\iota \circ \varphi)(a)u^* - (\iota \circ \psi)(a)\| &= \limsup_{\omega} \|v_n \varphi(a)v_n^* - \psi(a)\| \\ &\leq \limsup_{n \rightarrow \infty} \|v_n \varphi(a)v_n^* - \psi(a)\| = 0 \end{aligned}$$

for all a in A , and this completes the proof. \square

Proposition 6.2.6. *If A is a purely infinite, simple C^* -algebra, then so is A_ω for every free ultrafilter ω .*

Proof: We check property (i) in Proposition 4.1.1. Let a, b be non-zero positive elements in A_ω . Find positive elements a_n, b_n in A such that

$$a = \pi_\omega(a_1, a_2, a_3, \dots), \quad b = \pi_\omega(b_1, b_2, b_3, \dots).$$

We know from Lemma 6.2.3 that

$$\lim_{\omega} \|a_n\| = \|a\| > 0, \quad \lim_{\omega} \|b_n\| = \|b\| > 0.$$

We can therefore find X in ω such that $\|a_n\| \geq \|a\|/2$ and $\|b_n\| \leq 2\|b\|$ for all n in X . Use Lemma 4.1.7 to find y_n in A satisfying $y_n^* a_n y_n = b_n$ and

$$\|y_n\| \leq 2(\|b_n\|/\|a_n\|)^{1/2} \leq 4(\|b\|/\|a\|)^{1/2}, \quad n \in X.$$

Set $y_n = 0$ when n belongs to $\mathbb{N} \setminus X$. Then $y = (y_1, y_2, y_3, \dots)$ belongs to $\ell^\infty(A)$ and $\pi_\omega(y)^* a \pi_\omega(y) = b$. \square

6.3 Kirchberg's Embedding Theorems for Nuclear and Exact C^* -algebras

Following the article [84] by Kirchberg and Phillips we shall in this section prove Kirchberg's embedding theorem that all separable, exact C^* -algebras can be embedded into the Cuntz algebra \mathcal{O}_2 . A few places we cut corners and refer the reader to [84] for details, but I hope that the present exposition nonetheless will give the reader a good understanding of the proof.

The proposition below about excising states will be used in the proof of Proposition 6.3.3, which is one of two technical cornerstones in the proof of the embedding theorem. In the terminology of [1], a state ω on a unital C^* -algebra A is said to be *excised* by a net $\{h_\lambda\}$ of positive elements in A if $\|h_\lambda\| = \omega(h_\lambda) = 1$ for all λ and $\|h_\lambda^{1/2} a h_\lambda^{1/2} - \omega(a) h_\lambda\| \rightarrow 0$ for all a in A . It is shown in [1] that a state can be excised if and only if it belongs to the weak-* closure of the set of pure states on the C^* -algebra. We prove a special case of this result here:

Proposition 6.3.1 (Akemann–Anderson–Pedersen). *Let A be a unital, purely infinite, simple C^* -algebra and let ω be a state on A . For each finite subset F of A and for each $\varepsilon > 0$ there is a non-zero projection p in A with $\|pap - \omega(a)p\| < \varepsilon$ for all a in F .*

Proof: The set of pure states is weak- $*$ dense in the state space of an antiliminal, prime, unital C^* -algebra (see [35, Lemme 11.2.4]) and each simple, purely infinite C^* -algebra has these properties. By an approximation argument we can therefore assume that ω is a pure state. Let L be the left kernel of ω , i.e., the set of those a in A for which $\omega(a^*a) = 0$. The set $N = L \cap L^*$ is a hereditary sub- C^* -algebra of A .

Each purely infinite, simple C^* -algebra has real rank zero by Proposition 4.1.1, and so we can find an approximate unit $\{q_\lambda\}$ for N in which each q_λ is a projection. Observe that $\omega(q_\lambda) = 0$ for all λ . Put $p_\lambda = 1 - q_\lambda$. Then $\omega(p_\lambda) = 1$. We show that

$$p_\lambda(a - \omega(a) \cdot 1)p_\lambda = p_\lambda ap_\lambda - \omega(a)p_\lambda \rightarrow 0$$

for all a in A . The element $a - \omega(a) \cdot 1$ belongs to the kernel of ω , and this kernel is equal to $L + L^*$ because ω is assumed to be a pure state (see [107, 3.13.6]). Hence it suffices to show that $xp_\lambda \rightarrow 0$ for all x in L . But if x belongs to L , then x^*x belongs to N , and hence

$$\|xp_\lambda\|^2 = \|p_\lambda x^* xp_\lambda\| \leq \|x^* xp_\lambda\| = \|x^* x(1 - q_\lambda)\| \rightarrow 0.$$

□

Lemma 6.3.2. *Let ρ be a unital, completely positive map from $M_n(\mathbb{C})$ to a unital, properly infinite C^* -algebra A . Then there exist a (possibly non-unital) $*$ -homomorphism $\varphi: M_n(\mathbb{C}) \rightarrow A$ and an isometry t in A such that $\rho(x) = t^* \varphi(x) t$ for all x in $M_n(\mathbb{C})$.*

Proof: Choose a system $\{e_{ij}\}$ of matrix units for $M_n(\mathbb{C})$. We find an element v in $M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \otimes A$, a $*$ -homomorphism $\psi: M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \otimes A \rightarrow A$, and an isometry s in A satisfying

$$v^*(x \otimes 1 \otimes 1)v = e_{11} \otimes e_{11} \otimes \rho(x), \quad \psi(e_{11} \otimes e_{11} \otimes a) = sas^*, \quad (6.3.1)$$

for all $x \in M_n(\mathbb{C})$ and all $a \in A$. The elements v and s above will then necessarily satisfy $v^*v = e_{11} \otimes e_{11} \otimes 1$ and $ss^* = \psi(e_{11} \otimes e_{11} \otimes 1) = \psi(v^*v)$. Thus $t = \psi(v)s$ is an isometry. Let $\varphi: M_n(\mathbb{C}) \rightarrow A$ be the $*$ -homomorphism given by $\varphi(x) = \psi(x \otimes 1 \otimes 1)$. Then

$$t^* \varphi(x) t = s^* \psi(v^*(x \otimes 1 \otimes 1)v)s = s^* \psi(e_{11} \otimes e_{11} \otimes \rho(x))s = \rho(x)$$

for all x in $M_n(\mathbb{C})$ as desired.

We proceed to find v , ψ , and s . The element

$$y = (\text{id}_{M_n(\mathbb{C})} \otimes \rho) \left(\sum_{i,j=1}^n e_{ij} \otimes e_{ij} \right) = \sum_{i,j=1}^n e_{ij} \otimes \rho(e_{ij}) \in M_n(\mathbb{C}) \otimes A$$

is positive (because ρ is completely positive and $\sum_{i,j=1}^n e_{ij} \otimes e_{ij}$ is positive — being a multiple of a projection). It follows that y has a positive square root that can be written $y^{1/2} = \sum_{i,j=1}^n e_{ij} \otimes a_{ij}$ for suitable a_{ij} in A . Put

$$v = \sum_{i,j=1}^n e_{i1} \otimes e_{j1} \otimes a_{ji} \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \otimes A.$$

Then (by comparing the second term below with $y = (y^{1/2})^* y^{1/2}$) we get

$$v^*(e_{ij} \otimes 1 \otimes 1)v = e_{11} \otimes e_{11} \otimes \sum_{k=1}^n a_{ki}^* a_{kj} = e_{11} \otimes e_{11} \otimes \rho(e_{ij})$$

for all i, j , and hence the first identity in equation (6.3.1) holds.

Choose a system $\{g_{ij}\}_{i,j=1}^{n^2}$ of matrix units for $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ with $g_{11} = e_{11} \otimes e_{11}$. Because A is properly infinite we can find isometries u_1, u_2, \dots, u_{n^2} in A with orthogonal range projections (see Proposition 1.1.2). Put $s = u_1$ and let $\psi: M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \otimes A \rightarrow A$ be given by $\psi(g_{ij} \otimes a) = u_i a u_j^*$. Then ψ and s satisfy the second identity in equation (6.3.1). \square

The next result (cf. [84, Proposition 1.7]) gives a strong Stinespring type theorem for unital, completely positive maps on Kirchberg algebras.

Proposition 6.3.3. *Let A be a unital, purely infinite, simple C^* -algebra and let $\rho: A \rightarrow A$ be a nuclear, unital, completely positive map. Then for all finite subset F of A and for all $\varepsilon > 0$ there is an isometry s in A such that $\|s^* a s - \rho(a)\| \leq \varepsilon$ for all a in F .*

Proof: By the assumption that ρ is nuclear there is an approximate factorization

$$\begin{array}{ccc} A & \xrightarrow{\rho} & A \\ & \searrow \sigma \quad \nearrow \eta & \\ & M_n(\mathbb{C}) & \end{array}$$

where σ and η are unital, completely positive maps, such that $\eta \circ \sigma$ agrees with ρ on F to within any specified positive tolerance. We may therefore assume that $\rho = \eta \circ \sigma$. Use Lemma 6.3.2 to find a (possibly non-unital) $*$ -homomorphism $\varphi: M_n(\mathbb{C}) \rightarrow A$ and an isometry t in A such that $\eta(x) = t^* \varphi(x) t$ for all x in $M_n(\mathbb{C})$.

We proceed to find an element u in A such that $u^* a u$ is close to $(\varphi \circ \sigma)(a)$ for all a in F . Let $\{e_{ij}\}$ be the standard matrix units for $M_n(\mathbb{C})$ and let $\{\xi_1, \xi_2, \dots, \xi_n\}$ be the standard basis for \mathbb{C}^n . Define a function ω on $M_n(\mathbb{C}) \otimes A$ by

$$\omega\left(\sum_{i,j=1}^n e_{ij} \otimes a_{ij}\right) = \frac{1}{n} \sum_{i,j=1}^n \langle \sigma(a_{ij}) \xi_j, \xi_i \rangle.$$

Inspection shows that $\sigma(a) = n \sum_{i,j} \omega(e_{ij} \otimes a) e_{ij}$, and ω is a state because σ is completely positive (see [106, Theorem 5.1] and [84, Lemma 1.5]). By Proposition 6.3.1 there is a non-zero projection p in $M_n(\mathbb{C}) \otimes A$ such that $p(e_{ij} \otimes a)p$ is close to $\omega(e_{ij} \otimes a)p$ for all a in F . Since $M_n(\mathbb{C}) \otimes A$ is purely infinite it contains a partial isometry v satisfying $v^*v = e_{11} \otimes \varphi(e_{11})$ and $vv^* \leq p$. Write $v = \sum_{j=1}^n e_{j1} \otimes v_j$ with $v_j \in A$, and calculate

$$\begin{aligned} e_{11} \otimes v_i^* a v_j &= v^*(e_{ij} \otimes a)v = v^*p(e_{ij} \otimes a)p v \\ &\approx \omega(e_{ij} \otimes a)v^*p v = \omega(e_{ij} \otimes a)e_{11} \otimes \varphi(e_{11}), \end{aligned}$$

where \approx here means “close to”. Hence $v_i^* a v_j$ is close to $\omega(e_{ij} \otimes a)\varphi(e_{11})$ for all a in F . Put $u = \sqrt{n} \sum_{j=1}^n v_j \varphi(e_{1j})$. Then

$$u^* a u = n \sum_{i,j=1}^n \varphi(e_{i1}) v_i^* a v_j \varphi(e_{1j}) \approx n \sum_{i,j=1}^n \omega(e_{ij} \otimes a) \varphi(e_{ij}) = \varphi(\sigma(a))$$

for all a in F . We may assume that F contains the unit of A in which case u^*u is close to $\varphi(1_A)$ and $tt^* \leq \varphi(1_A)$. Perturb u slightly to make u^*u equal to $\varphi(1_A)$. Then $s = ut$ is an isometry, and

$$s^* a s \approx t^*(\varphi \circ \sigma)(a)t = (\eta \circ \sigma)(a) = \rho(a)$$

for all a in F as desired. \square

Proposition 6.3.3 and the next result (taken from [84, Lemma 1.10]) are the two technical key ingredients in the proofs of the embedding theorem and of the tensor product theorems. (Observe the role played by the crucial number n entering in condition (ii).)

Proposition 6.3.4 (Extension). *Let E be a finite dimensional operator system in a separable, unital, exact C^* -algebra A and let $\varepsilon > 0$. There is a natural number n (that depends on A , E , and ε) such that if B_1 and B_2 are separable, unital C^* -algebras and $\rho_1: E \rightarrow B_1$ and $\rho_2: E \rightarrow B_2$ are unital, completely positive maps that satisfy:*

- (i) ρ_1 is injective;
- (ii) $\|\text{id}_{M_n(\mathbb{C})} \otimes \rho_1^{-1}\| \leq 1 + \varepsilon/2$, where $\rho_1^{-1}: \rho_1(E) \rightarrow E$; and
- (iii) B_2 is nuclear;

then there is a unital, completely positive map $\eta: B_1 \rightarrow B_2$ such that $\|\eta \circ \rho_1 - \rho_2\| < \varepsilon$, i.e., such that the diagram

$$\begin{array}{ccc} & E & \\ \rho_1 \swarrow & & \searrow \rho_2 \\ B_1 & \xrightarrow{\eta} & B_2 \end{array}$$

commutes within ε on the unit ball of E .

Before giving its proof we mention the following corollary to Proposition 6.3.4 that underscores its usefulness:

Corollary 6.3.5. *Let A be a unital, separable, exact C^* -algebra, and let B_1 and B_2 be unital, separable C^* -algebras with B_2 nuclear.*

- (i) *For every pair of unital $*$ -homomorphisms $\varphi_1 : A \rightarrow B_1$ and $\varphi_2 : A \rightarrow B_2$, with φ_1 injective, there is a sequence $\{\eta_n\}$ of unital, completely positive maps from B_1 to B_2 such that $\eta_n \circ \varphi_1 \rightarrow \varphi_2$.
If, in addition, $B_1 = B_2 = B$ is a Kirchberg algebra, then there is a sequence $\{s_n\}$ of isometries in B such that $s_n^* \varphi_1(a) s_n \rightarrow \varphi_2(a)$ for all a in A .*
- (ii) *Let ω be a free ultrafilter on \mathbb{N} . Let $\rho_1, \rho_2, \rho_3, \dots$ be a sequence of unital, completely positive maps from A to B_1 such that the map $\rho : A \rightarrow (B_1)_\omega$ given by $\rho(a) = \pi_\omega(\rho_1(a), \rho_2(a), \dots)$ is an injective $*$ -homomorphism. Then for each sequence $\sigma_1, \sigma_2, \sigma_3, \dots$ of unital, completely positive maps from A to B_2 there is a sequence of unital, completely positive maps $\eta_n : B_1 \rightarrow B_2$ such that*

$$\lim_{\omega} \|(\eta_n \circ \rho_n)(a) - \sigma_n(a)\| = 0$$

for all a in A .

If, in addition, $B_1 = B_2 = B$ is a Kirchberg algebra, then there is a sequence $\{s_n\}$ of isometries in B such that

$$\lim_{\omega} \|s_n^* \rho_n(a) s_n - \sigma_n(a)\| = 0$$

for all a in A .

The first part of (i) is an immediate consequence of Proposition 6.3.4. To prove the first part of (ii) we must for each finite dimensional operator system E in A and for each $\varepsilon > 0$ find a subset X in ω and unital, completely positive maps $\eta_k : B_1 \rightarrow B_2$ such that

$$\|(\eta_k \circ \rho_k)|_E - \sigma_k|_E\| < \varepsilon$$

for all k in X . This will follow from Proposition 6.3.4 if for each n in \mathbb{N} we can find X in ω such that $\rho_k|_E$ is injective and $\|\text{id}_{M_n(\mathbb{C})} \otimes (\rho_k|_E)^{-1}\| \leq 1 + \varepsilon/2$ for all k in X . Both conditions will hold if $\|(\text{id}_{M_n(\mathbb{C})} \otimes \rho_k)(a)\| \geq (1 + \varepsilon/2)^{-1}$ for all a in the unit sphere of $M_n(\mathbb{C}) \otimes E$. The assumption in (ii) that ρ is an injective $*$ -homomorphism in combination with Lemma 6.2.3 imply that

$$\lim_{\omega} \|(\text{id}_{M_n(\mathbb{C})} \otimes \rho_k)(a)\| = \|a\|$$

for all a in $M_n(\mathbb{C}) \otimes E$. A compactness argument applied to the unit sphere of $M_n(\mathbb{C}) \otimes E$ now gives the existence of X .

The second parts of (i) and (ii) follow from Proposition 6.3.3.

Proof of Proposition 6.3.4: The proof is summarized in the diagram:

$$\begin{array}{ccccc}
 & E & \xrightarrow{\text{id}} & E & \\
 \rho_1 \swarrow & \downarrow \sigma_1 & \nearrow \eta_1 & \downarrow \sigma_2 & \searrow \rho_2 \\
 B_1 & \dashrightarrow \tau_1 \dashrightarrow M_n(\mathbb{C}) & \dashrightarrow \tau_2 \dashrightarrow M_r(\mathbb{C}) & \dashrightarrow \eta_2 \dashrightarrow & B_2
 \end{array} \quad (6.3.2)$$

where $\sigma_1, \sigma_2, \eta_2, \tau_1$, and τ_2 are unital, completely positive maps (to be determined) and η_1 is a unital, completely bounded map defined on the image of σ_1 and with $\|\eta_1\|_{\text{cb}} \leq 1 + \varepsilon/4$ (also to be determined). Once these maps have been found, making the diagram *almost* commutative, we can take η to be $\eta_2 \circ \tau_2 \circ \tau_1$. (More explicitly, the first triangle in diagram (6.3.2) will be made to commute within $\varepsilon/2$, the second triangle will commute exactly, and the third and the fourth triangle will be made to commute within $\varepsilon/4$.)

Second triangle. Use Corollary 6.1.12 to find a natural number n (depending on A, E , and ε), a unital, completely positive map $\sigma_1: E \rightarrow M_n(\mathbb{C})$ and a unital, completely bounded map $\eta_1: \sigma_1(E) \rightarrow E$ with $\|\eta_1\|_{\text{cb}} \leq 1 + \varepsilon/4$ and $\eta_1 \circ \sigma_1 = \text{id}_E$.

Fourth triangle. Since B_2 is nuclear, the unital, completely positive map $\rho_2: E \rightarrow B_2$ is nuclear and we can therefore find r and unital, completely positive maps σ_2 and η_2 as indicated in diagram (6.3.2) such that $\|\rho_2 - \eta_2 \circ \sigma_2\| \leq \varepsilon/4$.

Third triangle. Put $F = \sigma_1(E) \subseteq M_n(\mathbb{C})$. Use the modification of Wittstock's Extension Theorem (Lemma 6.1.7) on $\sigma_2 \circ \eta_1: F \rightarrow M_r(\mathbb{C})$, and the fact that $\|\sigma_2 \circ \eta_1\|_{\text{cb}} \leq 1 + \varepsilon/4$, to find a unital, completely positive map $\tau_2: M_n(\mathbb{C}) \rightarrow M_r(\mathbb{C})$ with $\|\tau_2|_F - \sigma_2 \circ \eta_1\|_{\text{cb}} \leq \varepsilon/4$.

First triangle. Put $G = \rho_1(E) \subseteq B_1$ and consider the unital, completely bounded map $\sigma_1 \circ \rho_1^{-1}: G \rightarrow M_n(\mathbb{C})$. The cb-norm of a linear map $D: G \rightarrow M_n(\mathbb{C})$ is equal to the operator norm of the induced map $D \otimes M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ (see [106, Proposition 7.9]). This entails that

$$\|\sigma_1 \circ \rho_1^{-1}\|_{\text{cb}} = \|\text{id}_{M_n(\mathbb{C})} \otimes (\sigma_1 \circ \rho_1^{-1})\| = \|\text{id}_{M_n(\mathbb{C})} \otimes \rho_1^{-1}\| \leq 1 + \varepsilon/2.$$

We can therefore apply Lemma 6.1.7 once again to obtain a unital, completely positive map $\tau_1: B_1 \rightarrow M_n(\mathbb{C})$ with $\|\tau_1|_G - \sigma_1 \circ \rho_1^{-1}\| \leq \varepsilon/2$. \square

Lemma 6.3.6. *Let D be a unital C^* -algebra, let A be a unital sub- C^* -algebra of D , and let s be an isometry in D . If the map $a \mapsto s^*as$, for $a \in A$, is multiplicative, then ss^* commutes with all elements in A ; and if $s^*as = a$ for all a in A , then s commutes with all elements in A .*

Proof: Put $p = ss^*$. Assume that $a \mapsto s^*as$ is multiplicative. Then for each unitary u in A ,

$$(pup)^*(pup) = s(s^*u^*s)(s^*us)s^* = s(s^*u^*us)s^* = ss^*ss^* = p.$$

Represent A on a Hilbert space \mathcal{H} . For each unit vector ξ in $p(\mathcal{H})$,

$$1 = \|u\xi\|^2 = \|up\xi\|^2 = \|pup\xi\|^2 + \|(1-p)up\xi\|^2 = 1 + \|(1-p)up\xi\|^2.$$

This shows that $(1-p)up = 0$. In a similar way we see that $(1-p)u^*p = 0$. These two identities imply that p commutes with u . It follows that p commutes with all elements in A .

If $s^*as = a$ for all a in A , then $as = aps = pas = ss^*as = sa$. \square

In the lemma below (taken from [84, Lemma 1.10]) we need an approximate version of Lemma 6.3.6. One can obtain this either directly or by applying the given proof of Lemma 6.3.6 to the limit algebra $(A)_\infty$.

Lemma 6.3.7 (A la Cantor–Bernstein). *Let A and B be unital C^* -algebras, let B_0 be a sub- C^* -algebra of B that contains the unit of B , and suppose that there is a unital embedding of \mathcal{C}_2 into the relative commutant $B \cap B'_0$. Let $\varphi, \psi: A \rightarrow B_0$ be unital $*$ -homomorphisms, and suppose that there are sequences $\{s_n\}_{n=1}^\infty$ and $\{t_n\}_{n=1}^\infty$ of isometries in B_0 such that*

$$\|s_n^*\varphi(a)s_n - \psi(a)\| \rightarrow 0, \quad \|t_n^*\psi(a)t_n - \varphi(a)\| \rightarrow 0$$

for all a in A . Then φ and ψ are approximately unitarily equivalent relatively to B .

Proof: Define projections $e_1(n), e_2(n)$ and $f_1(n), f_2(n), f_3(n)$ in B_0 by $e_1(n) = s_n s_n^*, f_1(n) = t_n t_n^*, e_2(n) = s_n f_1(n) s_n^*$, and $f_{j+1}(n) = t_n e_j(n) t_n^*$ for $j = 1, 2$. Set

$$\begin{aligned} p_1(n) &= 1 - e_1(n), & p_2(n) &= e_1(n) - e_2(n), & p_3(n) &= e_2(n), \\ q_1(n) &= 1 - f_1(n), & q_2(n) &= f_1(n) - f_2(n), \\ q_3(n) &= f_2(n) - f_3(n), & q_4(n) &= f_3(n). \end{aligned}$$

It follows from Lemma 6.3.6 (and the remarks below it) that $[e_1(n), \varphi(a)] \rightarrow 0$ and $[f_1(n), \psi(a)] \rightarrow 0$ for all a in A . Similarly, because $(s_n t_n)^* \varphi(a) (s_n t_n) \rightarrow \varphi(a)$ and $e_2(n) = (s_n t_n)(s_n t_n)^*$, we have $[e_2(n), \varphi(a)] \rightarrow 0$. In the same way we see that $[f_2(n), \psi(a)] \rightarrow 0$ and $[f_3(n), \psi(a)] \rightarrow 0$ as $n \rightarrow \infty$ for all a in A . It follows that

$$\lim_{n \rightarrow \infty} \|p_j(n)\varphi(a) - \varphi(a)p_j(n)\| = 0, \quad \lim_{n \rightarrow \infty} \|q_j(n)\psi(a) - \psi(a)q_j(n)\| = 0. \quad (6.3.3)$$

Put $v(n) = s_n q_1(n) (= p_2(n)s_n)$ and $w_j(n) = t_n^* q_{j+1}(n) (= p_j(n)t_n^*)$ for $j = 1, 2, 3$. Then $v(n)$ and $w_j(n)$ are partial isometries in B_0 and they satisfy

$$v(n)^* v(n) = q_1(n), \quad v(n)v(n)^* = p_2(n), \quad (6.3.4)$$

$$w_j(n)^* w_j(n) = q_{j+1}(n), \quad w_j(n)w_j(n)^* = p_j(n), \quad j = 1, 2, 3. \quad (6.3.5)$$

Use (6.3.3) to see that

$$\lim_{n \rightarrow \infty} \|\varphi(a)v(n) - v(n)\psi(a)\| = 0, \quad (6.3.6)$$

$$\lim_{n \rightarrow \infty} \|\varphi(a)w_j(n) - w_j(n)\psi(a)\| = 0, \quad j = 1, 2, 3. \quad (6.3.7)$$

Find isometries c_1, c_2 in $B \cap B'_0$ satisfying the Cuntz relation $c_1 c_1^* + c_2 c_2^* = 1$, and put

$$u_n = w_1(n) + w_2(n)c_1 + w_3(n) + v(n)c_2.$$

Use (6.3.4) and (6.3.5) to check that u_n is unitary, and use (6.3.6) and (6.3.7) to see that $\|\varphi(a)u_n - u_n\psi(a)\| \rightarrow 0$ for all a in A . \square

Theorem 6.3.8. *Let A be a unital, separable, exact C^* -algebra.*

- (i) *Let B be a simple, separable, unital, and nuclear C^* -algebra. Then any two unital, injective $*$ -homomorphisms $\varphi, \psi: A \rightarrow B \otimes \mathcal{O}_2$ are approximately unitarily equivalent.*
- (ii) *Any two unital, injective $*$ -homomorphisms $\varphi, \psi: A \rightarrow \mathcal{O}_2$ are approximately unitarily equivalent.*

Proof: (i). Note first that $B \otimes \mathcal{O}_2$ is a unital Kirchberg algebra (by Theorem 4.1.10). Use Corollary 6.3.5 (i) to find sequences $\{s_n\}_{n=1}^\infty$ and $\{t_n\}_{n=1}^\infty$ of isometries in $B \otimes \mathcal{O}_2$ with $\|s_n^* \varphi(a) s_n - \psi(a)\| \rightarrow 0$ and $\|t_n^* \psi(a) t_n - \varphi(a)\| \rightarrow 0$ for all a in A . Let $\iota: \mathcal{O}_2 \rightarrow \mathcal{O}_2 \otimes \mathcal{O}_2$ be given by $\iota(x) = x \otimes 1$ and take an isomorphism $\lambda: \mathcal{O}_2 \otimes \mathcal{O}_2 \rightarrow \mathcal{O}_2$ (cf. Theorem 5.2.1). It follows from Lemma 6.3.7 that $(\text{id}_B \otimes \iota) \circ \varphi$ and $(\text{id}_B \otimes \iota) \circ \psi$ are approximately unitarily equivalent (relatively to $B \otimes \mathcal{O}_2 \otimes \mathcal{O}_2$). Use Theorem 5.1.1 to see that $\lambda \circ \iota \approx_u \text{id}_{\mathcal{O}_2}$. We now have

$$\varphi \approx_u (\text{id}_B \otimes \lambda) \circ (\text{id}_B \otimes \iota) \circ \varphi \approx_u (\text{id}_B \otimes \lambda) \circ (\text{id}_B \otimes \iota) \circ \psi \approx_u \psi,$$

as desired.

(ii) follows from (i) and from Theorem 5.2.1. \square

It will be proved later (in Theorem 7.1.2) that $B \otimes \mathcal{O}_2$ is isomorphic to \mathcal{O}_2 when B is as in Theorem 6.3.8 (i). (Theorem 6.3.8 (i) goes into the proof of Theorem 7.1.2.)

The results obtained so far will now be used to embed separable, exact C^* -algebras into \mathcal{O}_2 .

Lemma 6.3.9. *Let A be a separable, unital, exact C^* -algebra, and suppose that there is a unital, injective $*$ -homomorphism $\varphi: A \rightarrow (\mathcal{O}_2)_\infty$ with a unital, completely positive lift $\rho: A \rightarrow \ell^\infty(\mathcal{O}_2)$, i.e., such that*

$$\begin{array}{ccc} & \ell^\infty(\mathcal{O}_2) & \\ \rho \nearrow & \downarrow \pi_\infty & \\ A & \xrightarrow{\varphi} & (\mathcal{O}_2)_\infty \end{array}$$

commutes. Then there is a unital, injective $$ -homomorphism from A into \mathcal{O}_2 .*

Proof: Write $\rho(a) = (\rho_1(a), \rho_2(a), \rho_3(a), \dots)$, where $\rho_k: A \rightarrow \mathcal{O}_2$ are unital, completely positive mappings. We apply Proposition 6.3.4 to this sequence as follows: Take an increasing sequence $\{E_j\}_{j=1}^\infty$ of operator spaces in A such that

$\bigcup_{j=1}^{\infty} E_j$ is dense in A . Let n_j be as in Proposition 6.3.4 corresponding to $E = E_j$, $\varepsilon = \varepsilon_j$, and $\delta = \varepsilon_j/2$ (where $\varepsilon_j > 0$ is to be determined later). Because φ is injective we can find an increasing sequence $\{k_j\}$ of integers such that ρ_{k_j} is injective on E_j and such that

$$\|(\text{id}_{M_{n_j}(\mathbb{C})} \otimes \rho_{k_j}|_{E_j})^{-1}\| \leq 1 + \varepsilon_j/2, \quad \|(\text{id}_{M_{n_j}(\mathbb{C})} \otimes \rho_{k_{j+1}}|_{E_j})^{-1}\| \leq 1 + \varepsilon_j/2.$$

Now, by Proposition 6.3.4, there exist unital, completely positive maps η_j and σ_j making the two diagrams

$$\begin{array}{ccc} & E_j & \\ \rho_{k_j} \swarrow & & \searrow \rho_{k_{j+1}} \\ \mathcal{O}_2 & \xrightarrow{\eta_j} & \mathcal{O}_2 \end{array} \quad \begin{array}{ccc} & E_j & \\ \rho_{k_{j+1}} \swarrow & & \searrow \rho_{k_j} \\ \mathcal{O}_2 & \xrightarrow{\sigma_j} & \mathcal{O}_2 \end{array}$$

are commutative within ε_j on the unit ball of E_j . Use next Proposition 6.3.3 on η_j and σ_j to find sequences $\{s_j\}$ and $\{t_j\}$ of isometries in \mathcal{O}_2 such that

$$\|s_j^* \rho_{k_j}(a) s_j - \rho_{k_{j+1}}(a)\| \leq \varepsilon_j \|a\|, \quad \|t_j^* \rho_{k_{j+1}}(a) t_j - \rho_{k_j}(a)\| \leq \varepsilon_j \|a\|$$

for all j and for all a in E_j .

A strengthened version of Lemma 6.3.7 (see [84, Lemma 1.10]) shows that if ε_j has been chosen small enough, then there is a sequence $\{u_j\}$ of unitaries in $\mathcal{O}_2 \otimes \mathcal{O}_2$ satisfying $\|u_j(\rho_{k_j}(a) \otimes 1)u_j^* - \rho_{k_{j-1}}(a) \otimes 1\| \leq 2^{-j}\|a\|$ in $\mathcal{O}_2 \otimes \mathcal{O}_2$ for all j and for all a in E_j . (To obtain this we must have a quantitative version of Lemma 6.3.7, and we must allow φ and ψ in Lemma 6.3.7 to be *almost multiplicative* unital, completely positive maps rather than actual $*$ -homomorphisms. Alternatively, one can pass to maps from A into $(\mathcal{O}_2)_{\infty}$ to obtain actual $*$ -homomorphisms from the unital, completely positive maps ρ_{k_j} .)

The limit

$$\psi(a) = \lim_{j \rightarrow \infty} \text{Ad}(u_1 u_2 \cdots u_j)(\rho_{k_j}(a) \otimes 1)$$

exists for all a in A . Because φ is an injective $*$ -homomorphism, the sequence $\{\rho_n\}$ of unital, completely positives is asymptotically multiplicative and asymptotically isometric, and ψ is therefore an injective $*$ -homomorphism from A into $\mathcal{O}_2 \otimes \mathcal{O}_2$. Use finally Theorem 5.2.1 to obtain an embedding of A into \mathcal{O}_2 . \square

Lemma 6.3.10. *Every quasidiagonal, separable, unital, exact C^* -algebra admits a unital embedding into \mathcal{O}_2*

Proof: For every quasidiagonal, separable, unital C^* -algebra A there is a commutative diagram

$$\begin{array}{ccc}
& \prod_{k=1}^{\infty} M_{n_k} & \xrightarrow{\iota} \ell^{\infty}(\mathcal{O}_2) \\
\rho \nearrow & \downarrow \pi & \downarrow \pi_{\infty} \\
A \xrightarrow{\varphi} & \prod_{k=1}^{\infty} M_{n_k} / \sum_{k=1}^{\infty} M_{n_k} & \xrightarrow{\bar{\iota}} (\mathcal{O}_2)_{\infty}
\end{array} \quad (6.3.8)$$

for some sequence $\{M_{n_k}\}_{k=1}^{\infty}$ of matrix algebras over \mathbb{C} , where φ is a unital embedding and ρ is a unital, completely positive lift of φ . (Each matrix algebra $M_n(\mathbb{C})$ has a unital embedding into \mathcal{O}_2 , and in this way we obtain the embeddings ι and $\bar{\iota}$ displayed in (6.3.8).) If A is exact, then A admits a unital embedding into \mathcal{O}_2 by Lemma 6.3.9 (and the commutative diagram (6.3.8)). \square

In the proof of the embedding theorem below we wish to embed a crossed product $B \rtimes_{\tau} \mathbb{Z}$ into \mathcal{O}_2 , where B is a (certain) unital C^* -algebra and τ is an automorphism of B . The crossed product $B \rtimes_{\tau} \mathbb{Z}$ is generated by B and a unitary u that implements the action τ . If D is a unital C^* -algebra, if $\varphi: B \rightarrow D$ is a unital $*$ -homomorphism, and v is a unitary element in D such that $v\varphi(b)v^* = \varphi(\tau(b))$ for all b in B , then there is a $*$ -homomorphism $\bar{\varphi}: B \rtimes_{\tau} \mathbb{Z} \rightarrow D$ satisfying $\bar{\varphi}|_B = \varphi$ and $\bar{\varphi}(u) = v$. If φ is injective, then the $*$ -homomorphism $\psi: B \rtimes_{\tau} \mathbb{Z} \rightarrow D \otimes C(\mathbb{T})$, given by $\psi(b) = \varphi(b) \otimes 1$ for b in B , and $\psi(u) = \bar{\varphi}(u) \otimes z$, where z is the canonical unitary generator for $C(\mathbb{T})$, is injective (see [107, Theorem 7.7.5]).

Theorem 6.3.11 (Kirchberg's Exact Embedding Theorem). *A separable C^* -algebra A is exact if and only if there is an injective $*$ -homomorphism $\iota: A \rightarrow \mathcal{O}_2$. The embedding ι can be chosen to be unital if A is unital.*

In other words, a separable C^* -algebra is exact if and only if it is isomorphic to a sub- C^* -algebra of the Cuntz algebra \mathcal{O}_2 .

Proof: The proof, taken from [84, Theorem 2.8], is a bootstrap argument where the general statement is obtained from the two previous lemmas through some manipulations with A . We give all details of the proof only in the case where A is nuclear — the general case follows the same line, but requires extra work.

Consider the unitization B of $C_0(\mathbb{R}, A)$, let τ be the automorphism of $C_0(\mathbb{R}, A)$ obtained by translation: $\tau(f)(t) = f(t+1)$, and extend τ to an automorphism of B . Then

$$\mathcal{H} \otimes A \cong C_0(\mathbb{R}, A) \rtimes_{\tau} \mathbb{Z} \subseteq B \rtimes_{\tau} \mathbb{Z}.$$

Since A embeds into $\mathcal{H} \otimes A$ it suffices to show that the crossed product $B \rtimes_{\tau} \mathbb{Z}$ embeds into \mathcal{O}_2 . (The latter claim in Theorem 6.3.11 about making ι unital will follow from the first part of the theorem because each (non-zero) corner $p\mathcal{O}_2p$ of \mathcal{O}_2 is isomorphic to \mathcal{O}_2 .)

The C^* -algebras B and $B \rtimes_{\tau} \mathbb{Z}$ are exact by Proposition 6.1.10. The C^* -algebra B is a sub- C^* -algebra of a C^* -algebra that is homotopy equivalent to 0, and so we can use [143, Theorem 5] by Voiculescu to conclude that B is quasidiagonal.

Lemma 6.3.10 now tells us that B embeds unittally into \mathcal{O}_2 , and we can therefore assume that B is a unital sub- C^* -algebra of \mathcal{O}_2 .

The two unital, injective $*$ -homomorphism $\iota, \iota \circ \tau: B \rightarrow \mathcal{O}_2$ are approximately unitarily equivalent by Theorem 6.3.8, and so we can find a sequence $\{v_n\}_{n=1}^\infty$ of unitaries in \mathcal{O}_2 such that $\|v_n b v_n^* - \tau(b)\| \rightarrow 0$ for all b in B .

Put $v_\infty = (v_1, v_2, \dots)$ in $\ell^\infty(\mathcal{O}_2)$, and put $v = \pi_\infty(v_\infty)$ in $(\mathcal{O}_2)_\infty$. With ι being the canonical inclusion of \mathcal{O}_2 into $(\mathcal{O}_2)_\infty$ we have $v\iota(b)v^* = \iota(\tau(b))$ for all b in B . Hence there is a $*$ -homomorphism $\psi: B \rtimes_\tau \mathbb{Z} \rightarrow (\mathcal{O}_2)_\infty$ that satisfies $\psi|_B = \iota|_B$ and $\psi(u) = v$, where u is the unitary in $B \rtimes_\tau \mathbb{Z}$ that implements the action of τ .

By the remarks preceding the formulation of the present theorem there is an *injective* unital $*$ -homomorphism from $B \rtimes_\tau \mathbb{Z}$ into $(\mathcal{O}_2)_\infty \otimes C(\mathbb{T})$. One can for example use the embedding $\mathcal{O}_2 \otimes \mathcal{O}_2 \rightarrow \mathcal{O}_2$ (from Theorem 5.2.1) to find a unital embedding of $(\mathcal{O}_2)_\infty \otimes C(\mathbb{T})$ into $(\mathcal{O}_2)_\infty$, thus supplying us with a unital, injective $*$ -homomorphism from $B \rtimes_\tau \mathbb{Z}$ into $(\mathcal{O}_2)_\infty$.

If A is nuclear, then so are B and $B \rtimes_\tau \mathbb{Z}$, and in this case we can use the Choi–Effros lifting theorem (Theorem 6.1.4) to find a unital, completely positive lift $\rho: B \rtimes_\tau \mathbb{Z} \rightarrow \ell^\infty(\mathcal{O}_2)$ of ψ . Lemma 6.3.9 now gives the desired embedding of $B \rtimes_\tau \mathbb{Z}$ into \mathcal{O}_2 .

If A is exact (and not nuclear), then one can give an ad hoc argument based on a lifting result of Effros and Haagerup, [43], to show that the embedding of $B \rtimes_\tau \mathbb{Z}$ into $(\mathcal{O}_2)_\infty$ has a unital, completely positive lift (see [84] for the details). \square

We state without proof the following characterization of separable, *nuclear* C^* -algebras from [79, Theorem A (ii)]. This theorem will not be used in the classification of Kirchberg algebras described in the following two sections.

Theorem 6.3.12 (Kirchberg Nuclear Embedding Theorem). *A separable C^* -algebra A is nuclear if and only if there is an injective $*$ -homomorphism $\iota: A \rightarrow \mathcal{O}_2$ and a conditional expectation from \mathcal{O}_2 onto $\iota(A)$.*

Chapter 7

Tensor Products by \mathcal{O}_2 and by \mathcal{O}_∞

The fundamental role of the Cuntz algebras \mathcal{O}_2 and \mathcal{O}_∞ is demonstrated in Kirchberg’s two tensor product theorems proved in this chapter. Recall that the zero C^* -algebra and \mathcal{O}_2 have isomorphic K -theory and that the complex numbers \mathbb{C} and \mathcal{O}_∞ have isomorphic K -theory. This should lead us to think of \mathcal{O}_2 and \mathcal{O}_∞ as the purely infinite version of (the tensorial) zero, respectively, of (the tensorial) unit. Indeed, for any C^* -algebra A , $A \otimes \mathcal{O}_2$ is KK -equivalent to \mathcal{O}_2 and $A \otimes \mathcal{O}_\infty$ is KK -equivalent to A . Kirchberg’s tensor product theorems say that these KK -equivalences are actually isomorphisms for particularly nice C^* -algebras A .

7.1 On $A \otimes \mathcal{O}_2$

The isomorphism theorem $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ below follows immediately from the embedding theorem (Theorem 6.3.11), from the uniqueness theorems (Theorems 5.1.1 and 6.3.8), and from an approximate intertwining (as in Corollary 2.3.4). Before formulating and proving this isomorphism we state and prove Proposition 7.1.1 below (from [84, Proposition 3.4]) — a result of independent interest that also is a key ingredient in the $A \otimes \mathcal{O}_\infty$ theorem in the next section. The proposition has a corollary that all unital Kirchberg algebras have plenty of central sequences. The relative commutant $A_\omega \cap A'$ is the C^* -algebra of asymptotically central sequences in A . (Recall the definition of ultrapowers A_ω from Section 6.2.)

Proposition 7.1.1 (Central Sequences in Kirchberg Algebras). *The relative commutant $A_\omega \cap A'$ is a purely infinite, unital, simple C^* -algebra whenever A is a unital Kirchberg algebra and ω is a free ultrafilter.*

Proof: By Proposition 4.1.1 it will suffice to show that for each positive element h in $A_\omega \cap A'$ of norm 1 there exists an isometry s in $A_\omega \cap A'$ with $s^*hs = 1$.

Put $X = \text{sp}(h) \subseteq [0, 1]$ and define $*$ -homomorphisms $\varphi, \psi: C(X) \otimes A \rightarrow A_\omega$ by

$$\varphi(f \otimes a) = f(h)a, \quad \psi(f \otimes a) = f(1)a, \quad f \in C(X), a \in A.$$

Because $C(X) \otimes A$ is nuclear, the Choi–Effros lifting theorem (Theorem 6.1.4) provides unital, completely positive lifts $\rho, \sigma: C(X) \otimes A \rightarrow \ell^\infty(A)$ of φ and ψ . Write

$$\rho(x) = (\rho_1(x), \rho_2(x), \rho_3(x), \dots), \quad \sigma(x) = (\sigma_1(x), \sigma_2(x), \sigma_3(x), \dots),$$

for x in $C(X) \otimes A$, where ρ_n and σ_n are unital, completely positive maps from $C(X) \otimes A$ into A . Since φ is injective and A is a Kirchberg algebra we can use Corollary 6.3.5 (ii) to find a sequence $\{s_n\}_{n=1}^\infty$ of isometries in A such that $\lim_\omega \|s_n^* \rho_n(x) s_n - \sigma_n(x)\| = 0$ for all x in $C(X) \otimes A$. Put $s = \pi_\omega(s_1, s_2, \dots) \in A_\omega$. Then s is an isometry and $s^* \varphi(x) s = \psi(x)$ for all x in $C(X) \otimes A$. It follows in particular that

$$s^*as = s^* \varphi(1 \otimes a) s = \psi(1 \otimes a) = a$$

for all a in A , whence s belongs to $A_\omega \cap A'$ by Lemma 6.3.6. Finally,

$$s^*hs = s^* \varphi(\iota \otimes 1) s = \psi(\iota \otimes 1) = 1,$$

where ι is the function $\iota(t) = t$. □

The theorem below is from [84, Theorem 3.7].

Theorem 7.1.2 (Kirchberg’s $A \otimes \mathcal{O}_2$ Theorem). *The tensor product $A \otimes \mathcal{O}_2$ is isomorphic to \mathcal{O}_2 if and only if A is a simple, separable, unital, and nuclear C^* -algebra.*

Proof: The “only if” part is trivial. To see the “if” part, suppose that A is a simple, separable, unital, and nuclear C^* -algebra. Then we have unital $*$ -homomorphisms

$$A \otimes \mathcal{O}_2 \xrightarrow{\varphi} \mathcal{O}_2 \xrightarrow{\iota} A \otimes \mathcal{O}_2,$$

where the existence of φ follows from Kirchberg’s embedding theorem (Theorem 6.3.11) and ι is the unital $*$ -homomorphism given by $\iota(x) = 1 \otimes x$. It follows from Theorem 5.1.1 that $\varphi \circ \iota \approx_u \text{id}_{\mathcal{O}_2}$ and it follows from Theorem 6.3.8 (i) that $\iota \circ \varphi \approx_u \text{id}_{A \otimes \mathcal{O}_2}$. The theorem now follows from Corollary 2.3.4 (approximate intertwining). \square

7.2 On $A \otimes \mathcal{O}_\infty$

To derive the second tensor product theorem we use an ultrapower analog of the approximate intertwining result Proposition 2.3.5. Recall that $\mathcal{M}(A)$ denotes the multiplier algebra of A . We adopt the convention $B \subseteq \mathcal{M}(B) \subseteq \mathcal{M}(B)_\omega$ and $A_\omega \subseteq B_\omega$ when A and B are C^* -algebras with $A \subseteq B$ and ω is a filter on \mathbb{N} .

Proposition 7.2.1 (Approximate Intertwining — Ultrapower Formulation).

Let A and B be separable C^* -algebras, let $\varphi: A \rightarrow B$ be an injective $*$ -homomorphism, and suppose that for some free ultrafilter ω on \mathbb{N} there is a sequence $\{v_n\}_{n=1}^\infty$ of unitaries in $\mathcal{M}(B)_\omega \cap \varphi(A)'$ such that

$$\lim_{n \rightarrow \infty} \text{dist}(v_n^* b v_n, \varphi(A)_\omega) = 0$$

for all b in B . Then A and B are isomorphic and there is an isomorphism $\psi: A \rightarrow B$ which is approximately unitarily equivalent to φ .

Proof: It suffices to show that the hypothesis above imply the hypothesis of Proposition 2.3.5. To this end let ω be a free ultrafilter and let $\{v_n\}_{n=1}^\infty$ be a sequence of unitaries in $\mathcal{M}(B)_\omega \cap \varphi(A)'$ such that the distance from $v_n^* b v_n$ to $\varphi(A)_\omega$ tends to 0 for all b in B . Let $\{a_1, a_2, \dots, a_N\}$ and $\{b_1, b_2, \dots, b_M\}$ be finite subsets of A and B , respectively, and let $\varepsilon > 0$. Take $v = v_k$ for a sufficiently large k and elements c_1, c_2, \dots, c_M in $\varphi(A)_\omega$ with $\|v^* b_j v - c_j\| \leq \varepsilon/2$. Next use Lemma 6.2.4 to write

$$v = \pi_\omega(u_1, u_2, u_3, \dots), \quad c_j = \pi_\omega(\varphi(a_{1,j}), \varphi(a_{2,j}), \varphi(a_{3,j}), \dots),$$

where $\{u_n\}$ is a sequence of unitaries in $\mathcal{M}(B)$, and where $\{a_{n,j}\}_{n=1}^\infty$ is a bounded sequence in A for each j . Since v commutes with $\varphi(A)$ and $\|v^* b_j v - c_j\| \leq \varepsilon/2$ we get

$$\lim_{\omega} \|u_n \varphi(a) - \varphi(a) u_n\| = 0, \quad \lim_{\omega} \|u_n^* b_j u_n - \varphi(a_{n,j})\| \leq \varepsilon/2$$

for all a in A and for $j = 1, 2, \dots, M$. It now follows from Lemma 6.2.3 that there are infinitely many k such that

$$\|u_k \varphi(a_i) - \varphi(a_i) u_k\| \leq \varepsilon, \quad \text{dist}(u_k^* b_j u_k, \varphi(A)) \leq \varepsilon$$

for $i = 1, 2, \dots, N$ and $j = 1, 2, \dots, M$. This shows that the hypothesis of Proposition 2.3.5 are satisfied. \square

Theorem 7.2.2. *Let A be a separable C^* -algebra and let B be a simple, separable, unital, and nuclear C^* -algebra. Then A is isomorphic to $A \otimes B$ if*

- (i) B admits a unital embedding into $\mathcal{M}(A)_\omega \cap A'$ for some free ultrafilter ω on \mathbb{N} ; and
- (ii) the two $*$ -homomorphisms $\alpha, \beta: B \rightarrow B \otimes B$ given by $\alpha(b) = b \otimes 1$ and $\beta(b) = 1 \otimes b$, $b \in B$, are approximately unitarily equivalent.

Remark 7.2.3.

- (a) Condition (i) in Theorem 7.2.2 is satisfied if there is a sequence of unital, injective $*$ -homomorphisms $\varphi_n: B \rightarrow \mathcal{M}(A)$ such that $\|\varphi_n(b)a - a\varphi_n(b)\| \rightarrow 0$ for all $a \in A$ and $b \in B$, since $b \mapsto \pi_\omega(\varphi_1(b), \varphi_2(b), \dots)$ then will be a unital embedding of B into $\mathcal{M}(A)_\omega \cap A'$.
- (b) If B satisfies condition (ii) in Theorem 7.2.2, then B must be nuclear and simple (see [84]).
- (c) The “if” in Theorem 7.2.2 can be replaced by “if and only if” if B is isomorphic to $\bigotimes_{n=1}^\infty B$. The C^* -algebras \mathcal{O}_2 and \mathcal{O}_∞ have this property by Corollary 5.2.4 and Theorem 7.2.6.
- (d) Condition (ii) in Theorem 7.2.2 is satisfied if and only if there is a sequence $\{u_n\}$ of unitaries in $B \otimes B$ such that $u_n(b \otimes 1)u_n^* \rightarrow 1 \otimes b$ for all b in B . This is colloquially referred to by saying that the “half flip on $B \otimes B$ is approximately inner”, and it is a weaker statement than approximate innerness of the flip on $B \otimes B$ given by $b_1 \otimes b_2 \mapsto b_2 \otimes b_1$.

Proof of Theorem 7.2.2: Let $\varphi: A \rightarrow A \otimes B$ be the injective $*$ -homomorphism $\varphi(a) = a \otimes 1_B$ with image $\varphi(A) = A \otimes \mathbb{C} \cdot 1_B$. We show that φ fulfills the hypothesis of Proposition 7.2.1. There is a unital $*$ -homomorphism $\alpha: B \rightarrow \mathcal{M}(\varphi(A))_\omega \cap \varphi(A)'$ by (i). Let $\beta: B \rightarrow \mathcal{M}(A \otimes B)_\omega \cap \varphi(A)'$ be given by $\beta(b) = 1_{\mathcal{M}(A)} \otimes b$. The images of α and β commute, so these two images generate the C^* -algebra $B \otimes B$, and we can conclude from (ii) that there is a sequence $\{v_n\}$ of unitaries in the sub- C^* -algebra $C^*(\alpha(B), \beta(B))$ of $\mathcal{M}(A \otimes B)_\omega \cap \varphi(A)'$ such that $v_n^* \beta(b) v_n \rightarrow \alpha(b)$ for all b in B . Hence

$$\lim_{n \rightarrow \infty} v_n^* (a \otimes b) v_n = \lim_{n \rightarrow \infty} v_n^* \varphi(a) \beta(b) v_n = \varphi(a) \alpha(b) \in \varphi(A)_\omega$$

for all a in A and all b in B . This shows that

$$\lim_{n \rightarrow \infty} \text{dist}(v_n^* c v_n, \varphi(A)_\omega) = 0$$

for all c in $A \otimes B$. The theorem now follows from Proposition 7.2.1. \square

Remark 7.2.4 ($\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ revisited). The isomorphism $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ (from Theorem 5.2.1) is a corollary to Theorem 7.2.2 with $A = B = \mathcal{O}_2$:

We must check conditions (i) and (ii) of Theorem 7.2.2. It follows from Lemma 5.2.3 that there is an asymptotically central sequence $\{\rho_n\}$ of unital $*$ -homomorphisms from \mathcal{O}_2 into itself, and hence (i) holds by Remark 7.2.3 (a). To check (ii) we must show that the two $*$ -homomorphisms $\mathcal{O}_2 \rightarrow \mathcal{O}_2 \otimes \mathcal{O}_2$, given by $x \mapsto x \otimes 1$ and $x \mapsto 1 \otimes x$, are approximately unitarily equivalent, but this is contained in Theorem 5.1.1.

The proof of Theorem 5.2.1 given in Section 5.2 is not much different from the proof described here.

We now apply Theorem 7.2.2 to the case $B = \mathcal{O}_\infty$ and obtain a characterization of those unital, separable C^* -algebras A for which $A \cong A \otimes \mathcal{O}_\infty$. That \mathcal{O}_∞ satisfies condition (ii) in Theorem 7.2.2 follows from H. Lin and Phillip's uniqueness theorem for \mathcal{O}_∞ from [95] (a result that has as a corollary that \mathcal{O}_∞ is classifiable; cf. Section 2.5):

Proposition 7.2.5 (Uniqueness for \mathcal{O}_∞). *Any two unital $*$ -homomorphisms φ, ψ from \mathcal{O}_∞ into a unital, purely infinite, simple C^* -algebra A are approximately unitarily equivalent.*

Proof: The first step is to establish the proposition in the case where $[1_A]_0 = 0$ in $K_0(A)$. Let s_1, s_2, \dots be the (canonical) generators of \mathcal{O}_∞ . For each (even) positive integer n the three projections

$$1_A - \sum_{j=1}^{n-1} \varphi(s_j s_j^*), \quad 1_A - \sum_{j=1}^{n-1} \psi(s_j s_j^*), \quad 1_A,$$

are non-zero, they represent the same element of $K_0(A)$ (namely the zero-element), and hence they are equivalent by Proposition 4.1.4. We can therefore find isometries t_n and r_n in A such that

$$\sum_{j=1}^{n-1} \varphi(s_j s_j^*) + t_n t_n^* = 1_A = \sum_{j=1}^{n-1} \psi(s_j s_j^*) + r_n r_n^*.$$

Define unital $*$ -homomorphisms $\varphi_n, \psi_n: \mathcal{O}_n \rightarrow A$ by mapping the canonical generators of \mathcal{O}_n to $(\varphi(s_1), \varphi(s_2), \dots, \varphi(s_{n-1}), t_n)$, respectively to $(\psi(s_1), \psi(s_2), \dots, \psi(s_{n-1}), r_n)$. It follows from Theorem 5.1.2 that φ_n and ψ_n are approximately unitarily equivalent if n is even and the K_1 -class of the unitary element

$$\sum_{j=1}^{n-1} \varphi(s_j) \psi(s_j)^* + t_n r_n^*$$

belongs to $(n-1)K_1(A)$. Upon replacing t_n with wt_n for a suitable unitary element w in the corner algebra $(t_n t_n^*)A(t_n t_n^*)$ this requirement can be met. We then have a

sequence $\{v_k\}_{k=1}^\infty$ of unitaries in A such that $v_k \varphi_n(x) v_k^* \rightarrow \psi_n(x)$ for all x in \mathcal{O}_n ; and in particular, $v_k \varphi(s_j) v_k^* \rightarrow \psi(s_j)$ for $j = 1, 2, \dots, n-1$. This proves the proposition in the special case where $[1_A]_0 = 0$.

We proceed to give the proof in the general case. Let ω be a free ultrafilter on \mathbb{N} , and consider the C^* -algebras $A \subset A_\omega$. The relative commutant $A_\omega \cap A'$ is simple and purely infinite by Proposition 7.1.1. In particular there are projections p, q, r in $A_\omega \cap A'$ satisfying $1 = p + q + r$ and $1 \sim p \sim q$ (relative to A_ω). Now $[p+r]_0 = [q+r]_0 = 0$ in $K_0(A_\omega \cap A')$. For each $*$ -homomorphism $\rho: \mathcal{O}_\infty \rightarrow A$ and for each projection e in $A_\omega \cap A'$, let $\rho_e: \mathcal{O}_\infty \rightarrow eA_\omega e$ be the unital $*$ -homomorphism given by $\rho_e(x) = \rho(x)e$.

It follows from the first part of the proof that $\varphi_{p+r} \approx_u \psi_{p+r}$ in $(p+r)A_\omega(p+r)$ and that $\varphi_q + \psi_r \approx_u \psi_{q+r}$ in $(q+r)A_\omega(q+r)$. Hence

$$\varphi = \varphi_{p+r} + \varphi_q \approx_u \psi_{p+r} + \varphi_q = (\varphi_q + \psi_r) + \psi_p \approx_u \psi_{q+r} + \psi_p = \psi \text{ in } A_\omega.$$

Use finally Lemma 6.2.5 to see that $\varphi \approx_u \psi$ in A . \square

Theorem 7.2.6 (Tensor Products with \mathcal{O}_∞).

- (i) Let A be a separable C^* -algebra. Then A is isomorphic to $A \otimes \mathcal{O}_\infty$ if and only if there is a unital embedding of \mathcal{O}_∞ into $\mathcal{M}(A)_\omega \cap A'$ for some free ultrafilter ω .
- (ii) Let A be a simple, separable, and nuclear C^* -algebra. Then A is isomorphic to $A \otimes \mathcal{O}_\infty$ if and only if A is purely infinite.
- (iii) $\mathcal{O}_\infty \cong \bigotimes_{n=1}^\infty \mathcal{O}_\infty$.

Proof: The “if” part of (i) is an immediate consequence of Theorem 7.2.2 and Proposition 7.2.5.

(ii). If A is simple, then $A \otimes \mathcal{O}_\infty$ is simple and purely infinite by Theorem 4.1.10. Suppose conversely that A is purely infinite (and simple, separable, and nuclear), i.e., that A is a Kirchberg algebra. By Zhang’s Dichotomy (Proposition 4.1.3), A is either unital or isomorphic to $A_0 \otimes \mathcal{K}$ for some unital Kirchberg algebra A_0 . It therefore suffices to prove the claim in the case where A is unital. Proposition 7.1.1 says that $A_\omega \cap A'$ is simple and purely infinite. In particular there is a unital embedding of \mathcal{O}_∞ into $A_\omega \cap A'$ (see Proposition 4.2.3). Hence $A \cong A \otimes \mathcal{O}_\infty$ by (i).

(iii). To prove this proceed as in Corollaries 5.1.5 and 5.2.4.

The “only if” part of (i). If A is isomorphic to $A \otimes \mathcal{O}_\infty$, then we can use (iii) to find a sequence of unital embeddings $\varphi_n: \mathcal{O}_\infty \rightarrow \mathcal{M}(A)$ such that $\varphi_n(x)a - a\varphi_n(x) \rightarrow 0$ for all x in \mathcal{O}_∞ and all a in A . This in turn will give a unital $*$ -homomorphism from \mathcal{O}_∞ into $\mathcal{M}(A)_\omega \cap A'$. \square

Corollary 7.2.7. Every Kirchberg algebra is approximately divisible (see Definition 3.1.10).

Proof: If B is an approximately divisible C^* -algebra, then so is $A \otimes B$ for every C^* -algebra B (with respect to any tensor product). By Theorem 7.2.6 (ii) and (iii) it therefore suffices to show that $\bigotimes_{k=1}^\infty \mathcal{O}_\infty$ is approximately divisible. To see

this, take for each fixed natural number n and for each k a unital embedding φ_k of $M_n(\mathbb{C}) \oplus M_{n+1}(\mathbb{C})$ into the k th tensor factor \mathcal{C}_∞ . Then $\{\varphi_k\}_{k=1}^\infty$ is an asymptotically central sequence of embeddings of $M_n(\mathbb{C}) \oplus M_{n+1}(\mathbb{C})$ into $\bigotimes_{k=1}^\infty \mathcal{C}_\infty$. \square

There are examples of (necessarily non-nuclear) separable, unital, simple, purely infinite C^* -algebras A such that A is not isomorphic to $A \otimes \mathcal{C}_\infty$ and A is not approximately divisible, see [41].

Chapter 8

Classification of Kirchberg Algebras

8.1 Equivalence and Addition of $*$ -homomorphisms

The classification theorem for Kirchberg algebras says that two Kirchberg algebras are stably isomorphic if they are KK -equivalent. By the approximate intertwining technique (see in particular Corollary 2.3.4) this will follow from the more refined statement that we shall refer to as the classification theorem for $*$ -homomorphisms between Kirchberg algebras: every element of $KK(A, B)$ is represented by a non-zero $*$ -homomorphism from A to B and if $\varphi, \psi: A \rightarrow B$ are two $*$ -homomorphisms representing the same element of $KK(A, B)$, then φ and ψ are asymptotically unitarily equivalent.

This, and a stronger statement, was proved independently by Kirchberg (Theorem 8.3.3) and by Phillips (Theorem 8.2.1). Phillips' work is based on the exact embedding theorem and the tensor product theorems from Chapters 6 and 7. We describe Phillips' approach in detail and we indicate some of the ideas in Kirchberg's approach. Another route to the classification theorem (for Kirchberg algebras in the UCT class \mathcal{N}) based on the embedding and of tensor product theorems was given by Dadarlat and Eilers in [37].

The isomorphism theorem and some of its consequences are treated in Section 8.4.

Recall from Definition 1.1.15 the notions of equivalences between $*$ -homomorphisms and their interrelations:

$$\begin{array}{ccccc} \varphi \sim_u \psi & \implies & \varphi \approx_{uh} \psi & \implies & \varphi \approx_u \psi \\ & & \downarrow & & \\ \varphi \sim_h \psi & \implies & KK(\varphi) = KK(\psi) & & \end{array} \quad (8.1.1)$$

If B is stable (in which case the unitary group of $\mathcal{M}(B)$ is connected), then

$$\varphi \sim_u \psi \implies \varphi \approx_{uh} \psi \implies \varphi \sim_h \psi \implies KK(\varphi) = KK(\psi). \quad (8.1.2)$$

If B is stable, then define the sum $\varphi \oplus \psi$ (occasionally denoted the *Cuntz sum*) of two $*$ -homomorphisms $\varphi, \psi: A \rightarrow B$ as follows. Because $\mathcal{M}(B)$ contains a copy of

Chapter 6

Voiculescu's Approximation Entropies

As mentioned in the introduction Voiculescu [V] has introduced entropies which are refinements of mean entropy and which provide a very nice technique to study entropy.

Let M be a hyperfinite von Neumann algebra with a faithful normal tracial state τ . Let $Pf(M)$ denote the family of finite subsets of M . Modifying the notation introduced before Lemma 2.9 we write $\omega \subset^\delta \mathcal{K}$ if $\omega \in Pf(M)$, $\mathcal{K} \subset M$ satisfy the condition that for each $x \in \omega$ there is a $y \in \mathcal{K}$ such that $\|x - y\|_2 < \delta$. Let further $\mathcal{F}(M)$ denote the family of finite dimensional C^* -subalgebras of M . As noted in Chapter 2, if $A \in \mathcal{F}(M)$ then $\text{rank } A$ is the dimension of a masa in A .

Definition 6.1. [V] If $\omega \in Pf(M)$, $\delta > 0$ put

$$r_\tau(\omega, \delta) = \inf\{\text{rank } A : A \in \mathcal{F}(M), \omega \subset^\delta A\},$$

called the δ -rank of ω .

Note that a slightly different choice for $r_\tau(\omega, \delta)$ would be to replace $\text{rank } A$ by $\exp(H_\tau(A))$, see [C5] and [G-S2].

Definition 6.2. [V] If α is a τ -invariant automorphism of M and $\delta > 0$, $\omega \in Pf(M)$ we put:

$$\begin{aligned} ha_\tau(\alpha, \omega, \delta) &= \limsup_{n \rightarrow \infty} \frac{1}{n} \log r_\tau\left(\bigcup_{j=0}^{n-1} \alpha^j(\omega), \delta\right) \\ ha_\tau(\alpha, \omega) &= \sup_{\delta > 0} ha_\tau(\alpha, \omega, \delta) \\ ha_\tau(\alpha) &= \sup\{ha_\tau(\alpha, \omega) : \omega \in Pf(M)\}. \end{aligned}$$

$ha_\tau(\alpha)$ is the *approximation entropy* of α .

An alternative is to take \liminf in the definition of $ha_\tau(\alpha, \omega, \delta)$. Then we get the *lower approximation entropy* $\ell ha_\tau(\alpha)$.

As for the previous entropies we have $ha_\tau(\alpha^k) = |k|ha_\tau(\alpha)$, $k \in \mathbb{Z}$. The proof that $ha_\tau(\alpha^{-1}) = ha_\tau(\alpha)$ is very easy; indeed

$$r_\tau\left(\bigcup_0^{n-1} \alpha^j(\omega), \delta\right) = r_\tau\left(\alpha^{-n+1}\left(\bigcup_0^{n-1} \alpha^j(\omega)\right), \delta\right) = r_\tau\left(\bigcup_0^{n-1} \alpha^{-j}(\omega), \delta\right).$$

The analogue of the Kolmogoroff-Sinai Theorem takes the following form.

Proposition 6.3. [V] *Let $\omega_j \in Pf(M)$, $j \in \mathbb{N}$, $\omega_1 \subset \omega_2 \subset \dots$ be a sequence such that $\bigcup_{j \in \mathbb{N}} \bigcup_{n \in \mathbb{Z}} \alpha^n(\omega_j)$ generates M as a von Neumann algebra. Then*

$$ha_\tau(\alpha) = \sup_{j \in \mathbb{N}} ha_\tau(\alpha, \omega_j) .$$

Proposition 6.4. [V] (i) *If $A \in \mathcal{F}(M)$ and $\omega \in Pf(M)$ generates A as a C^* -algebra then $H_\tau(A, \alpha) \leq \ell ha_\tau(\alpha, \omega)$.*
(ii) *$H(\alpha) \leq \ell ha_\tau(\alpha) \leq ha_\tau(\alpha)$.*

Proof. It suffices to show (i). Let $\varepsilon > 0$. By Lemma 2.9 there exists $\delta > 0$ such that if $B \in \mathcal{F}(M)$ satisfies $A \subset^\delta B$ then $H(A|B) < \varepsilon$. By hypothesis on ω there exists therefore $\delta_1 > 0$ such that if $\omega \subset^{\delta_1} B$ then $H(A|B) < \varepsilon$. This also implies that if $\alpha^j(\omega) \subset^{\delta_1} B$ then $H(\alpha^j(A)|B) < \varepsilon$. Put $r(n) = r_\tau(\bigcup_{j=0}^{n-1} \alpha^j(\omega), \delta_1)$. Then there exists $B \in \mathcal{F}(M)$ with $\text{rank } B = r(n)$ and $\alpha^j(A) \subset^{\delta_1} B$ for $0 \leq j \leq n-1$. Hence by Property (F) in Chapter 2,

$$\begin{aligned} H(A, \alpha(A), \dots, \alpha^{n-1}(A)) &\leq H(B) + \sum_{j=0}^{n-1} H(\alpha^j(A)|B) \\ &\leq \log r(n) + n\varepsilon , \end{aligned}$$

so that $H(A, \alpha) \leq ha_\tau(\alpha, \omega, \delta_1) + \varepsilon$, proving the proposition. \square

In general it can be quite difficult to know when an algebra B as in the above proof satisfies $\text{rank } B = r(n)$, hence to compute $r(n)$. A case when it is easy is that of the n -shift. In the notation of Remark 2.13 let $R = \bigotimes_{i \in \mathbb{Z}} (M_i, \tau_i)$ with $M_i = M_n(\mathbb{C})$, and α be the shift. Let $A = M_0 \in \mathcal{F}(R)$. Let, as is often done, ω be a complete set of matrix units for A . By Proposition 6.3 $ha_\tau(\alpha, \omega) = ha_\tau(\alpha)$. On the other hand $\bigcup_{j=0}^{k-1} \alpha^j(\omega) \subset \bigvee_{j=0}^{k-1} \alpha^j(A)$, which is a I_{n^k} -factor, hence

$$r_\tau\left(\bigcup_{j=0}^{n-1} \alpha^j(\omega), \delta\right) \leq n^k \quad \text{for all } \delta > 0 .$$

Thus, by the above and Proposition 6.4 $ha_\tau(\alpha) = ha_\tau(\alpha, \omega) \leq \log n = H(\alpha) \leq ha_\tau(\alpha)$, so $ha_\tau(\alpha) = \log n$.

One test for any definition of entropy is that it should coincide with the classical entropy in the abelian case. Via an application of the Shannon, Breiman, McMillan Theorem the approximation entropy does this [V].

We remarked in (2.6) that the entropy $H(\alpha)$ is superadditive on tensor products. For the approximation entropy the inequality goes the other way, i.e.

$$ha_{\tau_1 \otimes \tau_2}(\alpha_1 \otimes \alpha_2) \leq ha_{\tau_1}(\alpha_1) + ha_{\tau_2}(\alpha_2) .$$

Hence to show equality it suffices to show that the two entropies coincide. In Chapter 10 we shall look at such cases.

In the above treatment of the approximation entropy the trace played a minor role. If A is an AF-algebra we can do essentially the same, where we now replace the distance $\| \cdot \|_2$ with the operator norm. Then we get the entropy Voiculescu denotes by $hat(\alpha)$ – the *topological approximation entropy of α* .

The most flexible and therefore probably the most useful of Voiculescu's approximation entropies are the completely positive ones. We consider the von Neumann algebra definition first. Let (M, φ, α) be a W^* -dynamical system with M injective and φ faithful. Let $\|x\|_\varphi = \varphi(x^*x)^{1/2}$ be the φ -norm on M . Let

$$CPA(M, \varphi) = \{(\rho, \psi, B) : B \text{ a finite dimensional } C^*\text{-algebra, } \rho : M \rightarrow B, \\ \psi : B \rightarrow M \text{ are unital completely positive maps such that } \varphi \circ \psi \circ \rho = \varphi\} .$$

Definition 6.5. [V] If $\omega \in Pf(M)$ and $\delta > 0$ the *completely positive δ -rank* is

$$rcp_\varphi(\omega, \delta) \\ = \inf\{\text{rank } B : (\rho, \psi, B) \in CPA(M, \varphi), \|\psi \circ \rho(x) - x\|_\varphi < \delta \text{ for all } x \in \omega\} .$$

Then we continue as in Definition 6.2 to define the *completely positive approximation entropy $h_{cpa_\varphi}(\alpha)$* . Again we can prove much the same results as for the approximation entropy $ha_\tau(\alpha)$. Instead of using $\text{rank } B$ one can also use the entropy $S(\varphi \circ \psi)$ of the state $\varphi \circ \psi$ on B to obtain a slightly different entropy, see [C5, C7].

The C^* -algebra version of the above definition is like the corresponding entropy $hat(\alpha)$ independent of invariant states. Voiculescu defined this entropy for nuclear C^* -algebras, but later on Brown [Br1] saw that one can develop the theory for exact C^* -algebras.

Definition 6.6. [Br1] Let A be a C^* -algebra and $\pi : A \rightarrow B(H)$ a faithful $*$ -representation. Then

$$CPA(\pi, A) = \{(\rho, \psi, B) : \rho : A \rightarrow B, \psi : B \rightarrow B(H) \text{ are contractive} \\ \text{completely positive maps, } B \text{ is finite dimensional } C^*\text{-algebra}\}$$

Let $\omega \in Pf(A)$, $\delta > 0$. Then

$$rcp(\pi, \omega, \delta) = \inf\{\text{rank } B : (\rho, \psi, B) \in CPA(\pi, A), \text{ and} \\ \|\psi \circ \rho(x) - \pi(x)\| < \delta \text{ for all } x \in \omega\} .$$

It follows from [K] that the C^* -algebras for which this definition makes sense are the exact C^* -algebras. We shall therefore assume A is exact and define the *topological entropy of $\alpha \in \text{Aut}(A)$* , denoted by $ht(\pi, \alpha)$ as in Definition 6.2.

The first result to be proved is that the definition is independent of π , hence we can define

$$ht(\alpha) = ht(\pi, \alpha) ,$$

or if $A \subset B(H)$ as $ht(\text{id}_A, \alpha)$. The proof is a good illustration of the techniques involved. We may assume A is unital. Let $\pi_i : A \rightarrow B(H_i)$, $i = 1, 2$, be faithful $*$ -representations. Let $\omega \in Pf(A)$, $\delta > 0$. It suffices by symmetry to show

$$rcp(\pi_1, \omega, \delta) \geq rcp(\pi_2, \omega, \delta) . \quad (6.1)$$

Choose $(\rho, \psi, B) \in CPA(\pi_1, A)$ such that $\text{rank } B = rcp(\pi_1, \omega, \delta)$, and

$$\|\psi \circ \rho(x) - \pi_1(x)\| < \delta , \quad x \in \omega .$$

Consider the map $\pi_2 \circ \pi_1^{-1} : \pi_1(A) \rightarrow B(H_2)$. From Arveson's extension theorem for completely positive maps [Ar] there exists a unital completely positive map $\Phi : B(H_1) \rightarrow B(H_2)$ extending $\pi_2 \circ \pi_1^{-1}$. Thus we have $(\rho, \Phi \circ \psi, B) \in CPA(\pi_2, A)$ and $\|\Phi \circ \psi \circ \rho(x) - \pi_2(x)\| < \delta$ for $x \in \omega$, since $\pi_2(x) = \Phi \circ \pi_1(x)$. Thus (6.1) follows. \square

Again we can prove the basic properties of entropy. Note that monotonicity is an easy consequence of the fact that a C^* -subalgebra of an exact C^* -algebra is itself exact. The analogous result is not true for nuclear C^* -algebras. Another property which should be mentioned is that topological entropy does not yield new information for direct sums. Indeed, we have:

Proposition 6.7. [Br1] *If $A = \bigoplus_{i=1}^{\infty} A_i$, where each A_i is an exact C^* -algebra, and $\alpha = \bigoplus_{i=1}^{\infty} \alpha_i$, where $\alpha_i \in \text{Aut } A_i$, then $ht(\alpha) = \sup_i ht(\alpha_i)$.*

We conclude this chapter with a theorem which compares the entropies defined so far.

Theorem 6.8. [V] *(i) If (M, τ, α) is a W^* -dynamical system with τ a trace then $H_\tau(\alpha) \leq hcpa_\tau(\alpha) \leq ha_\tau(\alpha)$
(ii) [V] *If (A, ϕ, α) is a C^* -dynamical system with A an AF-algebra then $ht(\alpha) \leq hat(\alpha)$.
(iii) [V], [D2] *If in (ii) A is exact then $h_\phi(\alpha) \leq ht(\alpha)$.***

Chapter 7 Crossed Products

If (A, ϕ, α) is a C^* -dynamical system a natural problem is to compute the entropy of the extension of α to the crossed product $A \rtimes_\alpha \mathbb{Z}$. More generally, if G is a discrete

subgroup of $\text{Aut } A$ and $\beta \in \text{Aut } A$ commutes with G , compute the entropy of the extension of β to $A \times G$. The first positive result is due to Voiculescu [V], who showed that for an ergodic measure preserving Bernoulli transformation T on a Lebesgue probability space (X, B, μ) , $H(T) = H(\text{Ad } u_T)$, where u_T is the unitary operator in $L^\infty(X, B, \mu) \times_T \mathbb{Z}$ which implements T . Later on several extensions have appeared, see [Br1], [B-C], [D-S], [G-N2]. We first recall the definition of crossed products.

Let A be a unital C^* -algebra, G a discrete group, and $\alpha : G \rightarrow \text{Aut } A$ a group homomorphism. Let $\sigma : A \rightarrow B(H)$ be a faithful nondegenerate representation. Let

$$\pi : A \rightarrow B(\ell^2(G, H)) \cong B(\ell^2(G)) \otimes B(H)$$

be the representation given by

$$(\pi(x)\xi)(h) = \sigma(\alpha_{h^{-1}}(x))(\xi(h)) , \quad x \in A , \quad \xi \in \ell^2(G, H) , \quad h \in G ,$$

and let λ be the unitary representation of G on $\ell^2(G, H)$ given by

$$(\lambda_g \xi)(h) = \xi(g^{-1}h) , \quad \xi \in \ell^2(G, H) , \quad g, h \in G .$$

Then we have

$$\lambda_{g^{-1}} \pi(x) \lambda_g = \pi(\alpha_g(x)) , \quad x \in A , \quad g \in G .$$

The *reduced crossed product C^* -algebra* $A \rtimes_\alpha G$ is the norm closure of the linear span of the set $\{\pi(x)\lambda_g : x \in A, g \in G\}$. Up to isomorphism $A \rtimes_\alpha G$ is independent of the choice of σ , so for simplicity we assume henceforth that σ is the identity map. Let $\{\xi_h\}_{h \in G}$ be the standard orthonormal basis in $\ell^2(G)$, so $\xi_h(g) = \delta_{g,h}$, $g, h \in G$. Then if $\xi = \xi_h \otimes \psi$ with $\psi \in H$ we have

$$(\lambda_g \xi)(h) = \xi_{g^{-1}h} \otimes \psi = ((\ell_g \otimes 1)\xi)(h) ,$$

where ℓ_g is the left regular representation of G . Furthermore

$$\pi(x)\xi = \pi(x)(\xi_h \otimes \psi) = \xi_h \otimes \alpha_{g^{-1}}(x)\psi .$$

By the above, since we may consider A as a subalgebra of $B(\ell^2(G, H))$, we may also assume from the outset that α is implemented by a unitary representation $g \rightarrow U_g$, $g \in G$, of G . Thus we have

$$(1 \otimes U_g)^* \pi(x) \lambda_h (1 \otimes U_g) = \pi(\alpha_g(x)) \lambda_h .$$

For simplicity let us assume G is abelian – the argument works for G amenable. We follow the approach of [S-S] and [Br1]. Let $e_{p,q} \in B(\ell^2(G))$ denote the standard matrix units, i.e.

$$e_{p,q}(\xi_t) = \delta_{q,t} \xi_p ,$$

where $\delta_{q,t}$ is the Kronecker δ . Then we have

$$\pi(x)\lambda_g = \sum_{t \in G} e_{t,t-g} \otimes \alpha_{-t}(x), \quad x \in A, \quad g \in G.$$

In particular, if $\beta \in \text{Aut } A$, and we assume as before that $\beta = \text{Ad } v$ for a unitary $v \in B(H)$, then if β commutes with all α_g then

$$\begin{aligned} \text{Ad}(1 \otimes v)(\pi(x)\lambda_g) &= \sum e_{t,t-g} \otimes v \alpha_{-t}(x) v^* \\ &= \sum e_{t,t-g} \otimes \alpha_{-t}(\beta(x)) \\ &= \pi(\beta(x))\lambda_g. \end{aligned}$$

Thus β extends to an automorphism $\hat{\beta} = \text{Ad}(1 \otimes v)$ of $A \times_\alpha G$.

If $F \subset G$ is a finite set let P_F denote the orthogonal projection of $\ell^2(G)$ onto $\text{span}\{\xi_t : t \in F\}$. Then we find

$$(P_F \otimes 1)(\pi(x)\lambda_g)(P_F \otimes 1) = \sum_{t \in F \cap (F+g)} e_{t,t-g} \otimes \alpha_{-t}(x) \in M_F \otimes A,$$

where $M_F = P_F B(\ell^2(G)) P_F$.

In order to compute the entropy of $\hat{\beta}$ on $A \times_\alpha G$ the idea is now to start with a triple $(B, \rho, \psi) \in CPA(\text{id}_A, A)$ and extend it to a triple $(M_F \otimes B, \Phi, \Psi) \in CPA(\text{id}_{A \times_\alpha G}, A \times_\alpha G)$ such that we can control the estimates. If $f \in L^\infty(G)$ has support contained in F let m_f denote the corresponding multiplication operator on $\ell^2(G)$, and define

$$T_f(x) = \sum_{t \in G} \ell_g^* \otimes U_g(m_f \otimes 1)x((m_f^* \otimes 1)\ell_g \otimes U_g^*, \quad x \in B(\ell^2(G, H)).$$

Note that by amenability we can assume $\|f\|_2 = 1$ and $f * \tilde{f}(g_i)$ is close to 1 on a given set g_1, \dots, g_k determining F where $\tilde{f}(g) = \overline{f(-g)}$. With ρ and ψ as above we put

$$\Phi_F(x) = (P_F \otimes 1)x(P_F \otimes 1), \quad x \in B(\ell^2(G, H)).$$

Then $\Phi_F(A \times_\alpha G) \subset M_F \otimes A$, so that

$$(M_F \otimes B, (1 \otimes \rho) \circ \Phi_F, T_f \circ (1 \otimes \psi)) \in CPA(\text{id}_{A \times_\alpha G}, A \times_\alpha G)$$

is the desired triple extending (B, ρ, ψ) . Since $\text{rank}(M_F \otimes B) = \text{card } F \cdot \text{rank } B$, all that remains is to choose F with some care depending on a given set $\omega \in Pf(A \times_\alpha G)$, which we may suppose is of the form $\omega = \{\pi(x_i)\lambda_{g_i}; i = 1, 2, \dots, n\}$.

The above construction essentially works for all the different entropies considered, and even for $\beta \in \text{Aut } A$ commuting with all α_g when G is amenable.

Theorem 7.1. *Let A be a unital C^* -algebra, G a discrete amenable group and $\alpha : G \rightarrow \text{Aut } A$ a representation. Let $\beta \in \text{Aut } A$ commute with all α_g , $g \in G$. Let $\hat{\beta}$ be the natural extension of β to $\text{Aut}(A \times_\alpha G)$. Then we have*

- (i) [D-S], [C6]. If A is exact then $ht(\hat{\beta}) = ht(\beta)$.
(ii) [G-N2]. If A is an injective von Neumann algebra and φ a normal state which is both G - and β -invariant then, if φ is identified with its canonical extension to $A \times_{\alpha} G$,

$$h_{cpa_{\varphi}}(\hat{\beta}) = h_{cpa_{\varphi}}(\beta) , \quad \text{and} \quad h_{\varphi}(\hat{\beta}) = h_{\varphi}(\beta) .$$

Note that when $A = L^{\infty}(X, B, \mu)$ and $\beta = \alpha_1 = \alpha_T$, $G = \mathbb{Z}$, the theorem implies the result of Voiculescu alluded to in the first paragraph of the chapter. When G is abelian and $\beta = \alpha_g$ some $g \in G$, part (i) was proved by Brown [Br1]. A variation of (ii) can also be found in [B-C].

Sometimes one can prove results on operator algebras by representing them as crossed products, see e.g. the proof of Corollary 5.8. Another example is \mathcal{O}_{∞} – the universal C^* -algebra generated by isometries $\{S_i\}_{i \in \mathbb{Z}}$ which satisfy the relation

$$\sum_{i=-r}^r S_i S_i^* \leq 1 \quad \text{for all } r \in \mathbb{N} .$$

Every bijection $\alpha : \mathbb{Z} \rightarrow \mathbb{Z}$ defines an automorphism, also denoted by α , of \mathcal{O}_{∞} by $\alpha(S_i) = S_{\alpha(i)}$. By [Cu] there exist an AF-algebra B , $\Phi \in \text{Aut } B$, an imbedding $\pi : \mathcal{O}_{\infty} \rightarrow B \times_{\Phi} \mathbb{Z}$, and a projection $p \in B$, such that $\pi(\mathcal{O}_{\infty}) = p(B \times_{\Phi} \mathbb{Z})p$. By using techniques similar to those used to prove Theorem 7.1 we have

Theorem 7.2. [B-C] *If $\alpha \in \text{Aut } \mathcal{O}_{\infty}$ is induced by a bijective function $\alpha : \mathbb{Z} \rightarrow \mathbb{Z}$ then $ht(\alpha) = 0$. In particular, if φ is an α -invariant state on \mathcal{O}_{∞} then $h_{\varphi}(\alpha) = 0$.*

Note that the last statement follows from the first and Theorem 6.8 since $h_{\varphi}(\alpha) \leq ht(\alpha)$, since \mathcal{O}_{∞} is nuclear. For a closely related result see [C-N]. This theorem is the first we shall encounter, which shows that if a C^* -dynamical system (A, φ, α) is highly nonabelian then the entropy of α tends to be small.

A problem related to the above is the computation of the entropy of the canonical endomorphism Φ of the C^* -algebra \mathcal{O}_n of Cuntz [Cu], which is the C^* -algebra generated by n isometries S_1, \dots, S_n such that $\sum_{i=1}^n S_i S_i^* = 1$. Analogously to \mathcal{O}_{∞} , \mathcal{O}_n can be written as a crossed product $B \times_{\sigma} \mathbb{N}$, where $B = \bigotimes_{i \in \mathbb{N}} M_i$ with $M_i = M_n(\mathbb{C})$, σ is the shift to the right, and φ the canonical state extending the trace on B . The canonical endomorphism Φ is defined by

$$\Phi(x) = \sum_{i=1}^n S_i x S_i^* , \quad x \in \mathcal{O}_n .$$

It is a simple task to extend the entropies h_{φ} and ht to endomorphisms. We have

Theorem 7.3. [C4] *The canonical endomorphism Φ on \mathcal{O}_n satisfies*

$$ht(\Phi) = h_{\varphi}(\Phi) = \log n .$$

The result has a natural extension to the Cuntz-Krieger algebra \mathcal{O}_A defined by an irreducible $n \times n$ matrix which is not a permutation matrix. Then we have [Bo-Go]

$$ht(\alpha) = \log r(A) ,$$

where $r(A)$ is the spectral radius of A . For further extensions see [PWY].

Chapter 8

Free Products

In Theorem 7.2 we saw that the shift on \mathcal{O}_∞ has entropy zero. The first example of a highly nonabelian dynamical system where the entropy is zero, was the shift on the Π_1 -factor $L(\mathbb{F}_\infty)$ obtained from the left regular representation of the free group in infinite number of generators [S1]. This phenomenon was rather surprising because the shift is so ergodic that there is no globally invariant injective von Neumann subalgebra except for the scalars. We shall in the present chapter study extensions of the above result.

If (A, φ) denotes a unital C*-algebra with a state φ and $A_\iota, \iota \in I$, is a C*-subalgebra of A with $1 \in A_\iota$, we say $(A_\iota)_{\iota \in I}$ is a *free family* if $\varphi(a_1 a_2 \dots a_n) = 0$ whenever $a_i \in A_{\iota(i)}$ with $\iota(i) \neq \iota(i+1)$, $1 \leq i \leq n-1$, and $\varphi(a_i) = 0$, $1 \leq i \leq n$. If the A_ι generate A and the GNS-representation of φ is faithful, we say (A, φ) is the free product of the $(A_\iota, \varphi_\iota)_{\iota \in I}$, where $\varphi_\iota = \varphi|_{A_\iota}$, and use the notation

$$(A, \varphi) = (*A_\iota, *\varphi_\iota)_{\iota \in I} ,$$

see [VDN] for details and also for an explicit construction of the GNS-representation of φ on the full Fock space. If σ is a permutation of I such that there is an isomorphism $\alpha_\iota : A_\iota \rightarrow A_{\sigma(\iota)}$ satisfying $\varphi_{\sigma(\iota)} \circ \alpha_\iota = \varphi_\iota$, then there is a unique φ -invariant automorphism $\alpha \in \text{Aut}(A)$ such that $\alpha(x) = \alpha_\iota(x)$ for $x \in A_\iota$. Of particular interest is the free shift, which is the automorphism α of $(*A_n, *\varphi_n)_{n \in \mathbb{Z}}$, where $A_n = A_0$, $\varphi_n = \varphi_0$, and $\alpha_n = \text{id}$, obtained from the shift $\sigma : n \rightarrow n+1$ of \mathbb{Z} .

To compute the entropy of α there are two approaches; the first is to show that $ht(\alpha) = 0$ and use the inequalities in Theorem 6.8 to conclude that the other entropies are zero. This works if A_0 is exact, because then $A = *A_\iota$ is exact [D1].

Theorem 8.1. [D2], [BDS] *In the above notation assume $A_n = A_0$ is exact, and let α be the free shift on $(A, \varphi) = (*A_n, *\varphi_n)_{n \in \mathbb{Z}}$. Then the topological entropy $ht(\alpha) = 0$.*

The result is even true when the free product is a reduced amalgamated free product over a finite dimensional C*-algebra and for more general automorphisms than the free shift. The result can be applied to recover Theorem 7.2, because one

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