2. 幂等元的提升

2. Lifting Idempotents

2.1. 为了从特征为素数 p 提升一个幂等元到特征为零, 有效的方法是使用一个特征为零的完备的离散赋值环使得其根上的商环是特征为 p 的域. 事实上, 完备化条件是充分的但不必要的. 下面, 我考虑这些问题.

2.1. In order to lift idempotents from characteristic p to characteristic zero, the safest method is to work over a complete discrete valuation ring of characteristic zero with a residue field of characteristic p. Yet, as a matter of fact, the completeness is a sufficient but not necessary condition. In this section, we will discuss on this question.

2.2. 设 $\mathcal{O}$ 是一个离散赋值环使得 $k = \mathcal{O}/J(\mathcal{O})$ 是特征为 p 的域, 其中 $J(\mathcal{O})$ 是 $\mathcal{O}$ 的根基, 并且 $\mathcal{O}$ 的分式域 $\mathcal{K}$ 是为特征零的域. 也就是说, 有一个满射的群同态 $\vartheta\colon \mathcal{K}^* \to \mathbb{Z}$ 使得

2.2. Let $\mathcal{O}$ be a discrete valuation ring such that $k = \mathcal{O}/J(\mathcal{O})$ is a field of characteristic p, where $J(\mathcal{O})$ denotes the radical of $\mathcal{O}$, and that its field of quotients $\mathcal{K}$ has characteristic zero. That is to say, we have a surjective group homomorphism $\vartheta\colon \mathcal{K}^* \to \mathbb{Z}$ fulfilling

2.2.1 $$\{\lambda \in \mathcal{K}^* \mid \vartheta(\lambda) \geq 0\} = \mathcal{O} - \{0\}\,;$$

特别是, 有 $\pi \in \mathcal{O}$ 使得 $J(\mathcal{O}) = \pi\mathcal{O}$. 请注意, 如果 $\mathcal{K}'$ 是 $\mathcal{K}$ 的 Galois 扩张, 那么从范数映射 $\mathfrak{N}\colon \mathcal{K}'^* \to \mathcal{K}^*$ 不难定义一个满射的群同态 $\vartheta'\colon \mathcal{K}'^* \to \mathbb{Z}$; 进一步, 不难证明下面的集合

in particular, there is $\pi \in \mathcal{O}$ such that $J(\mathcal{O}) = \pi\mathcal{O}$. Note that, if $\mathcal{K}'$ is a Galois extension of $\mathcal{K}$, then from the norm map $\mathfrak{N}\colon \mathcal{K}'^* \to \mathcal{K}^*$ it is not difficult to define a surjective group homomorphism $\vartheta'\colon \mathcal{K}'^* \to \mathbb{Z}$; moreover, it is easy to check that the set

2.2.2 $$\{\lambda' \in \mathcal{K}'^* \mid \mathfrak{N}(\lambda') \in \mathcal{O}\} \cup \{0\}$$

就是 $\mathcal{K}'$ 中 $\mathcal{O}$ 上的整元构成的环 $\mathcal{O}'$; 也就是说, $\mathcal{O}'$ 也是离散赋值环.

coincides with the integral closure $\mathcal{O}'$ of $\mathcal{O}$ in $\mathcal{K}'$; in other words, $\mathcal{O}'$ still is a discrete valuation ring.

2.3. 设 $\{\lambda_n\}_{n\in\mathbb{N}}$ 是一个 $\mathcal{K}$ 的元素的序列; 如果存在 $\lambda \in \mathcal{K}$ 使得对任意 $n \in \mathbb{N}$ 有 $\lambda - \lambda_n \in \pi^{n+1}\mathcal{O}$, 那么我们说 $\{\lambda_n\}_{n\in\mathbb{N}}$ 有极限 λ, 并且我们记 $\lambda = \lim_{n\to\infty}\{\lambda_n\}$ (请注意, 这个条件比通常的收敛条件更强). 例子: 如果对任意 $n \in \mathbb{N}$ 有 $\lambda_n = \lambda$, 这个序列当然有极限 λ. 有极限的序列满足所谓 Cauchy 条件;

2.3. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{K}$; whenever there is $\lambda \in \mathcal{K}$ such that we have $\lambda - \lambda_n \in \pi^{n+1}\mathcal{O}$ for any $n \in \mathbb{N}$, we say that the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ has the *limit* λ, and we write $\lambda = \lim_{n\to\infty}\{\lambda_n\}$ (note that this condition is stronger than the usual one). For instance, if for any $n \in \mathbb{N}$ we have $\lambda_n = \lambda$, this sequence obviously has the limit λ. The sequences having a limit fulfill the so-called Cauchy condition; in particular, for any

特别是, 对任意 $n \in \mathbb{N}$ 有

$n \in \mathbb{N}$, we have

2.3.1 $$\lambda_{n+1} - \lambda_n \in \pi^{n+1}\mathcal{O}$$

(事实上, 上面的条件比 Cauchy 条件更强). 反过来, 如果满足这个条件的 $\mathcal{K}$-序列都有极限, 那么我们说 $\mathcal{K}$ 和 $\mathcal{O}$ 是完备的.

(actually, this condition is stronger than Cauchy's condition). Conversely, if all the $\mathcal{K}$-sequences fulfilling this condition have a limit then we say that $\mathcal{K}$ and $\mathcal{O}$ are *complete*.

2.4. 不难造一个包含 $\mathcal{K}$ 的完备域 $\hat{\mathcal{K}}$. 我们在所有满足条件 2.3.1 的 $\mathcal{K}$-序列里考虑下面的等价关系: 如果该集合里的两个序列 $\{\lambda_n\}_{n\in\mathbb{N}}$ 和 $\{\mu_n\}_{n\in\mathbb{N}}$ 满足

2.4. It is not difficult to construct a complete field $\hat{\mathcal{K}}$ containing $\mathcal{K}$. We consider the following equivalence relationship in the set of all the sequences fulfilling condition 2.3.1: if two sequences $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\mu_n\}_{n\in\mathbb{N}}$ of this set fulfill

2.4.1 $$\mu_n - \lambda_n \in \pi^{n+1}\mathcal{O}$$

就说这两个序列是等价的. 例子: 固定 $\ell \in \mathbb{N}$, 如果 $\mu_n = \lambda_\ell$ 或 λ_n 当 $n \le \ell$ 或 $n \ge \ell$, 那么这两个序列是等价的; 又, $\{\lambda_{\ell+n}\}_{n\in\mathbb{N}}$ 也满足条件 2.3.1 并 $\{\lambda_n\}_{n\in\mathbb{N}}$ 和它是等价的. 请注意, 如果 $\{\lambda_n\}_{n\in\mathbb{N}}$ 没有极限 0, 那么对适当的 ℓ, λ_ℓ 不属于 $\pi^{\ell+1}\mathcal{O}$; 此时, 从条件 2.3.1 可推出 $\vartheta(\lambda_{\ell+n}) = \vartheta(\lambda_\ell)$, 其中 $n \in \mathbb{N}$ (cf. 2.2.1).

we say that they are equivalent. For instance, fixing $\ell \in \mathbb{N}$, if $\mu_n = \lambda_\ell$ or λ_n according to $n \le \ell$ or $n \ge \ell$ then these sequences are equivalent; moreover, $\{\lambda_{\ell+n}\}_{n\in\mathbb{N}}$ also fulfills condition 2.3.1 and clearly is equivalent to $\{\lambda_n\}_{n\in\mathbb{N}}$. Note that, if $\{\lambda_n\}_{n\in\mathbb{N}}$ has not the limit 0 then, for a suitable ℓ, λ_ℓ does not belong to $\pi^{\ell+1}\mathcal{O}$; thus, from condition 2.3.1 we get $\vartheta(\lambda_{\ell+n}) = \vartheta(\lambda_\ell)$ for any $n \in \mathbb{N}$ (cf. 2.2.1).

2.5. 设 $\hat{\mathcal{K}}$ 是所有满足条件 2.3.1 的 $\mathcal{K}$-序列的等价类的集合; 显然, 序列的加法在 $\hat{\mathcal{K}}$ 里决定一个交换群结构. 而且, 设 $\hat{\lambda} = \widetilde{\{\lambda_n\}}_{n\in\mathbb{N}}$ 与 $\hat{\mu} = \widetilde{\{\mu_n\}}_{n\in\mathbb{N}}$ 是 $\hat{\mathcal{K}}$ 的两个非等于零元素; 我们能假定对任意 $n \in \mathbb{N}$ 有 $\vartheta(\lambda_n) = \vartheta(\lambda_\circ)$ 与 $\vartheta(\mu_n) = \vartheta(\mu_\circ)$; 取 $\ell \in \mathbb{N}$ 满足 $-\vartheta(\lambda_\circ) \le \ell$ 与 $-\vartheta(\mu_\circ) \le \ell$; 此时, 因为下面的差

2.5. Let $\hat{\mathcal{K}}$ be the set of equivalent classes of sequences of $\mathcal{K}$ which fulfill condition 2.3.1; clearly, the usual sum of sequences determines a structure of commutative group in $\hat{\mathcal{K}}$. Moreover, let $\hat{\lambda} = \widetilde{\{\lambda_n\}}_{n\in\mathbb{N}}$ and $\hat{\mu} = \widetilde{\{\mu_n\}}_{n\in\mathbb{N}}$ be two nonzero elements of $\hat{\mathcal{K}}$; by the remarks above, we may assume that, for any $n \in \mathbb{N}$, we have $\vartheta(\lambda_n) = \vartheta(\lambda_\circ)$ and $\vartheta(\mu_n) = \vartheta(\mu_\circ)$; choose ℓ fulfilling $-\vartheta(\lambda_\circ) \le \ell$ and $-\vartheta(\mu_\circ) \le \ell$; at that point, since the difference

2.5.1 $$\lambda_{n+1}\mu_{n+1} - \lambda_n\mu_n = \lambda_{n+1}(\mu_{n+1} - \mu_n) + (\lambda_{n+1} - \lambda_n)\mu_n\,,$$

显然属于 $\pi^{n+1-\ell}\mathcal{O}$, 所以序列 $\{\lambda_{\ell+n}\,\mu_{\ell+n}\}_{n\in\mathbb{N}}$ 满足2.3.1; 那么, 不难验证 $\{\lambda_{\ell+n}\,\mu_{\ell+n}\}_{n\in\mathbb{N}}$ 的等价类,记 $\hat{\lambda}\hat{\mu}$,不依赖我们的选择.

belongs to $\pi^{n+1-\ell}\mathcal{O}$, $\{\lambda_{\ell+n}\,\mu_{\ell+n}\}_{n\in\mathbb{N}}$ fufills 2.3.1; then, it is not difficut to check that the equivalent class of $\{\lambda_{\ell+n}\,\mu_{\ell+n}\}_{n\in\mathbb{N}}$, noted $\hat{\lambda}\hat{\mu}$, does not depend on our choice.

2.6. 这个乘法远算在 $\hat{\mathcal{K}}-\{0\}$ 里决定一个交换群结构; 确实, 不难验证结合律. 另一方面, 令 v 记 $\vartheta(\lambda_\circ)$ 与 0 中较大的数; 不难验证有

2.6. This operation determines a commutative group structure in $\hat{\mathcal{K}}-\{0\}$; indeed, the associativity is easily checked. On the other hand, denote by v the biggest of $\vartheta(\lambda_\circ)$ and 0; we clearly have

2.6.1 $$\lambda_{v+n}=\lambda_v(1+\sum_{i=1}^{n}\delta_i\pi^i)$$

其中 $n\in\mathbb{N}$ 与 $\delta_i\in\mathcal{O}$; 用归纳法我们能定义下面的序列 $\{\mu_n\}_{n\in\mathbb{N}}$

where $n\in\mathbb{N}$ and $\delta_i\in\mathcal{O}$; arguing by induction, we can define the following sequence $\{\mu_n\}_{n\in\mathbb{N}}$

2.6.2 $$\mu_n=1+\sum_{i=1}^{n}\varepsilon_i\pi^i\,,\quad \varepsilon_i=-\delta_i-\sum_{j=1}^{i-1}\delta_j\varepsilon_{i-j}\,;$$

因为对任意 $i\in\mathbb{N}$ 有 $\varepsilon_i\in\mathcal{O}$, 所以它满足条件 2.3.1; 此时, 不难验证其等价类乘以 λ_v^{-1} 常数序列的等价类就是 $\hat{\lambda}$ 的逆元素.

since we have $\varepsilon_i\in\mathcal{O}$ for any $i\in\mathbb{N}$, this sequence fulfills condition 2.3.1; thus, it is easy to check that its equivalent class multiplied by the equivalent class of the constant sequence λ_v^{-1} is the inverse of $\hat{\lambda}$.

2.7. 当然, 我定义 $0\hat{\lambda}=0=\hat{\lambda}\,0$; 那么不难验证分配律. 这样, 就证明了 $\hat{\mathcal{K}}$ 是一个域, 而只要把 $\lambda\in\mathcal{K}$ 与 λ 常数序列的等价类等同一致, $\mathcal{K}$ 就是一个 $\hat{\mathcal{K}}$ 的子域. 在 $\hat{\mathcal{K}}$ 里, 设 $\hat{\mathcal{O}}$ 是所有 $\mathcal{O}$-序列的等价类的集合; 显然, $\hat{\mathcal{O}}$ 是 $\hat{\mathcal{K}}$ 的子环, 并且它包含 $\mathcal{O}$.

2.7. Obviously, we set $0\hat{\lambda}=0=\hat{\lambda}\,0$; now, it is not difficult to check the distributivity. Hence, we have proved that $\hat{\mathcal{K}}$ is a field, and it suffices to identify $\lambda\in\mathcal{K}$ with the equivalent class of the constant sequence λ to get $\mathcal{K}$ as a subfield of $\hat{\mathcal{K}}$. Let $\hat{\mathcal{O}}$ be the subset of $\hat{\mathcal{K}}$ of all the equivalent classes of sequences in $\mathcal{O}$; clearly, $\hat{\mathcal{O}}$ is a subring of $\hat{\mathcal{K}}$ which contains $\mathcal{O}$.

2.8. 而且, 我们能把 $\mathcal{K}$ 的离散赋值 $\vartheta\colon\mathcal{K}^*\to\mathbb{Z}$ 扩张为一个 $\hat{\mathcal{K}}$ 的离散赋值 $\hat{\vartheta}$ 如下: 设 $\hat{\lambda}=\widetilde{\{\lambda_n\}}_{n\in\mathbb{N}}$ 是一个 $\hat{\mathcal{K}}^*$ 的元素; 别忘了, 我们能假定对任

2.8. Moreover, we can extend the discrete valuation $\vartheta\colon\mathcal{K}^*\to\mathbb{Z}$ of $\mathcal{K}$ to a discrete valuation $\hat{\vartheta}$ of $\hat{\mathcal{K}}$ as follows: let $\hat{\lambda}=\widetilde{\{\lambda_n\}}_{n\in\mathbb{N}}$ be an element of $\hat{\mathcal{K}}^*$; recall that we may assume that we have $\vartheta(\lambda_n)=\vartheta(\lambda_\circ)$ for any

意 $n \in \mathbb{N}$ 有 $\vartheta(\lambda_n) = \vartheta(\lambda_\circ)$; 那么就令 $\hat{\theta}(\hat{\lambda}) = \theta(\lambda_\circ)$; 特别是, $\hat{\theta}(\hat{\lambda}) \geq 0$ 当且仅当 $\hat{\lambda} \in \hat{\mathcal{O}}$.

$n \in \mathbb{N}$; in this case, we just set $\hat{\theta}(\hat{\lambda}) = \theta(\lambda_\circ)$; in particular, note that we have $\hat{\theta}(\hat{\lambda}) \geq 0$ if and only if $\hat{\lambda} \in \hat{\mathcal{O}}$.

2.9. 最后, 我断言 $\hat{\mathcal{K}}$ 和 $\hat{\mathcal{O}}$ 是完备的. 设 $\{\hat{\lambda}_n\}_{n\in\mathbb{N}}$ 是 $\hat{\mathcal{K}}$ 的一个元素的序列使得对任意 $n \in \mathbb{N}$ 有

2.9. Finally, we claim that $\hat{\mathcal{K}}$ and $\hat{\mathcal{O}}$ are complete. Let $\{\hat{\lambda}_n\}_{n\in\mathbb{N}}$ be a sequence of elements of $\hat{\mathcal{K}}$ such that, for any $n \in \mathbb{N}$, we have

2.9.1 $$\hat{\lambda}_{n+1} - \hat{\lambda}_n \in \pi^{n+1}\hat{\mathcal{O}};$$

这样, 如果 $\hat{\lambda}_n = \widetilde{\{\lambda_{n,m}\}}_{m\in\mathbb{N}}$, 对适当的 $\mu_{n,m} \in \mathcal{O}$, 其中 $n, m \in \mathbb{N}$, 还有

thus, if $\hat{\lambda}_n = \widetilde{\{\lambda_{n,m}\}}_{m\in\mathbb{N}}$ then, for a suitable choice of $\mu_{n,m} \in \mathcal{O}$, where $n, m \in \mathbb{N}$, we still have

2.9.2 $$(\lambda_{n+1,m} - \lambda_{n,m}) - \pi^{n+1}\mu_{n,m} \in \pi^{m+1}\mathcal{O};$$

别忘了, 由 2.4 的子里, 能假定对任意 $m \leq n$ 有 $\lambda_{n,m} = \lambda_{n,n+1}$; 此时, 元素 $\lambda_{n+1,m} - \lambda_{n,m}$ 属于 $\pi^{n+1}\mathcal{O}$, 其中 $m, n \in \mathbb{N}$; 因为这些序列满足条件 2.3.1, 所以对任意 $n \in \mathbb{N}$ 下面的差

recall that, according to the example in 2.4, we may assume that $\lambda_{n,m} = \lambda_{n,n+1}$ for any $m \leq n$; then, for any $m, n \in \mathbb{N}$, we obtain $\lambda_{n+1,m} - \lambda_{n,m} \in \pi^{n+1}\mathcal{O}$; since all these sequences fulfill condition 2.3.1, for any $n \in \mathbb{N}$, the difference

2.9.3 $$\lambda_{n+1,n+1} - \lambda_{n,n} = \lambda_{n+1,n+1} - \lambda_{n+1,n} + \lambda_{n+1,n} - \lambda_{n,n};$$

属于 $\pi^{n+1}\mathcal{O}$; 这样, $\{\lambda_{n,n}\}_{n\in\mathbb{N}}$ 满足 2.3.1, 从而这个序列决定 $\hat{\mathcal{K}}$ 的一个元素 $\hat{\lambda}$.

belongs to $\pi^{n+1}\mathcal{O}$; thus, $\{\lambda_{n,n}\}_{n\in\mathbb{N}}$ fulfills condition 2.3.1 and therefore determines an element $\hat{\lambda}$ of $\hat{\mathcal{K}}$.

2.10. 进一步, 对任意 $m, n \in \mathbb{N}$ 只要 $m \geq n$ 就有

2.10. Moreover, for any $m, n \in \mathbb{N}$, it suffices that $m \geq n$ to get

2.10.1 $$\lambda_{m,m} - \lambda_{n,m} = \sum_{i=n}^{m-1} (\lambda_{i+1,m} - \lambda_{i,m}) \in \pi^{n+1}\mathcal{O};$$

所以, 只要把上面的例子应用到序列 $\{\lambda_{m,m}\}_{m\in\mathbb{N}}$, 就 $\hat{\lambda} - \hat{\lambda}_n$ 属于 $\pi^{n+1}\hat{\mathcal{O}}$, 其中 $n \in \mathbb{N}$; 也就是说, 序列 $\{\hat{\lambda}_n\}_{n\in\mathbb{N}}$ 有极限 $\hat{\lambda}$. 请注意, 如果 $\{\lambda_n\}_{n\in\mathbb{N}}$ 在 $\mathcal{K}$ 中是满足条件 2.3.1 的序列, 那么 $\{\lambda_n\}_{n\in\mathbb{N}}$ 的等价类在 $\hat{\mathcal{K}}$ 中就是这个序列的极限; 所以只要 $\mathcal{K}$ 是完备的, 就得到 $\hat{\mathcal{K}} = \mathcal{K}$.

hence, it suffices to apply the example above to the sequence $\{\lambda_{m,m}\}_{m\in\mathbb{N}}$ to get that $\hat{\lambda} - \hat{\lambda}_n$ belongs to $\pi^{n+1}\hat{\mathcal{O}}$ for any $n \in \mathbb{N}$; that is, the sequence $\{\hat{\lambda}_n\}_{n\in\mathbb{N}}$ has the limit $\hat{\lambda}$. Note that, if $\{\lambda_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}$ which fulfills condition 2.3.1, then the element of $\hat{\mathcal{K}}$ determined by the equivalent class of $\{\lambda_n\}_{n\in\mathbb{N}}$ actually is the limit of this sequence; hence, if $\mathcal{K}$ is complete then we simply get $\hat{\mathcal{K}} = \mathcal{K}$.

2.11. **我们总是假定 $\mathcal{O}$-代数都是有限秩的自由 $\mathcal{O}$-模.** 设 A 是一个 $\mathcal{O}$-代数; 令 J 记 A 的根. 已经知道, 如果 $\mathcal{O}$ 是完备的, 那么 A-序列 $\{a_n\}_{n\in\mathbb{N}}$ 只要满足 $a_{n+1}-a_n\in J^{n+1}$, 其中 $n\in\mathbb{N}$, 它就有极限; 也就是说, 存在 $a\in A$ 使得对任意 $n\in\mathbb{N}$ 有 $a-a_n\in J^{n+1}$. 确实, 对适当的 $r\in\mathbb{N}$ 有 $J^r\subset\pi.A$, 从而对任意 $n\in\mathbb{N}$ 下面的差

2.11. *We always assume that the $\mathcal{O}$-algebras are $\mathcal{O}$-free $\mathcal{O}$-modules of finite rank.* Let A be an $\mathcal{O}$-algebra; denote by J the radical of A. It is well-known that if $\mathcal{O}$ is complete then it suffices that a sequence $\{a_n\}_{n\in\mathbb{N}}$ in A fulfills $a_{n+1}-a_n\in J^{n+1}$ for any $n\in\mathbb{N}$, to guarantee that it has a limit; precisely, in that case there is $a\in A$ such that $a-a_n\in J^{n+1}$ for any $n\in\mathbb{N}$. Indeed, we have $J^r\subset\pi.A$ for a suitable $r\in\mathbb{N}$ and therefore, for any $n\in\mathbb{N}$, the difference

2.11.1 $$a_{(r+1)(n+1)}-a_{(r+1)n}=\sum_{i=0}^{r}(a_{(r+1)n+i+1}-a_{(r+1)n+i})$$

属于 $J^{(r+1)n+1}$; 因为 A 是有限秩的自由 $\mathcal{O}$-模并对任意 $n\geq r$ 有 $J^{(r+1)n+1}\subset\pi^{n+1}\cdot A$, 所以存在 $a\in A$ 使得对任意 $n\geq r$ 有

belongs to $J^{(r+1)n+1}$; since A is an $\mathcal{O}$-free $\mathcal{O}$-module of finite rank and, for any $n\geq r$, we have $J^{(r+1)n+1}\subset\pi^{n+1}\cdot A$, there is $a\in A$ such that, for any $n\geq r$, we have

2.11.2 $$a-a_{(r+1)n}\in\pi^{n+1}.A\subset J^{n+1}\,,$$

从而还有

and therefore we also have

2.11.3 $$a-a_n=a-a_{(r+1)n}+\sum_{i=n+1}^{(r+1)n}(a_i-a_{i-1})\in J^{n+1}\,;$$

进一步, 对任意 $n\leq r$ 也得到

moreover, for any $n\in\mathbb{N}$, we still obtain

2.11.4 $$a-a_n=a-a_r+\sum_{i=n}^{r-1}(a_{i+1}-a_i)\in J^{n+1}\,.$$

2.12. 特别是, 如果 A' 是 $\mathcal{O}$-代数并 f 是 $\mathcal{O}$-代数同态从 A 到 A' 使得对任意 $n\in\mathbb{N}$ 有 $f(a_{n+1})-f(a_n)\in J'^{n+1}$ 其中 J' 是 A' 的根基,那么 $f(a)-f(a_{rn})$ 属于 $\pi^{n+1}.A'\subset J'^{n+1}$, 从而得到 $f(a)-f(a_n)\in J'^{n+1}$; 也就是说, $\{f(a_n)\}_{n\in\mathbb{N}}$ 有极限 $f(a)$. 而且如果 $r\in J$ 那么 $\{\sum_{\ell=0}^{n}r^\ell\}_{n\in\mathbb{N}}$ 有极限, 记为 $\sum_{\ell\in\mathbb{N}}r^\ell$; 这个极限就是 $1-r$ 的逆元素; 从而, A^* 包含 $1+J$.

2.12. In particular, if A' is an $\mathcal{O}$-algebra and f an $\mathcal{O}$-algebra homomorphism from A to A' such that, for any $n\in\mathbb{N}$, we have $f(a_{n+1})-f(a_n)\in J'^{n+1}$, where J' is the radical of A', then $f(a)-f(a_{rn})$ belongs to $\pi^{n+1}.A'\subset J'^{n+1}$ and therefore we get $f(a)-f(a_n)\in J'^{n+1}$; in other terms, $\{f(a_n)\}_{n\in\mathbb{N}}$ has the limit $f(a)$. Moreover, if $r\in J$ then the sequence $\{\sum_{\ell=0}^{n}r^\ell\}_{n\in\mathbb{N}}$ has a limit, noted $\sum_{\ell\in\mathbb{N}}r^\ell$; this limit actually coincides with the inverse of $1-r$; consequently, A^* contains $1+J$. Note that,

请注意, 如果 $\mathcal{K}'$ 是 $\mathcal{K}$ 的 Galois 扩张, 并 $\mathcal{O}'$ 是 $\mathcal{K}'$ 中 $\mathcal{O}$ 上的整元的环 $\mathcal{O}'$, 那么 $\mathcal{O}'$ 与 $\mathcal{K}'$ 也是完备的.

if $\mathcal{K}'$ is a Galois extension of $\mathcal{K}$, and $\mathcal{O}'$ is the integral closure of $\mathcal{O}$ in $\mathcal{K}'$, then $\mathcal{O}'$ and $\mathcal{K}'$ are complete too.

定理 2.13. 假定 $\mathcal{O}$ 是完备的. 设 A 是一个交换 $\mathcal{O}$-代数并设 I 是 A 的理想. 令 J 记 A 的根基并 $s: A \to A/J$ 记自然的映射, 再令

Theorem 2.13. *Assume that $\mathcal{O}$ is complete. Let A be a commutative $\mathcal{O}$-algebra and I an ideal of A. Denote by J the radical of A and by $s: A \to A/J$ the canonical map, and set* †

2.13.1
$$幂^{p^{\aleph}}(I) = \bigcap_{n\in\mathbb{N}} (\{a^{p^n} \mid a \in I\} + J^{n+1}),$$
$$幂^{p^{\aleph}}\big(s(I)\big) = \bigcap_{n\in\mathbb{N}} \{s(a)^{p^n} \mid a \in I\}.$$

那么 s 决定一个从 幂$^{p^{\aleph}}(I)$ 到 幂$^{p^{\aleph}}\big(s(I)\big)$ 的双射; 而且, 有

Then, s determines a bijection between 幂$^{p^{\aleph}}(I)$ and 幂$^{p^{\aleph}}\big(s(I)\big)$; moreover, we have

2.13.2
$$幂^{p^{\aleph}}(I) \cdot 幂^{p^{\aleph}}(I) \subset 幂^{p^{\aleph}}(I).$$

证明: 请注意, 如果 $a, b \in A$ 满足 $s(a) = s(b)$, 那么对任意 $n \in \mathbb{N}$ 有 $a^{p^n} - b^{p^n} \in J^{n+1}$; 确实, 只要使用归纳法, 我们就能假定 $n > 1$ 并且 $c = a^{p^{n-1}} - b^{p^{n-1}}$ 属于 J^n; 显然有

Proof: First of all, note that if $a, b \in A$ fulfill the equality $s(a) = s(b)$ then, for any $n \in \mathbb{N}$, we have $a^{p^n} - b^{p^n} \in J^{n+1}$; indeed, we argue by induction on n and may assume that $n > 1$ and that the element $c = a^{p^{n-1}} - b^{p^{n-1}}$ belongs to J^n; clearly, we have

2.13.3
$$a^{p^n} = (b^{p^{n-1}} + c)^p \in b^{p^n} + pJ^n + J^{np} \subset b^{p^n} + J^{n+1}.$$

现在, 如果 $a, b \in$ 幂$^{p^{\aleph}}(I)$ 那么对任意 $n \in \mathbb{N}$ 存在 $a_n, b_n \in I$ 与 $r_n, s_n \in J^{n+1}$ 使得

Now, if $a, b \in$ 幂$^{p^{\aleph}}(I)$ then for any $n \in \mathbb{N}$, we can find $a_n, b_n \in I$ and $r_n, s_n \in J^{n+1}$ such that

2.13.4
$$a = (a_n)^{p^n} + r_n, \quad b = (b_n)^{p^n} + s_n;$$

特别是, ab 属于下面的交

in particular, ab belongs to the intersection

2.13.5
$$\bigcap_{n\in\mathbb{N}} \big((a_n b_n)^{p^n} + J^{n+1}\big);$$

从而 ab 也属于 幂$^{p^{\aleph}}(I)$. 而且, 如果有 $s(a) = s(b)$ 那么还有

hence, ab still belongs to 幂$^{p^{\aleph}}(I)$. Moreover, if we have $s(a) = s(b)$ then $s(a_n)^{p^n} = s(b_n)^{p^n}$

† The Chinese character 幂 is pronounced "mi" as in "middle" and means "power" or "exponent".

$s(a_n)^{p^n} = s(b_n)^{p^n}$, 从而 $0 = s(a_n-b_n)^{p^n}$, 这是因为 A/J 是特征 p 的. 另一方面, 因为 A/J 是域的直积所以 $s(a_n) = s(b_n)$; 这样, 得到

and therefore we still have $s(a_n - b_n)^{p^n} = 0$ since A/J has characteristic p. On the other hand, since A/J is a direct product of fields, we also get $s(a_n) = s(b_n)$; thus, we obtain

2.13.6 $$a - b \in \bigcap_{m\in\mathbb{N}} J^m \subset \bigcap_{m\in\mathbb{N}} \pi^m\cdot A = \{0\}\,.$$

所以 s 决定一个单映射从 幂$^{p^{\aleph}}(I)$ 到 幂$^{p^{\aleph}}\big(s(I)\big)$; 我们要证明这个映射也是满射. 设 $\bar a$ 是一个幂$^{p^{\aleph}}\big(s(I)\big)$的元素; 也就是说, 对任意 $n\in\mathbb{N}$ 存在 $a_n\in I$ 使得 $s(a_n)^{p^n} = \bar a$; 特别是, 有

Hence, s determines an injective map from 幂$^{p^{\aleph}}(I)$ to 幂$^{p^{\aleph}}\big(s(I)\big)$; we will prove that this map is surjective too. Let $\bar a$ be an element of 幂$^{p^{\aleph}}\big(s(I)\big)$; explicitly, this means that, for any $n\in\mathbb{N}$, there is $a_n\in I$ such that $s(a_n)^{p^n} = \bar a$; in particular, we get

2.13.7 $$s\big((a_{n+1})^p\big)^{p^n} = \bar a = s(a_n)^{p^n}$$

仍然因为 A/J 就是特征为 p 的域的直积, 所以我们得到 $s\big((a_{n+1})^p\big) = s(a_n)$; 从而, 也得到

and therefore, since A/J is a direct product of fields of characteristic p, we have $s\big((a_{n+1})^p\big) = s(a_n)$; consequently, we also obtain

2.13.8 $$(a_{n+1})^{p^{n+1}} - (a_n)^{p^n} = \big((a_{n+1})^p\big)^{p^n} - (a_n)^{p^n} \in J^{n+1}\,.$$

那么存在 $a\in I$ 使得对任意 $n>0$ 有 $a-(a_n)^{p^n}\in J^{n+1}$ (见 2.11), 从而得到

Then, there is $a\in I$ such that, for any $n\in\mathbb{N}$, we have $a-(a_n)^{p^n}\in J^{n+1}$ (see 2.11), and therefore we get

2.13.9 $$s(a) = \big(s(a_n)\big)^{p^n} = \bar a\,;$$

也就是说, $s(a)$ 属于幂$^{p^{\aleph}}\big(s(I)\big)$.

that is to say, $s(a)$ belongs to 幂$^{p^{\aleph}}\big(s(I)\big)$.

推论 2.14. 假定 $\mathcal{O}$ 是完备的. 设 A 是 $\mathcal{O}$-代数并设 I 是一个 A 的理想. 令 J 记 A 的根, 再令 $s\colon A\to\bar A$ 为自然的映射, 其中 $\bar A = A/J$. 对任意幂等元 $\bar\imath\in\bar I$, 其中 $\bar I = s(I)$, 存在一个幂等元 $i\in A$ 使得 $s(i)=\bar\imath$. 而且, 如果 $i'\in A$ 是一个幂等元使得 $s(i')=\bar\imath$ 有 $a\in A^*$ 使得 $i'=i^a$.

Corollary 2.14. *Assume that $\mathcal{O}$ is complete. Let A be an $\mathcal{O}$-algebra and I an ideal of A. Denote by J the radical of A and by $s\colon A \longrightarrow \bar A$ the canonical map, where $\bar A = A/J$. For any idempotent $\bar\imath\in\bar I$, where $\bar I = s(I)$, there exists an idempotent $i\in A$ such that $s(i)=\bar\imath$. Moreover, if $i'\in A$ is an idempotent such that $s(i')=\bar\imath$, then there exists $a\in A^*$ such that $i'=i^a$.*

证明: 显然有 $a\in A$ 使得 $s(a)=\bar\imath$; 令 $B=\sum_{n\in\mathbb{N}}\mathcal{O}\cdot a^n$;

Proof: Clearly there is $a\in A$ such that $s(a)=\bar\imath$; set $B=\sum_{n\in\mathbb{N}}\mathcal{O}\cdot a^n$; then B is a

那么 B 是 A 的子代数, 而有 $B\cap J = J(B)$; 所以, 只要用 B, $B\cap I$ 分别代替 A, I 就能假定 A 是交换的. 那么, 由定理 2.13,既然 $\bar\imath$ 属于幂$^{p^{\aleph}}(s(I))$ 所以存在 $i\in$ 幂$^{p^{\aleph}}(I)$ 满足 $s(i)=\bar\imath$, 还满足 $s(i^2)=\bar\imath^2=\bar\imath$; 仍使用定理 2.13, 既然 i^2 也属于幂$^{p^{\aleph}}(I)$, 所以 $i^2=i$. 最后, 如果 $i'\in A$ 是一个幂等元使得 $s(i')=\bar\imath$, 那么令 $a=ii'+(1-i)(1-i')$; 一方面, 显然 $ia=ai'$, 另一方面有

subalgebra of A and we have $B\cap J=J(B)$; consequently, up to the replacement of A and I by B and $B\cap I$, we may assume that A is commutative. Then, according to Theorem 2.13, since $\bar\imath$ belongs to 幂$^{p^{\aleph}}\big(s(I)\big)$, there exists $i\in$ 幂$^{p^{\aleph}}(I)$ fulfilling $s(i)=\bar\imath$, and in particular $s(i^2)=\bar\imath^2=\bar\imath$; according to Theorem 2.13 again, since i^2 also belongs to 幂$^{p^{\aleph}}(I)$, we get $i^2=i$. Finally, if $i'\in A$ is an idempotent fulfilling $s(i')=\bar\imath$, then consider the element $a=ii'+(1-i)(1-i')$; on the one hand, we clearly have $ia=ai'$; on the other hand, we get

2.14.1 $$s(a)=\bar\imath^2+(1-\bar\imath)^2=1\,,$$

从而, 还有 $a\in 1+J\subset A^*$ (见 2.11).

and therefore we still get $a\in 1+J\subset A^*$ (see 2.11).

2.15. 一般的说, 因为一个 $\mathcal{O}$-代数 A 是自由 $\mathcal{O}$-模, 我们能把 $a\in A$ 与 $1\otimes a\in\hat{\mathcal{O}}\otimes_{\mathcal{O}}A$ 或者 $1\otimes a\in\mathcal{K}\otimes_{\mathcal{O}}A$ 或者 $1\otimes a\in\hat{\mathcal{K}}\otimes_{\mathcal{O}}A$ 都等同一致. 而且, 既然 $\mathcal{K}\cap\hat{\mathcal{O}}=\mathcal{O}$ 显然

2.15. As a general rule, since any $\mathcal{O}$-algebra A is an $\mathcal{O}$-free $\mathcal{O}$-module, we can identify $a\in A$ with $1\otimes a\in\hat{\mathcal{O}}\otimes_{\mathcal{O}}A$, $1\otimes a\in\mathcal{K}\otimes_{\mathcal{O}}A$ or $1\otimes a\in\hat{\mathcal{K}}\otimes_{\mathcal{O}}A$. Moreover, since we have $\mathcal{K}\cap\hat{\mathcal{O}}=\mathcal{O}$, we clearly get

2.15.1 $$(\mathcal{K}\otimes_{\mathcal{O}}A)\cap(\hat{\mathcal{O}}\otimes_{\mathcal{O}}A)=A\,.$$

命题 2.16. 设 A 是一个 $\mathcal{O}$-代数. 如果每一个 $\mathcal{K}\otimes_{\mathcal{O}}A$ 的本原幂等元在 $\hat{\mathcal{K}}\otimes_{\mathcal{O}}A$ 里也是本原的, 那么每一个 A 的本原幂等元在 $\hat{\mathcal{O}}\otimes_{\mathcal{O}}A$ 里也是本原的.

Proposition 2.16. *Let A be an $\mathcal{O}$-algebra. If any primitive idempotent of $\mathcal{K}\otimes_{\mathcal{O}}A$ remains primitive in $\hat{\mathcal{K}}\otimes_{\mathcal{O}}A$, then any primitive idempotent of A still remains primitive in $\hat{\mathcal{O}}\otimes_{\mathcal{O}}A$.*

证明: 设 i 是一个 A 的本原幂等元; 只要用 iAi 代替 A 就能假定 A 的单元素是本原的. 那么, 设 $\hat\imath$ 是一个 $\hat{A}=\hat{\mathcal{O}}\otimes_{\mathcal{O}}A$ 的不等于零幂等元并且设 $\hat{J}'$ 和 $\hat{J}''$ 是两个 $\hat{\mathcal{K}}\otimes_{\hat{\mathcal{O}}}\hat{A}$ 的相互正交本原幂等元的集合使得

Proof: Let i be a primitive idempotent of A; up to the replacement of A by iAi, we may assume that the unity element of A is primitive. Then, let $\hat\imath$ be a nonzero idempotent of $\hat{A}=\hat{\mathcal{O}}\otimes_{\mathcal{O}}A$, and consider two sets $\hat{J}'$ and $\hat{J}''$ of pairwise orthogonal primitive idempotents such that

2.16.1 $$\sum_{\hat\jmath\in\hat{J}'}\hat\jmath=\hat\imath\,,\qquad\sum_{\hat\jmath\in\hat{J}''}\hat\jmath=1-\hat\imath\,.$$

另一方面, 由我们的假设, 一个满足 $\sum_{j\in J} j=1$ 的 $\mathcal{K}\otimes_{\mathcal{O}}A$ 的相互正交本原幂等元的集合 J 在 $\hat{\mathcal{K}}\otimes_{\mathcal{O}}A$ 里也是本原幂等元的集合. 从而, 由下面的引理 2.17, 存在一个双映射 $\tau\colon J\to\hat{J}$ 其中 $\hat{J}=\hat{J}'\cup\hat{J}''$, 和 $(\hat{\mathcal{K}}\otimes_{\mathcal{O}}A)^*$ 的元素 $\hat{a}$ 使得对任意 $j\in J$ 有 $\tau(j)=j^{\hat{a}}$; 令 $J'=(\tau)^{-1}(\hat{J}')$ 与 $i=\sum_{j\in J'} j$, 就特别有 $\hat{\imath}=i^{\hat{a}}$. 事实上, 我们能假定 $\hat{a}\in\hat{A}$; 那么选择 $h,\ell\in\mathbb{N}$ 使得

On the other hand, from our hypothesis, a set of pairwise orthogonal primitive idempotents of $\mathcal{K}\otimes_{\mathcal{O}}A$ such that $\sum_{j\in J} j = 1$ remains a set of primitive idempotents in $\hat{\mathcal{K}}\otimes_{\mathcal{O}}A$. Moreover, according to Lemma 2.17 below, there are a bijective map $\tau\colon J\to\hat{J}$, where $\hat{J}=\hat{J}'\cup\hat{J}''$, and an element $\hat{a}$ in $(\hat{\mathcal{K}}\otimes_{\mathcal{O}}A)^*$ such that $\tau(j)=j^{\hat{a}}$ for any $j\in J$; in particular, setting $J'=(\tau)^{-1}(\hat{J}')$ and $i=\sum_{j\in J'} j$, we get $\hat{\imath}=i^{\hat{a}}$. Obviously, we may assume that $\hat{a}$ belongs to $\hat{A}$; then, consider $h,\ell\in\mathbb{N}$ such that

2.16.2 $$\hat{a}^{-1}\in\pi^{-h}\cdot\hat{A}\,,\quad i\in\pi^{-\ell}\cdot A\,.$$

设 $\{a_n\}_{n\in\mathbb{N}}$ 与 $\{b_n\}_{n\in\mathbb{N}}$ 是两个 A 的序列使得

Choose two sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ of elements of A such that

2.16.3 $$\hat{a}=\lim_{n\to\infty}\{a_n\}\,,\quad \pi^h\cdot\hat{a}^{-1}=\lim_{n\to\infty}\{b_n\}\,;$$

也就是说, 对任意 $n\in\mathbb{N}$ 元素 $\hat{a}-a_n$ 和 $\pi^h\cdot\hat{a}^{-1}-b_n$ 都属于 $\pi^{n+1}\cdot\hat{A}$. 特别是, 使 $m\geq 2h+\ell$ 固定, 显然有

in other terms, for any $n\in\mathbb{N}$, the elements $\hat{a}-a_n$ and $\pi^h\cdot\hat{a}^{-1}-b_n$ belong to $\pi^{n+1}\cdot\hat{A}$. In particular, choosing $m\geq 2h+\ell$, we have

2.16.4 $$\begin{aligned}1&=\big(a_m+(\hat{a}-a_m)\big)\big(\pi^{-h}\cdot b_m+(\hat{a}^{-1}-\pi^{-h}\cdot b_m)\big)\\&=\pi^{-h}\cdot a_m b_m+c\end{aligned}$$

其中 c 属于 $\pi^{m+1-h}\cdot\hat{A}$, 还属于 $\mathcal{K}\otimes_{\mathcal{O}}A$, 这是因为 $a_m b_m$ 属于 A; 从而有

where c belongs to $\pi^{m+1-h}\cdot\hat{A}$ and, at the same time, belongs to $\mathcal{K}\otimes_{\mathcal{O}}A$ since $a_m b_m$ belongs to A; consequently, we have

2.16.5 $$c\in\pi^{m+1-h}\cdot\hat{A}\cap(\mathcal{K}\otimes_{\mathcal{O}}A)=\pi^{m+1-h}\cdot A$$

并且 A 的元素 $1-c$ 在 A 中是可逆的; 所以 a_m 在 $\mathcal{K}\otimes_{\mathcal{O}}A$ 中也是可逆的, 更精确有

and therefore the element $1-c$ is inversible in A; hence, a_m is inversible in $\mathcal{K}\otimes_{\mathcal{O}}A$ and explicitly we have

2.16.6 $$\begin{aligned}(a_m)^{-1}&=\pi^{-h}\cdot b_m(1-c)^{-1}\\&=\pi^{-h}\cdot b_m+\sum_{n=1}^{\infty}\pi^{-h}\cdot b_m c^n\,;\end{aligned}$$

这样$a_m\, i\,(a_m)^{-1}$属于$\mathcal{K}\otimes_{\mathcal{O}} A$, 同时也属于$\hat{A}$, 这是因为

thus, $a_m\, i\,(a_m)^{-1}$ belongs to $\mathcal{K}\otimes_{\mathcal{O}} A$ and, simultaneously, it belongs to $\hat{A}$ since

2.16.7 $$\hat{\imath} - a_m\, i\,(a_m)^{-1} = (\hat{a} - a_m)\, i\,(\hat{a}^{-1}) + a_m\, i\,\big(\hat{a}^{-1} - (a_m)^{-1}\big)\,;$$

仍由于等式$\hat{A}\cap(\mathcal{K}\otimes_{\mathcal{O}} A) = A$, 这个幂等元 $a_m\, i\,(a_m)^{-1}$ 属于A, 从而它是单位元; 所以$\hat{\imath}$也是单位元.

hence, since $\hat{A}\cap(\mathcal{K}\otimes_{\mathcal{O}} A) = A$, the idempotent $a_m\, i\,(a_m)^{-1}$ belongs to A, and therefore it coincides with the unity element; thus, $\hat{\imath}$ coincides with unity element too.

引理 2.17. **设$\mathcal{L}$是一个域并设B是一个有限维数的$\mathcal{L}$-代数. 如果J与J'都是两个相互正交本原幂等元的集合使得$\sum_{j\in J} j = 1 = \sum_{j'\in J} j'$, 那么存在$b\in B^*$使得$J' = J^b$.**

Lemma 2.17. *Let $\mathcal{L}$ be a field and B an $\mathcal{L}$-algebra of finite dimension. If J and J' are two sets of pairwise orthogonal primitive idempotents of B such that $\sum_{j\in J} j = 1 = \sum_{j'\in J} j'$, then there exists $b\in B^*$ such that $J' = J^b$.*

证明: 如果B的根基就是零那么, 只要使用 Wedderburn 定理, 就完成证明. 这样, 能假定B的根基非零, 那么有一个B的非零理想N使得$N^2=\{0\}$; 令$\bar{B} = B/N$, 再令$\bar{b}$记$b\in B$的像; 首先指出, 如果i是B的本原幂等元, 那么$\bar{\imath}$也是本原的; 这是因为, 如果$0\neq \ell\in iBi$满足$\bar{\ell}^2 = \bar{\ell}$, 那么得到$0=(\ell^2-\ell)^2 = \ell^4 - 2\ell^3 + \ell^2$, 而不难验证这个式子可推出

Proof: If the radical of B is zero, it suffices to apply the Wedderburn Theorem to prove the statement. Thus, we may assume that the radical of B is not zero, and then B has a nonzero ideal N such that $N^2=\{0\}$; set $\bar{B} = B/N$ and denote by $\bar{b}$ the image of $b\in B$; first of all, we claim that if i is a primitive idempotent of B then $\bar{\imath}$ still is primitive; indeed, if we choose $0\neq \ell\in iBi$ fulfilling $\bar{\ell}^2 = \bar{\ell}$, then we get $0=(\ell^2-\ell)^2 = \ell^4 - 2\ell^3 + \ell^2$ according to our choice of N and, from this equality, it is not difficult to check that we have

2.17.1 $$\ell^n = (n-2)\ell^3 - (n-3)\ell^2$$

其中$n\geq 2$, 还可推出$3\ell^2-2\ell^3$是一个幂等元, 从而有$i = 3\ell^2-2\ell^3$. 因为$\bar{\imath}=3\bar{\ell}^2-2\bar{\ell}^3=\bar{\ell}$所以$\bar{\imath}$也是本原的.

for any $n\geq 2$, which easily implies that $3\ell^2-2\ell^3$ is an idempotent and therefore we get $3\ell^2-2\ell^3 = i$; since we have $\bar{\imath} = 3\bar{\ell}^2-2\bar{\ell}^3 = \bar{\ell}$, $\bar{\imath}$ is a primitive idempotent.

现在, 对维数$\dim_{\mathcal{L}}(B)$使用归纳法; 由上面的结果, $\bar{J}$与$\bar{J}'$是两个$\bar{B}$的相互正交本原幂等元的集合使得$\sum_{\bar{\jmath}\in\bar{J}}\bar{\jmath} = 1 = \sum_{\bar{\jmath}'\in\bar{J}}\bar{\jmath}'$; 而且, B^*显然包含$1+N$; 所以, 自然映射$B^*\to\bar{B}^*$是满射;

Now, we argue by induction on $\dim_{\mathcal{L}}(B)$; according to the previous argument, $\bar{J}$ and $\bar{J}'$ are two sets of pairwise orthogonal primitive idempotents of $\bar{B}$ such that $\sum_{\bar{\jmath}\in\bar{J}}\bar{\jmath} = 1 = \sum_{\bar{\jmath}'\in\bar{J}}\bar{\jmath}'$; moreover, B^* clearly contains $1+N$; consequently, the canonical map $B^*\to\bar{B}^*$ is surjective;

这样, 存在 $b \in B^*$ 和一个双射 $\tau\colon J \to J'$ 使得对任意 $j \in J$ 有 $\overline{\tau(j)} = \overline{j^b}$. 那么, 考虑 $c = \sum_{j\in J} jb\tau(j)$; 一方面可得到

thus, there are $b \in B^*$ and a bijective map $\tau\colon J \to J'$ such that, for any $j \in J$, we have $\overline{\tau(j)} = \overline{j^b}$. Then, consider the element $c = \sum_{j\in J} jb\tau(j)$; on the one hand, we have

2.17.2
$$\bar c = \sum_{j\in J} \bar\jmath \bar b \overline{j^b} = \bar b\,,$$

从而 c 是可逆的; 另一方面对任意 $j \in J$ 有 $jc = jb\tau(j) = c\tau(j)$.

and therefore c is inversible; on the other hand, we have $jc = jb\tau(j) = c\tau(j)$ for any $j \in J$.

推论 2.18. **设 A 是 $\mathcal{O}$-代数. 令 J 记 A 的根基并 $s\colon A \to A/J = \bar A$ 记自然的映射. 假定 $\mathcal{K} \otimes_{\mathcal{O}} A$ 的每一个本原幂等元在 $\hat{\mathcal{K}} \otimes_{\mathcal{O}} A$ 里也是本原的. 那么对任意 $\bar A$ 的幂等元 $\bar\imath$ 存在一个幂等元 $i \in A$ 使得 $s(i) = \bar\imath$. 而且, 如果 $i' \in A$ 也是一个幂等元使得 $s(i') = \bar\imath$, 就存在 $a \in A^*$ 使得 $i' = i^a$**

Corollary 2.18. *Let A be an $\mathcal{O}$-algebra. Denote by J the radical of A and by $s\colon A \to \bar A = A/J$ the canonical map. Assume that any primitive idempotent of $\mathcal{K} \otimes_{\mathcal{O}} A$ remains primitive in $\hat{\mathcal{K}} \otimes_{\mathcal{O}} A$. Then, for any idempotent $\bar\imath$ of $\bar A$, there is an idempotent $i \in A$ such that $s(i) = \bar\imath$. Moreover, if $i' \in A$ is also an idempotent such that $s(i') = \bar\imath$, then there is $a \in A^*$ such that $i' = i^a$.*

证明: 设 J 是一个 A 的相互正交本原幂等元的集合使得 $\sum_{j\in J} j = 1$ 并且设 $\bar J'$ 和 $\bar J''$ 是两个 $\bar A$ 的相互正交本原幂等元的集合使得

Proof: Let J be a set of pairwise orthogonal primitive idempotents of A such that $\sum_{j\in J} j = 1$, and choose two sets $\bar J'$ and $\bar J''$ of pairwise orthogonal primitive idempotents of $\bar A$ such that

2.18.1
$$\sum_{\bar\jmath\in \bar J'} \bar\jmath = \bar\imath\,, \qquad \sum_{\bar\jmath\in \bar J''} \bar\jmath = 1 - \bar\imath\,.$$

由命题 2.16, 对任意 $j \in J$, 幂等元 $s(j)$ 在 $\bar A$ 里也是本原的; 那么, 由引理 2.17, 存在双映射 $\tau\colon J \to \bar J' \cup \bar J''$ 和 $\bar a \in \bar A^*$ 使得对任意 $j \in J$ 有 $\tau(j) = s(j)^{\bar a}$. 可是, 任意满足 $s(a) = \bar a$ 的元素 $a \in A$ 是可逆的, 这是因为, 由 Nakayama 引理, 从 $\bar A \bar a = \bar A$ 可推出 $Aa = A$. 所以有

According to Proposition 2.16, the idempotent $s(j)$ is also primitive in $\bar A$ for any $j \in J$; then, according to Lemma 2.17, there are an element $\bar a \in \bar A^*$ and a bijective map $\tau\colon J \to \bar J' \cup \bar J''$ such that we have $\tau(j) = s(j)^{\bar a}$ for any $j \in J$. But any element $a \in A$ such that $s(a) = \bar a$ is inversible since, according to the Nakayama Lemma, from $\bar A \bar a = \bar A$, we can deduce that $Aa = A$. Hence, we have

2.18.2
$$s\Big(\sum_{j\in (\tau)^{-1}(\bar J')} j^a\Big) = \bar\imath\,.$$

最后, 如果 i 与 i' 是两个 A 的幂等元使得 $s(i)=\bar{\imath}=s(i')$, 那么一方面 $c=ii'+(1-i)(1-i')$ 也是可逆的, 因为从 $s(c)=1$ 可推出 $Ac=A$; 另一方面, 显然有 $ic=ii'=ci'$.

Finally, if i and i' are two idempotents of A such that $s(i)=\bar{\imath}=s(i')$ then, on the one hand, $c=ii'+(1-i)(1-i')$ is inversible since $s(c)=1$ implies that $Ac=A$; on the other hand, clearly we have $ic=ii'=ci'$.

Springer

http://www.springer.com/978-3-540-43514-3

Blocks of Finite Groups
The Hyperfocal Subalgebra of a Block
Puig, L.
2002, V, 215 p., Hardcover
ISBN: 978-3-540-43514-3