11 Bézier techniques for triangular patches

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Just as univariate polynomials, multivariate polynomials correspond uniquely to symmetric multiaffine polynomials. With symmetric polynomials, it is a simple task to derive algorithms which evaluate, degree elevate, reparametrize, or subdivide a triangular surface in Bézier representation. The generalization of the techniques described for univariate polynomials in Chapter 3 is straightforward.

11.1 Symmetric polynomials

Every polynomial surface $\mathbf{b}(\mathbf{x})$ of total degree $\leq n$ can be associated with a unique n-affine **symmetric polynomial** $\mathbf{b}[\mathbf{x}_1 \ldots \mathbf{x}_n]$ over $\mathbb{R}^2$ having the following three properties.

- $\mathbf{b}[\mathbf{x}_1 \ldots \mathbf{x}_n]$ *agrees with* $\mathbf{b}(\mathbf{x})$ *on its* **diagonal**, *i.e.,*

$$\mathbf{b}[\mathbf{x} \ldots \mathbf{x}] = \mathbf{b}(\mathbf{x}) \ .$$

- $\mathbf{b}[\mathbf{x}_1 \ldots \mathbf{x}_n]$ *is* **symmetric** *in its variables, i.e., for any permutation* $(\mathbf{y}_1 \ldots \mathbf{y}_n)$ *of* $(\mathbf{x}_1 \ldots \mathbf{x}_n)$, *we obtain*

$$\mathbf{b}[\mathbf{y}_1 \ldots \mathbf{y}_n] = \mathbf{b}[\mathbf{x}_1 \ldots \mathbf{x}_n] \ .$$

- $\mathbf{b}[\mathbf{x}_1 \ldots \mathbf{x}_n]$ *is* **affine** *in each variable, i.e.,*

$$\mathbf{b}[(\alpha\mathbf{x} + (1-\alpha)\mathbf{y})\,\mathbf{x}_2 \,\ldots\, \mathbf{x}_n] = \alpha\mathbf{b}[\mathbf{x}\,\mathbf{x}_2 \,\ldots\, \mathbf{x}_n] + (1-\alpha)\mathbf{b}[\mathbf{y}\,\mathbf{x}_2 \,\ldots\, \mathbf{x}_n] \enspace .$$

The symmetric polynomial $\mathbf{b}[\mathbf{x}_1 \,\ldots\, \mathbf{x}_n]$ is also referred to as the **polar form** [Casteljau '85] or **blossom** [Ramshaw '87] of $\mathbf{b}(\mathbf{x})$.

To show that such symmetric polynomials exist for all polynomials, it suffices to consider basis polynomials and to derive explicit representations of their symmetric forms. Any linear combination

$$\mathbf{b}(\mathbf{x}) = \sum_{\mathbb{i}} \mathbf{c}_{\mathbb{i}}\, C_{\mathbb{i}}(\mathbf{x})$$

of nth degree polynomials $C_{\mathbb{i}}(\mathbf{x})$ with polar forms $C_{\mathbb{i}}[\mathbf{x}_1 \,\ldots\, \mathbf{x}_n]$, where $\mathbb{i} = (i,j,k) \geq \mathbb{o}$ and $|\mathbb{i}| = n$, has the symmetric polynomial

$$\mathbf{b}[\mathbf{x}_1 \,\ldots\, \mathbf{x}_n] = \sum_{\mathbb{i}} \mathbf{c}_{\mathbb{i}}\, C_{\mathbb{i}}[\mathbf{x}_1 \,\ldots\, \mathbf{x}_n] \enspace ,$$

which clearly satisfies the three properties above.

Note that the diagonal $\mathbf{b}[\mathbf{x} \ldots \mathbf{x}]$ can be of lower degree than n, although $\mathbf{b}[\mathbf{x}_1 \,\ldots\, \mathbf{x}_n]$ depends on n variables.

In case the $C_{\mathbb{i}}$ are the weighted monomials $A^n_{ij}(x,y) = \binom{n}{\mathbb{i}} x^i y^j$, we obtain the **elementary symmetric polynomials**

$$A^n_{ij}[\mathbf{x}_1 \,\ldots\, \mathbf{x}_n] = \sum_{\substack{\alpha<\cdots<\beta \\ \gamma<\cdots<\delta}} x_\alpha \overset{i}{\ldots} x_\beta\; y_\gamma \overset{j}{\ldots} y_\delta \enspace ,$$

where $\mathbf{x}_\alpha = (x_\alpha, y_\alpha)$ and $\alpha, \ldots, \beta, \gamma, \ldots, \delta$ are $i+j$ distinct integers between 1 and n. Obviously, the three characterizing properties of the polar form are satisfied.

In case the $C_{\mathbb{i}}$ are the Bernstein polynomials $B^n_{\mathbb{i}}(\mathbb{u}) = \binom{n}{\mathbb{i}} \mathbb{u}^{\mathbb{i}}$, we obtain

$$B^n_{\mathbb{i}}[\mathbb{u}_1 \,\ldots\, \mathbb{u}_n] = \sum_{\substack{\alpha<\cdots<\beta \\ \gamma<\cdots<\delta \\ \varepsilon<\cdots<\varphi}} u_\alpha \overset{i}{\cdots} u_\beta\; v_\gamma \overset{j}{\cdots} v_\delta\; w_\varepsilon \overset{k}{\cdots} w_\varphi \enspace ,$$

where $u_\alpha, v_\alpha, w_\alpha$ are the coordinates of $\mathbb{u}_\alpha$ and $(\alpha, \ldots, \beta, \gamma, \ldots, \delta, \varepsilon, \ldots, \varphi)$ is a permutation of $(1, \ldots, n)$. Again, one can easily check the three properties above.

Remark 1: The symmetric polynomials $B^n_{\mathbb{i}}[\mathbb{u}_1 \,\ldots\, \mathbb{u}_n]$ satisfy the recursion

$$B^n_{\mathbb{i}}[\mathbb{u}_1 \,\ldots\, \mathbb{u}_n] = u_1 B^{n-1}_{\mathbb{i}-\mathbb{e}_1}[\mathbb{u}_2 \,\ldots\, \mathbb{u}_n] + v_1 B^{n-1}_{\mathbb{i}-\mathbb{e}_2}[\mathbb{u}_2 \,\ldots\, \mathbb{u}_n] + w_1 B^{n-1}_{\mathbb{i}-\mathbb{e}_3}[\mathbb{u}_2 \,\ldots\, \mathbb{u}_n].$$

Remark 2: The barycentric coordinate vector $\mathbb{u}$ and the affine coordinate vector $\mathbf{x}$ are related by two transformations, given by

$$\mathbf{x} = \mathbf{x}(\mathbb{u}) = [\mathbf{a}_0\mathbf{a}_1\mathbf{a}_2]\mathbb{u} \quad \text{and} \quad \mathbb{u} = \mathbb{u}(\mathbf{x}) = \mathbb{p} + \mathbb{v}\mathbf{x} \ .$$

Since these transformations are affine, one can transform a polar form $\mathbf{a}[\mathbf{x}_1 \ldots \mathbf{x}_n]$, given by affine coordinates, to the corresponding polar form $\mathbf{b}[\mathbb{u}_1 \ldots \mathbb{u}_n] = \mathbf{a}[A\mathbb{u}_1 \ldots A\mathbb{u}_n]$ and vice versa, i.e., $\mathbf{a}[\mathbf{x}_1 \ldots \mathbf{x}_n] = \mathbf{b}[\mathbb{u}(\mathbf{x}_1) \ldots \mathbb{u}(\mathbf{x}_n)]$.

11.2 The main theorem

The uniqueness of the symmetric polynomials and their relationship to the Bézier representation is given by the following extension of the **main theorem**.

> *For every polynomial surface $\mathbf{b}(\mathbf{x})$ of degree $\le n$, there exists only one symmetric n-variate multiaffine polynomial $\mathbf{b}[\mathbf{x}_1 \ \ldots \ \mathbf{x}_n]$ with diagonal $\mathbf{b}[\mathbf{x} \ \ldots \ \mathbf{x}] = \mathbf{b}(\mathbf{x})$, and the points*
>
> $$\mathbf{b}^0_{\mathbb{i}} = \mathbf{b}[\mathbf{p} \overset{i}{\ldots} \mathbf{p}\ \mathbf{q} \overset{j}{\ldots} \mathbf{q}\ \mathbf{r} \overset{k}{\ldots} \mathbf{r}]$$
>
> *are the Bézier points of $\mathbf{b}(\mathbf{x})$ over $\mathbf{pqr}$.*

Proof: Consider the points

$$\mathbf{b}^l_{\mathbb{i}} = \mathbf{b}[\mathbf{p} \overset{i}{\ldots} \mathbf{p}\ \mathbf{q} \overset{j}{\ldots} \mathbf{q}\ \mathbf{r} \overset{k}{\ldots} \mathbf{r}\ \mathbf{x}_1 \overset{l}{\ldots} \mathbf{x}_l] \ , \qquad i+j+k+l = n \ .$$

Since $\mathbf{b}^n_{\mathbb{o}} = \mathbf{b}[\mathbf{x}_1 \ \ldots \ \mathbf{x}_n]$ is symmetric and multiaffine, it can be computed from the points $\mathbf{b}^0_{\mathbb{i}}$ by the recursion formula

$$\mathbf{b}^l_{\mathbb{i}} = u_l \mathbf{b}^{l-1}_{\mathbb{i}+\mathbb{e}_1} + v_l \mathbf{b}^{l-1}_{\mathbb{i}+\mathbb{e}_2} + w_l \mathbf{b}^{l-1}_{\mathbb{i}+\mathbb{e}_3} \ , \tag{1}$$

where u_l, v_l, w_l are the barycentric coordinates of $\mathbf{x}_l$ with respect to $\mathbf{pq}$, see Figure 11.1, where the points $\mathbf{b}[\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3]$ are labelled by their arguments $\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3$. Thus, different symmetric multiaffine maps must differ at some argument $[\mathbf{p} \overset{i}{\ldots} \mathbf{p}\ \mathbf{q} \overset{j}{\ldots} \mathbf{q}\ \mathbf{r} \overset{k}{\ldots} \mathbf{r}]$.

If all $\mathbf{x}_l$ equal $\mathbf{x}$, then the recursion formula above reduces to de Casteljau's algorithm for the computation of $\mathbf{b}(\mathbf{x})$. Consequently, since the Bézier representation is unique, the points $\mathbf{b}^0_{\mathbb{i}}$ are the Bézier points of $\mathbf{b}(\mathbf{x})$ over $\mathbf{pqr}$ and, furthermore, there can be only one symmetric n-affine polynomial with the diagonal $\mathbf{b}(\mathbf{x})$. $\diamond$

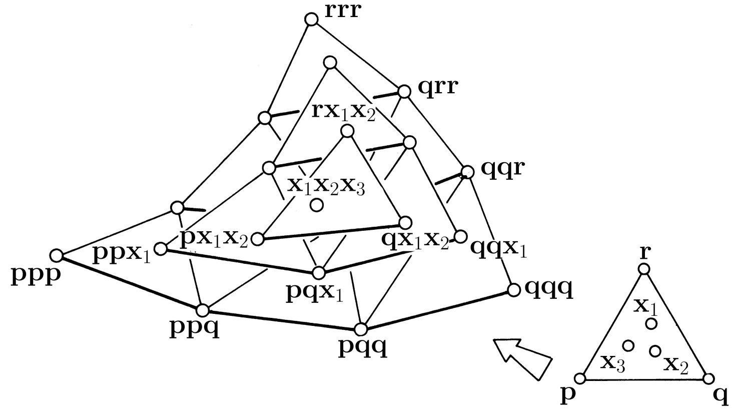

Figure 11.1: The generalized de Casteljau algorithm.

11.3 Subdivision and reparametrization

Recursion formula (1), which is illustrated in Figure 11.1, reveals another important property of de Casteljau's algorithm. The computation of $\mathbf{b}(\mathbf{x})$ generates also the Bézier points

$$\mathbf{b}[\mathbf{p} \overset{i}{\ldots} \mathbf{p}\, \mathbf{q} \overset{j}{\ldots} \mathbf{q}\, \mathbf{x} \overset{k}{\ldots} \mathbf{x}], \quad \mathbf{b}[\mathbf{p} \overset{i}{\ldots} \mathbf{p}\, \mathbf{x} \overset{j}{\ldots} \mathbf{x}\, \mathbf{r} \overset{k}{\ldots} \mathbf{r}] \ ,$$

and

$$\mathbf{b}[\mathbf{x} \overset{i}{\ldots} \mathbf{x}\, \mathbf{q} \overset{j}{\ldots} \mathbf{q}\, \mathbf{r} \overset{k}{\ldots} \mathbf{r}]$$

of $\mathbf{b}$ over $\mathbf{pqx}$, $\mathbf{pxr}$, and $\mathbf{xqr}$, respectively. Figure 11.2 shows an example for $n = 3$.

The Bézier nets of $\mathbf{b}(\mathbf{x})$ over $\mathbf{pqx}$, $\mathbf{pxr}$, and $\mathbf{xqr}$ form one connected net. It is folded if $\mathbf{x}$ lies outside $\mathbf{pqr}$. The computation of this composed net will be referred to as the **subdivision** of the Bézier net over $\mathbf{pqr}$ in $\mathbf{x}$.

One can compute the Bézier net of a polynomial surface $\mathbf{b}$ over a second triangle $\mathbf{xyz}$ by repeated subdivision from the net over $\mathbf{pq}$, see [Prautzsch '84a, Boehm et al. '84]. First one subdivides the net over $\mathbf{pq}$ in $\mathbf{x}$, then one subdivides the net over $\mathbf{xqr}$ in $\mathbf{y}$, and, finally, one subdivides the net over $\mathbf{xyr}$ in $\mathbf{z}$, see Figure 11.3.

A permutation of $\mathbf{pq}$ and $\mathbf{xyz}$ results in a different construction. If possible, one should subdivide at interior points in order to avoid non-convex combinations.

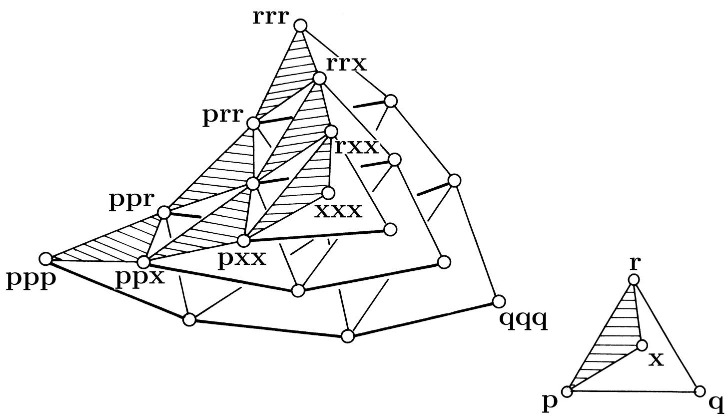

Figure 11.2: Subdividing a Bézier net.

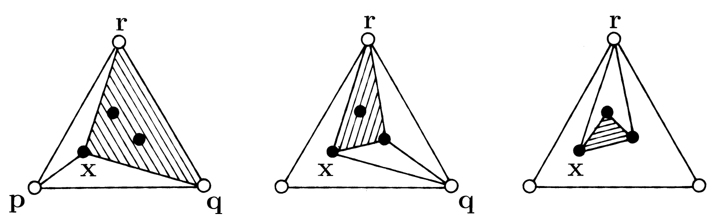

Figure 11.3: Reparametrization by repeated subdivision.

Figure 11.4 shows a situation, where it is impossible to avoid non-convex combinations with the above construction, no matter how one permutes **pq** and **xyz**.

Remark 3: The construction requires one to compute $3 \cdot \binom{n+3}{3} = O(n^3)$ affine combinations.

Remark 4: Every single Bézier point $\mathbf{b}[\mathbf{x} \overset{i}{\ldots} \mathbf{x}\,\mathbf{y} \overset{j}{\ldots} \mathbf{y}\,\mathbf{z} \overset{k}{\ldots} \mathbf{z}]$ of $\mathbf{b}$ over **xyz** can also be computed by the generalized de Casteljau algorithm, see Figure 11.1. The affine combinations computed by this algorithm are convex if $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ all lie in the triangle **pq**.

Remark 5: To compute the Bézier net over **xyz** by $\binom{n+2}{2}$ applications of the generalized de Casteljau algorithm, one needs to compute $\binom{n+2}{2} \cdot \binom{n+3}{3} = O(n^5)$ affine combinations.

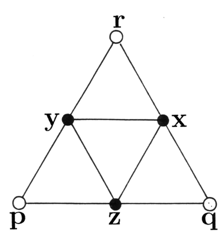

Figure 11.4: Special reference triangles.

11.4 Convergence under subdivision

The Bézier net of $\mathbf{b}(\mathbf{x})$ over a triangle $\mathbf{pq}$ is a good approximation of the patch $\mathbf{b}$ if the triangle is sufficiently small. To make this statement precise, let $\mathbf{pq}$ be any triangle in some fixed bounded region and let h be its diameter. Furthermore, let

$$\mathbf{i} = \mathbf{p}\frac{i}{n} + \mathbf{q}\frac{j}{n} + \mathbf{r}\frac{k}{n}$$

represent the point with barycentric coordinates $\mathbb{i}/n$. Then,

there is a constant M not depending on $\mathbf{pq}$ such that

$$\max_{\mathbb{i}} \|\mathbf{b}(\mathbf{i}) - \mathbf{b}_{\mathbb{i}}\| \leq Mh^2 \ .$$

For a proof, let D be the differential of $\mathbf{b}[\mathbf{x}\,\mathbf{i}\ldots\mathbf{i}] = \cdots = \mathbf{b}[\mathbf{i}\ldots\mathbf{i}\,\mathbf{x}]$ at $\mathbf{x} = \mathbf{i}$. Expanding the symmetric polynomial $\mathbf{b}[\mathbf{x}_1 \ldots \mathbf{x}_n]$ around $[\mathbf{i}\ldots\mathbf{i}]$, we obtain

$$\begin{aligned} \mathbf{b}_{\mathbb{i}} &= \mathbf{b}[\mathbf{i}\ldots\mathbf{i}] + iD[\mathbf{p}-\mathbf{i}] + jD[\mathbf{q}-\mathbf{i}] + kD[\mathbf{r}-\mathbf{i}] + O(h^2) \\ &= \mathbf{b}(\mathbf{i}) + O(h^2) \ , \end{aligned}$$

which concludes the proof. $\diamond$

An application of this approximation property is discussed in the following section.

11.5 Surface generation

As a consequence of section 11.4, repeated subdivision of a Bézier net produces arbitrarily good approximations of the underlying surface. We discuss three subdivision strategies.

(1) Subdividing triangles at their centers, as illustrated in Figure 11.5, leaves

the maximum diameter of the reference triangles unchanged. Hence, the Bézier nets over these triangulations do **not** converge to the surface.

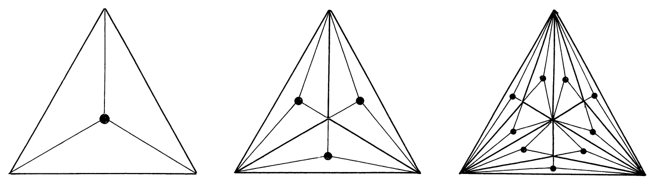

Figure 11.5: Subdivision at centers.

(2) Subdividing every triangle uniformly, as illustrated in Figures 11.6 and 11.7, generates a sequence of Bézier nets which converges to the surface.

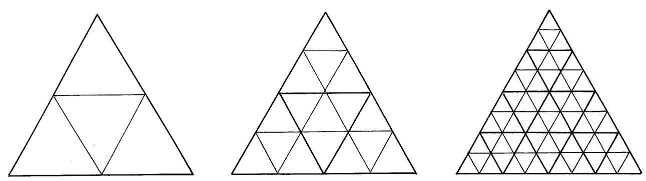

Figure 11.6: Uniform subdivision.

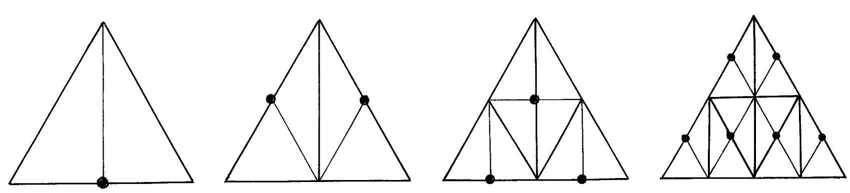

Figure 11.7: Repeated bisection.

(3) The uniform subdivision scheme shown in Figure 11.6 is either expensive to compute or uses non-convex combinations, see 11.3. Thus, for the purpose of surface generations, it is best to use the strategy illustrated in Figure 11.7. This refinement is inexpensive to compute, and one needs to evaluate only convex combinations.

Comparisons with other surface generation methods reveal that subdivision according to Figure 11.7 provides the fastest method known [Peters '94].

11.6 The symmetric polynomial of the derivative

The directional derivative $D_{\mathbb{v}}\mathbf{b}(\mathbb{u})$ of a polynomial surface with respect to a direction $\mathbb{v} = [v_0\, v_1\, v_2]^t$, $|\mathbb{v}| = v_0 + v_1 + v_2 = 0$, can also be written in terms of the symmetric polynomial $\mathbf{b}[\mathbb{u}_1 \ldots \mathbb{u}_n]$.

From 10.5, or simply by differentiating the symmetric polynomial, it follows that

$$\begin{aligned} D_{\mathbb{v}}\mathbf{b}(\mathbb{u}) &= n(v_0\mathbf{b}[\mathbb{e}_0\mathbb{u} \ldots \mathbb{u}] + v_1\mathbf{b}[\mathbb{e}_1\mathbb{u} \ldots \mathbb{u}] + v_2\mathbf{b}[\mathbb{e}_2\mathbb{u} \ldots \mathbb{u}]) \\ &= n\mathbf{b}[\mathbb{v}\ \mathbb{u} \ldots \mathbb{u}] \ . \end{aligned}$$

Obviously, $n\mathbf{b}[\mathbb{v}\mathbb{u}_2 \ldots \mathbb{u}_n]$ represents the $(n-1)$-affine symmetric polynomial of $D_{\mathbb{v}}\mathbf{b}(\mathbb{u})$.

The $\mathbb{u}_i$ represent points while $\mathbb{v}$ represents a vector with respect to the reference triangle. Further, $\mathbf{b}[\mathbb{v}\ \mathbb{u}_2 \ldots \mathbb{u}_n]$ represents a vector which is affine in $\mathbb{u}_2, \ldots, \mathbb{u}_n$ and linear in $\mathbb{v}$.

Repeating this process, we obtain the symmetric polynomial of any mixed directional derivative $\mathbf{c}(\mathbb{u}) = D_{\mathbb{v}_r} \ldots D_{\mathbb{v}_1}\mathbf{b}(\mathbb{u})$ with respect to r vectors $\mathbb{v}_1, \ldots, \mathbb{v}_r$,

$$\mathbf{c}[\mathbb{u}_{r+1} \ldots \mathbb{u}_n] = \frac{n!}{(n-r)!}\mathbf{b}[\mathbb{v}_1 \ldots \mathbb{v}_r\, \mathbb{u}_{r+1} \ldots \mathbb{u}_n] \ .$$

11.7 Simple C^r joints

Subdivision provides a convenient tool to describe certain differentiability conditions of two polynomial surfaces $\mathbf{b}(\mathbf{x})$ and $\mathbf{c}(\mathbf{x})$ given by their Bézier points $\mathbf{b}_{\mathbb{i}}$ and $\mathbf{c}_{\mathbb{i}}$ over $\mathbf{pq}$ and $\mathbf{sqr}$, respectively, see Figure 11.8.

From 10.5, it follows that the derivatives up to order r over the line $\mathbf{qr}$ determine, and are determined, by the Bézier points $\mathbf{b}_{\mathbb{i}}$ and $\mathbf{c}_{\mathbb{i}}$ for $i = 0, \ldots, r$. This leads to Farin's version of Stärk's theorem, see [Farin '86, p. 98] and [Sabin '77, p. 85].

> *The derivatives of* $\mathbf{b}$ *and* $\mathbf{c}$ *up to order* r *agree over* $\mathbf{qr}$ *if and only if the first* $r+1$ *rows of Bézier points of* $\mathbf{b}$ *and* $\mathbf{c}$ *over* $\mathbf{sqr}$ *agree, i.e.,* $\mathbf{b}[\mathbf{s} \overset{i}{\ldots} \mathbf{s}\, \mathbf{q} \overset{j}{\ldots} \mathbf{q}\, \mathbf{r} \overset{k}{\ldots} \mathbf{r}] = \mathbf{c}_{\mathbb{i}}$, $i = 0, \ldots, r$.

Over $\mathbf{pqr}$ and $\mathbf{sqr}$, the polynomial $\mathbf{b}[\mathbf{x} \overset{r}{\ldots} \mathbf{x}\mathbf{q} \overset{l}{\ldots} \mathbf{q}\mathbf{r} \overset{n-r-l}{\ldots} \mathbf{r}]$ has the Bézier points $\mathbf{b}_{\mathbb{i}}$ and $\mathbf{c}_{\mathbb{i}}$, respectively, where $i \le r$, $j \le l$ and $k \ge n - r - l$. The $\mathbf{c}_{\mathbb{i}}$ can be computed from the $\mathbf{b}_{\mathbb{i}}$ using de Casteljau's algorithm, see 11.3 and Figure 11.8 and 11.9.

Using the main theorem 11.2, this can be rephrased in the following way.

The derivatives of $\mathbf{b}$ *and* $\mathbf{c}$ *up to order* r *agree over* $\mathbf{qr}$ *if and only if for all* $l = 0, \ldots, n - r$ *the two polynomials* $\mathbf{b}[\mathbf{x} \overset{r}{\ldots} \mathbf{xq} \overset{l}{\ldots} \mathbf{qr} \overset{n-r-l}{\ldots} \mathbf{r}]$ *and* $\mathbf{c}[\mathbf{x} \overset{r}{\ldots} \mathbf{xq} \overset{l}{\ldots} \mathbf{qr} \overset{n-r-l}{\ldots} \mathbf{r}]$ *are equal.*

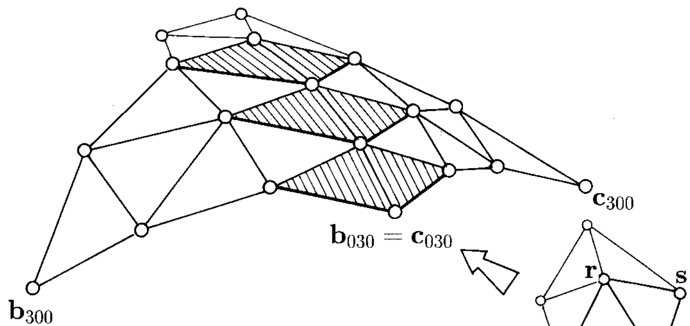

Figure 11.8: Sabin's simple C^1 joint.

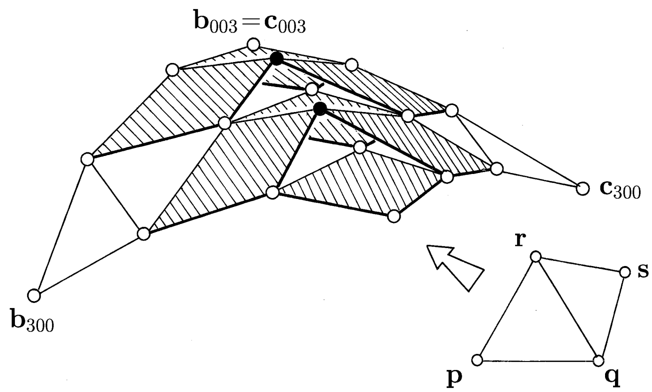

Figure 11.9: Farin's simple C^2 joint.

Remark 6: The shaded quadrilaterals in Figure 11.8 and 11.9 are different affine images of the quadrilateral **pqrs**. Consequently, any k triangular patches $\mathbf{b}^i(\mathbf{x}),\ i = 1, \ldots, m$, enclosing a common vertex have simple C^1 joints at this vertex if and only if their parameter triangles form an m-gon that is an affine image of the m-gon formed by the respective corner triangles of the associated Bézier nets.

Remark 7: Since two polynomials are equal if and only if their polar forms are equal, $\mathbf{b}(\mathbf{x})$ and $\mathbf{c}(\mathbf{x})$ have identical derivatives up to order r over the line $\mathbf{qr}$ if and only if their polar forms satisfy the equation

$$\mathbf{b}[\mathbf{x}_1 \ldots \mathbf{x}_r\, \mathbf{p} \overset{j}{\ldots} \mathbf{p}\, \mathbf{q} \overset{k}{\ldots} \mathbf{q}] = \mathbf{c}[\mathbf{x}_1 \ldots \mathbf{x}_r\, \mathbf{p} \overset{j}{\ldots} \mathbf{p}\, \mathbf{q} \overset{k}{\ldots} \mathbf{q}]$$

for arbitrary variables $\mathbf{x}_1, \ldots, \mathbf{x}_r$ and for all j and k with $r + j + k = n$. This condition is used in [Lai '91] to characterize multivariate C^r splines over arbitrary triangulations.

11.8 Degree elevation

A polynomial surface of degree n also has a Bézier representation of any degree m higher than n. As in the case of curves, a conversion to a higher degree representation is called degree **elevation**.

Given an nth degree Bézier representation,

$$\mathbf{b}(\mathbf{x}) = \sum \mathbf{b}_{\mathbb{i}} B^n_{\mathbb{i}}(\mathbb{u}) \ , \quad \mathbf{x} = [\mathbf{pqr}]\mathbb{u} \ ,$$

of some polynomial surface $\mathbf{b}(\mathbb{u})$ over a triangle $\mathbf{pqr}$, we show how to obtain its Bézier representation of degree $n + 1$. In analogy to the derivation for curves in 3.11, we use the symmetric polynomial $\mathbf{b}[\mathbf{x}_1 \ldots \mathbf{x}_n]$ of $\mathbf{b}(\mathbf{x})$. The polynomial

$$\mathbf{c}[\mathbf{x}_0 \ldots \mathbf{x}_n] = \frac{1}{n+1} \sum_{l=0}^{n} \mathbf{b}[\mathbf{x}_0 \ldots \mathbf{x}_l^* \ldots \mathbf{x}_n]$$

is multiaffine, symmetric and agrees with $\mathbf{b}(\mathbf{x})$ on its diagonal. Hence, due to the main theorem in 11.2, it follows that the points

$$\mathbf{b}_{\mathbb{j}} = \mathbf{c}[\mathbf{p} \overset{j_0}{\ldots} \mathbf{p}\, \mathbf{q} \overset{j_1}{\ldots} \mathbf{q}\, \mathbf{r} \overset{j_2}{\ldots} \mathbf{r}]$$

are the Bézier points of $\mathbf{b}(\mathbf{x})$ over $\mathbf{pqr}$ in its representation for degree $n + 1$. Consequently,

$$\begin{aligned} \mathbf{b}_{\mathbb{j}} = \quad & \frac{j_0}{n+1} \mathbf{b}[\mathbf{p} \overset{j_0-1}{\ldots} \mathbf{p}\, \mathbf{q} \overset{j_1}{\ldots} \mathbf{q}\, \mathbf{r} \overset{j_2}{\ldots} \mathbf{r}] \\ + \quad & \frac{j_1}{n+1} \mathbf{b}[\mathbf{p} \overset{j_0}{\ldots} \mathbf{p}\, \mathbf{q} \overset{j_1-1}{\ldots} \mathbf{q}\, \mathbf{r} \overset{j_2}{\ldots} \mathbf{r}] \\ + \quad & \frac{j_2}{n+1} \mathbf{b}[\mathbf{p} \overset{j_0}{\ldots} \mathbf{p}\, \mathbf{q} \overset{j_1}{\ldots} \mathbf{q}\, \mathbf{r} \overset{j_2-1}{\ldots} \mathbf{r}] \ . \end{aligned}$$

Figure 11.10 illustrates the associated construction for $n = 2$.

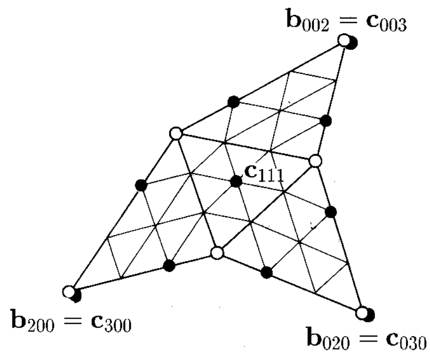

Figure 11.10: Degree elevation by one.

11.9 Convergence under degree elevation

Repeating the degree elevation process, we obtain any higher degree representation

$$\mathbf{b}(\mathbf{x}) = \sum \mathbf{d}_{\mathbb{k}} B^m_{\mathbb{k}}(\mathbb{u}) \ , \quad m > n \ .$$

The new Bézier points $\mathbf{d}_{\mathbb{k}}$ can be expressed in terms of the points $\mathbf{b}_{\mathbb{i}}$. With $r = m - n$, we write

$$\mathbf{b}(\mathbf{x}) = \sum_{|\mathbb{i}|=n} \mathbf{b}_{\mathbb{i}} B^n_{\mathbb{i}} \cdot 1$$

as

$$\begin{aligned} \mathbf{b}(\mathbf{x}) &= \sum_{|\mathbb{i}|=n} \mathbf{b}_{\mathbb{i}} B^n_{\mathbb{i}} \cdot \sum_{|\mathbb{j}|=r} B^r_{\mathbb{j}} \\ &= \sum_{|\mathbb{k}|=m} \left(\sum_{|\mathbb{i}|=n} \mathbf{b}_{\mathbb{i}} \beta_{\mathbb{i}\mathbb{k}} \right) B^m_{\mathbb{k}} \ , \end{aligned}$$

where

$$\beta_{\mathbb{i}\mathbb{k}} = \frac{B^n_{\mathbb{i}} B^r_{\mathbb{k}-\mathbb{i}}}{B^m_{\mathbb{k}}} = \frac{\binom{n}{\mathbb{i}}\binom{r}{\mathbb{k}-\mathbb{i}}}{\binom{m}{\mathbb{k}}} \ .$$

Thus, we obtain Zhou's formula

$$\mathbf{d}_{\mathbb{k}} = \sum_{|\mathbb{i}|=n} \mathbf{b}_{\mathbb{i}} \beta_{\mathbb{i}\mathbb{k}} \ ,$$

see [Farin '86, de Boor '87]. Let $\mathbb{i} = (i_1 i_2 i_3)$ and $\mathbb{k} = (k_1 k_2 k_3)$. Then, $\beta_{\mathbb{i}\mathbb{k}}$ can be written as

$$\binom{n}{\mathbb{i}}\left(\frac{k_1}{m}\cdots\frac{k_1-i_1+1}{m-i_1+1}\right)\left(\frac{k_2}{m-i_1}\cdots\frac{k_2-i_2+1}{m-i_2-i_2+1}\right)\left(\frac{k_3}{m-i_1-i_2}\cdots\frac{k_3-i_3+1}{m-n+1}\right),$$

from which we conclude that

$$\beta_{\mathbb{i}\mathbb{k}} \le \binom{n}{\mathbb{i}}\left(\frac{k_1}{m}+\frac{n}{m-n}\right)^{i_1}\left(\frac{k_2}{m}+\frac{n}{m-n}\right)^{i_2}\left(\frac{k_3}{m}+\frac{n}{m-n}\right)^{i_3}$$

and

$$\beta_{\mathbb{i}\mathbb{k}} \ge \binom{n}{\mathbb{i}}\left(\frac{k_1}{m}-\frac{n}{m}\right)^{i_1}\left(\frac{k_2}{m}-\frac{n}{m}\right)^{i_2}\left(\frac{k_3}{m}-\frac{n}{m}\right)^{i_3} \quad .$$

Thus, we find

$$\beta_{\mathbb{i}\mathbb{k}} = B^n_{\mathbb{i}}(\mathbb{k}/m) + O(1/m)$$

and, therefore,

$$\mathbf{d}_{\mathbb{k}} = \sum \mathbf{b}_{\mathbb{i}} B^n_{\mathbb{i}}(\mathbb{k}/m) + O(1/m) \quad .$$

Consequently, the mth degree Bézier nets of $\mathbf{b}(\mathbf{x})$ converge to $\mathbf{b}(\mathbf{x})$ linearly in $1/m$, see [Farin '79, Trump & Prautzsch '96].

11.10 Conversion to tensor product Bézier representation

Let $\mathbf{b}(\mathbf{x})$ be a bivariate polynomial with polar form $\mathbf{b}[\mathbf{x}_1 \ldots \mathbf{x}_n]$, and let $\mathbf{x}_{st} = \mathbf{x}(s,t)$ be any biaffine map that maps the unit square $[0,1]^2$ onto a convex quadrilateral, see Figure 11.11. Then the reparametrized polynomial

$$\mathbf{c}(s,t) = \mathbf{b}(\mathbf{x}(s,t))$$

is a tensor product polynomial of degree (n,n) in (s,t). Its tensor product polar form is given by

$$\mathbf{c}[s_1 \ldots s_n, t_1 \ldots t_n] = \frac{1}{n!}\sum_{\tau} \mathbf{b}[\mathbf{x}(s_1,\tau_1)\ldots\mathbf{x}(s_n,\tau_n)] \ ,$$

where the sum extends over all permutations $(\tau_1,\ldots,\tau_n)$ of $(t_1,\ldots,t_n)$. To verify this, one checks that $\mathbf{c}$ satisfies the three characterizing properties: the diagonal, symmetry and affinity property.

Knowing the tensor product polar form, we can apply the main theorem 9.2 to obtain the Bézier points of $\mathbf{c}(s,t)$ over $[0,1]^2$. These are the points

$$\mathbf{c}_{ij} = \mathbf{c}[0 \overset{n-i}{\ldots} 0 1 \overset{i}{\ldots} 1, 0 \overset{n-j}{\ldots} 0 1 \overset{j}{\ldots} 1] \ ,$$

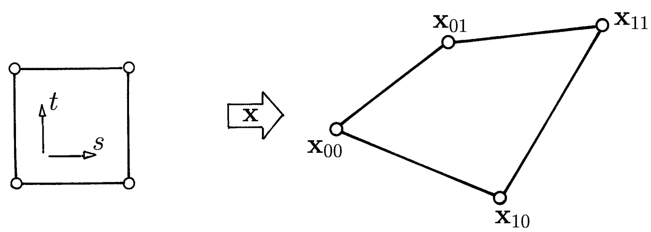

Figure 11.11: A biaffine reparametrization.

which can be written as

$$\mathbf{c}_{ij} = \sum_{k=0}^{j} \beta_{ijk} \mathbf{b}[\mathbf{x}_{00} \overset{n+k-i-j}{\dots} \mathbf{x}_{00}\mathbf{x}_{01} \overset{j-k}{\dots} \mathbf{x}_{01}\mathbf{x}_{10} \overset{i-k}{\dots} \mathbf{x}_{10}\mathbf{x}_{11} \overset{k}{\dots} \mathbf{x}_{11}] \ ,$$

with $n!\beta_{ijk}$ the number of permutations of $(0 \overset{n-j}{\dots} 0 1 \overset{j}{\dots} 1)$ such that exactly k ones sit in the last i positions. Hence,

$$\begin{aligned}
\beta_{ijk} &= \frac{1}{n!} \cdot \binom{j}{k} i \dots (i+1-k) \cdot (n-i) \dots (n-i+1-j+k) \cdot (n-j)! \\
&= \frac{k!}{n!} \binom{i}{k} \binom{j}{k} \frac{(n-i)!(n-j)!}{(n+k-i-j)!} \ .
\end{aligned}$$

From 11.3, we recall that the points

$$\mathbf{b}[\mathbf{x}_{00} \dots \mathbf{x}_{00}\mathbf{x}_{01} \dots \mathbf{x}_{01}\mathbf{x}_{10} \dots \mathbf{x}_{10}\mathbf{x}_{11} \dots \mathbf{x}_{11}]$$

arise when we subdivide the Bézier net of $\mathbf{b}(\mathbf{x})$ over $\mathbf{x}_{00}\mathbf{x}_{01}\mathbf{x}_{10}$ in $\mathbf{x}_{11}$, see also [DeRose et al. '93].

11.11 Conversion to triangular Bézier representation

A tensor product polynomial $\mathbf{b}(x, y)$ of degree (m, n) is of total degree $\leq m + n$. Thus, it has a Bézier representation of degree $l = m + n$ over any triangle $\mathbf{pqr}$. To compute it, let $\mathbf{b}[x_1 \dots x_m, y_1 \dots y_n]$ be the tensor product polar form of $\mathbf{b}(x, y)$. Then, the (non-tensor product) polar form of $\mathbf{b}(\mathbf{x})$ is given by

$$\mathbf{c}[\mathbf{x}_1 \dots \mathbf{x}_l] = \frac{1}{l!} \sum \mathbf{b}[x_{i_1} \dots x_{i_m}, y_{i_{m+1}} \dots y_{i_l}] \ ,$$

where the sum extends over all permutations $(i_1 \dots i_l)$ of $(1 \dots l)$ and $\mathbf{x}_i = (x_i, y_i)$. Namely, $\mathbf{c}$ obviously satisfies the three characterizing properties: the diagonal, symmetry and affinity property.

The Bézier points of $\mathbf{b}(\mathbf{x})$ over a triangle with vertices $\mathbf{p} = [p_1, p_2]^t$, $\mathbf{q} = [q_1, q_2]^t$ and $\mathbf{r} = [r_1, r_2]^t$ can be obtained by the main theorem 11.2.

They are the points

$$\begin{aligned}\mathbf{b}_{ijk} &= \mathbf{c}[\mathbf{p} \overset{i}{\ldots} \mathbf{pq} \overset{j}{\ldots} \mathbf{qr} \overset{k}{\ldots} \mathbf{r}] \\ &= \sum_{\substack{\alpha+\beta+\gamma=m \\ (\alpha,\beta,\gamma)\le(i,j,k)}} \delta^{ijk}_{\alpha\beta\gamma} \mathbf{b}[p_1 \overset{\alpha}{\ldots} p_1 q_1 \overset{\beta}{\ldots} q_1 r_1 \overset{\gamma}{\ldots} r_1, \\ &\qquad\qquad p_2 \overset{i-\alpha}{\ldots} p_2 q_2 \overset{j-\beta}{\ldots} q_2 r_2 \overset{k-\gamma}{\ldots} r_2] \ ,\end{aligned}$$

where

$$\delta^{ijk}_{\alpha\beta\gamma} = \binom{i}{\alpha}\binom{j}{\beta}\binom{k}{\gamma} m!n! \ .$$

Namely, there are $\binom{i}{\alpha}$ choices to pick α from i many p_1's etc., and there are $m!$ permutations of the first m variables $p_1, \overset{\alpha}{\ldots}, p_1, q_1, \overset{\beta}{\ldots}, q_1, r_1 \overset{\gamma}{\ldots}, r_1$ and $n!$ permutations of the last n variables. $\diamond$

From 9.5 we recall that the points

$$\mathbf{b}[p_1 \ldots p_1 q_1 \ldots q_1 r_1 \ldots r_1, p_2 \ldots p_2 q_2 \ldots q_2 r_2 \ldots r_2]$$

arise in the tensor product de Casteljau algorithm used to compute $\mathbf{b}(\mathbf{r})$ from the Bézier points over $[\mathbf{p}, \mathbf{q}]$.

11.12 Problems

1 Consider a functional polynomial with convex Bézier polyhedron. Show that degree elevation and uniform subdivision, as illustrated in Figure 11.6 preserve the convexity of the Bézier polyhedron.

2 There are parametric Bézier patches for which the statement in Problem 1 does not hold. Provide an example.

3 Generalize the intersection algorithm presented in Section 3.7 to triangular Bézier patches.

4 Let $\mathbf{v}_1 = \mathbf{q}-\mathbf{p}$ and $\mathbf{v}_2 = \mathbf{r}-\mathbf{q}$. Show that a Bézier net $\mathbf{b}_{\mathbb{i}}$ over $\mathbf{pq}$ is planar if all $\Delta_{\mathbf{v}_1}\Delta_{\mathbf{v}_1}\mathbf{b}_{\mathbb{i}}, \Delta_{\mathbf{v}_1}\Delta_{\mathbf{v}_2}\mathbf{b}_{\mathbb{i}}, \Delta_{\mathbf{v}_2}\Delta_{\mathbf{v}_2}\mathbf{b}_{\mathbb{i}}$ are zero. Thus, the maximum of these differences is a measure for the flatness of a triangular Bézier patch.

5 Storing triangular Bézier nets should be dealt with efficiently storage- and accesswise. An efficient way to deal with these Bézier nets is to store the points $\mathbf{b}_{ijk},\ i+j+k=n$, in a linear array, say

$$\mathbf{a}[L] = \mathbf{b}_{ijk} \ \ ,$$

where $L = L(i,j)$ runs from 1 to $\binom{n+2}{2}$.

Fast access is guaranteed if the function L is stored in a matrix $L = [L_{ij}]$. Provide an explicit formula for a function L used in Problem 5.

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