EPR-BELL TESTS WITH UNSHARP OBSERVABLES AND RELATIVISTIC QUANTUM MEASUREMENT

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Abstract. The Einstein-Podolsky-Rosen argument will be revisited and its validity checked for the case of realistic, imperfect measurements described in terms of positive operator valued measures. It is shown how Bell's inequalities can be satisfied if the degree of unsharpness in the observables involved is sufficiently large. The EPR contradiction between the assumptions of (usharp) reality, locality and the validity of quantum mechanics is resolved by maintaining realism and quantum mechanics and weakening locality. The form of nonlocality allowed is that of *objectification-at-a-distance*. Problems associated with the quantum description of localised, spacelike separated measurements are briefly discussed.

1. Introduction

In this contribution I will review the analysis of the Einstein-Podolsky-Rosen argument [19], Bell's inequalities [5] and of associated experiments for spins in terms of positive operator valued measures (in short: POVMs). Specifically, I will explore the relation between the Clauser-Horne-Shimony-Holt (CHSH) [16] inequality and a fundamental *classicality* property of observables—their *coexistence*; this leads to the question whether for macroscopic systems a relatively 'small' amount of unsharpness may suffice to ensure (and explain) the practical impossibility of exhibiting nonclassical features such as those represented by Bell-type inequalities.

I will present a derivation of Bell's inequalities for unsharp spins which follows a reconstruction by Mittelstaedt and Stachow [32] of the original EPR argument. In this treatment, the Bell inequalities follow from a conjunction of two assumptions, *(unsharp) reality* and *locality*, applied in the context of the quantum mechanics of an entangled pair of spins. Since the reality assumption can be consistently incorporated into the quantum formalism, it is locality that is incompatible with the latter. However, a contradiction only arises when the degree of unsharpness of the spins is not too high; otherwise the nonlocality of quantum mechanics cannot be detected with such observables. The contradiction can be resolved if the locality assumption is weakened so as to allow for a benign form of nonlocality: one has to accept that (unsharp) objectification can occur over spacelike distances or between dynamically separated parts of a system. Note that this argument is *not* about the supplementation of quantum mechanics with hidden

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variables but exhibits only the inevitability and nature of quantum nonlocality. But it does raise the question of a consistent description of the process of measurement for extended, entangled systems and for localised measurements in spacelike separated regions of spacetime.

A note on terminology may be in place. The term 'Bell inequality' refers, strictly speaking, to the inequality originally exhibited by Bell for pair probabilities associated with a triple of spin observables. Bell's argument made explicit use of the strict correlation between certain pairs of observables in the spin singlet state and thereby ignored the unavoidable experimental imprecisions. In order to provide a derivation of Bell's theorem without any recourse to quantum mechanical properties while taking into account experimental imperfections, Clauser, Horne, Shimony and Holt considered the case of a quadruple of spins, one pair each pertaining to one of the two particles involved. The ensuing inequality is known as 'CHSH inequality'. I will follow the widespread practice of referring to the latter as (Bell-)CHSH or simply Bell inequality.

2. Bell-CHSH inequalities, joint probabilities and coexistence

The role of CHSH inequalities as *classicality conditions* has been systematically studied by Pitowsky [36] and by Beltrametti and Maczynski in the early 1990s [6, 7, 8]. This was preceded by the observation due to Fine [20, 21] that a full set of Bell or CHSH inequalities is equivalent to the existence of triple or quadruple joint probability distributions. The concept of POVM as a joint observable for EPR-Bell observables was considered by Abu-Zeid and deMuynck in 1984 [1], with the conclusion that the violation of Bell inequalities reflects the nonexistence of such joint observables in the case of noncommuting sharp spin observables. The issue of formulating and exploring the meaning and role of Bell-type inequalities for unsharp spins has to my knowledge been addressed first by Busch in 1985 [11]; this was taken up and generalised by Kar and Roy in 1996 [28]. This part of my contribution will draw on the valuable review of Kar and Roy [29].

In this section I will exhibit the relationship between operator Bell inequalities and coexistence, showing that in the EPR context the latter implies the former but not conversely. This stands in contrast to the situation discussed by Fine and others, who showed that a set of Bell inequalities forms a necessary and sufficient condition for a family of pair probabilities to be embeddable into a quadruple joint probability. To explain the reason for this discrepancy, it will be helpful to briefly review Fine's theorem.

2.1. FINE'S THEOREM

In an EPR-Bell experiment on a correlated pair of spin 1/2 systems, one measures pairs of random variables $(\{a_k, a_{\bar{k}}\}, \{b_\ell, b_{\bar{\ell}}\})$, where $k \in \{1,2\}$, $\bar{k} \in \{\bar{1},\bar{2}\}$ and $\ell \in \{3,4\}$, $\bar{\ell} \in \{\bar{3},\bar{4}\}$ label two variables (spin observables) of system A and B, respectively. This gives rise to sets of frequencies which are to approach probabilities provided in a theoretical model of the experiment:

$$\begin{gathered} p_1,\ p_{\bar{1}},\ p_2,\ p_{\bar{2}},\ p_3,\ p_{\bar{3}},\ p_4,\ p_{\bar{4}}, \\ p_{13},\ p_{1\bar{3}},\ p_{\bar{1}3},\ p_{\bar{1}\bar{3}},\ p_{14},\ \dots,\ p_{23},\dots,\ p_{24},\ \dots,\ p_{\bar{2}\bar{4}}. \end{gathered} \tag{1}$$

Fine's theorem establishes a set of Bell-CHSH inequalities as a necessary and sufficient condition for this set of probabilities to be embeddable into a single classical probability model, that is, for the existence of a quadruple joint probability measure such that the single and pair probabilities arise as marginals.

Theorem 1. *For a system of probabilities (1) to be embeddable into a quadruple joint probability distribution* $\{p_{1234}, p_{123\bar{4}}, \ldots, p_{\bar{1}\bar{2}\bar{3}\bar{4}}\}$ *it is necessary and sufficient that the following set of Bell-CHSH inequalities holds:*

$$\begin{aligned}
0 &\leqslant p_{1\bar{3}} + p_{\bar{1}4} - p_{24} + p_{23} \leqslant 1\,, \\
0 &\leqslant p_{1\bar{4}} + p_{\bar{1}3} - p_{23} + p_{24} \leqslant 1\,, \\
0 &\leqslant p_{2\bar{3}} + p_{\bar{2}4} - p_{14} + p_{13} \leqslant 1\,, \\
0 &\leqslant p_{2\bar{4}} + p_{\bar{2}3} - p_{13} + p_{14} \leqslant 1\,,
\end{aligned} \tag{2}$$

or equivalently:

$$\begin{aligned}
0 &\leqslant p_1 + p_4 - p_{13} - p_{14} - p_{24} + p_{23} \leqslant 1\,, \\
0 &\leqslant p_1 + p_3 - p_{13} - p_{14} - p_{23} + p_{24} \leqslant 1\,, \\
0 &\leqslant p_2 + p_4 - p_{23} - p_{14} - p_{24} + p_{13} \leqslant 1\,, \\
0 &\leqslant p_2 + p_3 - p_{13} - p_{23} - p_{24} + p_{14} \leqslant 1\,.
\end{aligned} \tag{3}$$

We sketch the first steps of the proof. We introduce short-hands for the sought-for 4-probabilities:

$$\begin{array}{llll}
p_{1234} = a & p_{1\bar{2}34} = e & p_{\bar{1}234} = k & p_{\bar{1}\bar{2}34} = p \\
p_{123\bar{4}} = b & p_{1\bar{2}3\bar{4}} = f & p_{\bar{1}23\bar{4}} = \ell & p_{\bar{1}\bar{2}3\bar{4}} = q \\
p_{12\bar{3}4} = c & p_{1\bar{2}\bar{3}4} = g & p_{\bar{1}2\bar{3}4} = m & p_{\bar{1}\bar{2}\bar{3}4} = r \\
p_{12\bar{3}\bar{4}} = d & p_{1\bar{2}\bar{3}\bar{4}} = h & p_{\bar{1}2\bar{3}\bar{4}} = n & p_{\bar{1}\bar{2}\bar{3}\bar{4}} = s
\end{array} \tag{4}$$

Next we use these to reproduce the pair probabilities:

$$\begin{array}{ll}
p_{13} = a+b+e+f & p_{\bar{1}3} = k+\ell+p+q \\
p_{1\bar{3}} = c+d+g+h & p_{\bar{1}\bar{3}} = m+n+r+s \\
p_{14} = a+c+e+g & p_{\bar{1}4} = k+m+p+r \\
p_{1\bar{4}} = b+d+f+h & p_{\bar{1}\bar{4}} = \ell+n+q+s \\
p_{23} = a+b+k+\ell & p_{\bar{2}3} = e+f+p+q \\
p_{2\bar{3}} = c+d+m+n & p_{\bar{2}\bar{3}} = g+h+r+s \\
p_{24} = a+c+k+m & p_{\bar{2}4} = e+g+p+r \\
p_{2\bar{4}} = b+d+\ell+n & p_{\bar{2}\bar{4}} = f+h+q+s
\end{array} \tag{5}$$

We have to establish a minimum subset of $\{a, b, \ldots, s\}$ such that all other numbers can be expressed in terms of these and the given marginality relations. Start with a, b, c, d

assumed given. This yields:

$$
\begin{array}{ll}
e+f=p_{13}-a-b & p+q=p_{\bar{1}3}-k-\ell=p_{\bar{1}3}-p_{23}+a+b \\
g+h=p_{1\bar{3}}-c-d & r+s=p_{\bar{1}\bar{3}}-m-n=p_{\bar{1}\bar{3}}-p_{2\bar{3}}+c+d \\
e+g=p_{14}-a-c & p+r=p_{\bar{1}4}-k-m=p_{\bar{1}4}-p_{24}+a+c \\
f+h=p_{1\bar{4}}-b-d & q+s=p_{\bar{1}\bar{4}}-\ell-n=p_{\bar{1}\bar{4}}-p_{2\bar{4}}+b+d \\
k+\ell=p_{23}-a-b & p+q=p_{\bar{2}3}-e-f=p_{\bar{2}3}-p_{13}+a+b \\
m+n=p_{2\bar{3}}-c-d & r+s=p_{\bar{2}\bar{3}}-g-h=p_{\bar{2}\bar{3}}-p_{1\bar{3}}+c+d \\
k+m=p_{24}-a-c & p+r=p_{\bar{2}4}-e-g=p_{\bar{2}4}-p_{14}+a+c \\
\ell+n=p_{2\bar{4}}-b-d & q+s=p_{\bar{2}\bar{4}}-f-h=p_{\bar{2}\bar{4}}-p_{1\bar{4}}+b+d
\end{array}
\tag{6}
$$

Next, consider e,k,p given:

$$
\begin{aligned}
f &= p_{13}-a-b-e \\
g &= p_{14}-a-c-e \\
h &= p_{1\bar{3}}-c-d-(p_{14}-a-c-e)=p_{1\bar{3}}-p_{14}+a+e-d \\
\ell &= p_{23}-a-b-k \\
m &= p_{24}-a-c-k \\
n &= p_{2\bar{3}}-c-d-(p_{24}-a-c-k)=p_{2\bar{3}}-p_{24}+a+k-d \\
q &= p_{\bar{1}3}-p_{23}+a+b-p \\
r &= p_{\bar{1}4}-p_{24}+a+c-p \\
s &= p_{\bar{2}\bar{3}}-p_{1\bar{3}}+c+d-(p_{\bar{1}4}-p_{24}+a+c-p) \\
&= p_{\bar{2}\bar{3}}-p_{1\bar{3}}-p_{\bar{1}4}+p_{24}+d+p-a
\end{aligned}
\tag{7}
$$

As a check, we can see that

$$
a+b+\cdots+r+s=p_3+p_{\bar{3}}=1. \tag{8}
$$

The task is to ensure that all numbers $a,b,\ldots,r,s$ are nonnegative. Hence:

$$
\begin{aligned}
a &\geqslant 0,\ b\geqslant 0,\ c\geqslant 0,\ d\geqslant 0,\ e\geqslant 0,\ f\geqslant 0,\ g\geqslant 0,\ h\geqslant 0, \\
k &\geqslant 0,\ \ell\geqslant 0,\ m\geqslant 0,\ n\geqslant 0,\ p\geqslant 0,\ q\geqslant 0,\ r\geqslant 0,\ s\geqslant 0.
\end{aligned}
\tag{9}
$$

Inserting the expressions for the pair probabilities into the Bell inequality (2) and using the positivity (9) readily confirms the validity of the Bell inequality, given the existence of the quadruple joint probabilities (4-jpd). This constitutes the necessity part of the proof. Next one wants to see that a sufficient set of Bell inequalities ensures the existence of a 4-jpd. Thus one has to ensure that numbers $a,b,c,d,e,k,p\geqslant 0$ can be found such that (5) holds and all remaining numbers f,g,h,ℓ,m,n,q,r,s, which are determined by the first seven numbers, are nonnegative.

The nine inequalities $f,g,h,\ell,m,n,q,r,s\geq 0$ can be organised as follows, using (7):

$$
\begin{aligned}
p_{14}-p_{1\bar{3}}+d &\leqslant a+e\leqslant \min\{p_{13}-b,p_{14}-c\} \\
p_{24}-p_{2\bar{3}}+d &\leqslant a+k\leqslant \min\{p_{23}-b,p_{24}-c\} \\
p_{1\bar{3}}+p_{\bar{1}4}-p_{\bar{2}\bar{3}}-p_{24}-d &\leqslant p-a\leqslant \min\{p_{\bar{1}3}-p_{23}+b,p_{\bar{1}4}-p_{24}+c\}
\end{aligned}
\tag{10}
$$

This system, together with the inequalities $a,b,c,d,e,k,p \geqslant 0$, leads eventually to a set of inequalities for b, c and d, hence these numbers must lie in the intersection of a number of intervals. The condition that these intervals are nonempty finally entails the CHSH inequalities. Then one can choose $b,c,d \geq 0$ to lie in their respective intervals, and this enables one to choose $a,e,k,p \geq 0$ satisfying (10), which ensure the nonnegativity of the remaining nine constants.

2.2. COEXISTENCE AND BELL-CHSH INEQUALITIES FOR SPIN $\frac{1}{2}$

In recent years there has been increasing interest in the use of POVMs for tests of Bell-type inequalities as an indication of nonlocal quantum correlations (e.g., [4, 24, 37, 40, 42]). There are nonseparable mixed states for which the Bell-CHSH inequalities are violated not for the usual pairs of sharp spins but only for suitable families of unsharp observables. This situation is one illustration of the fact that optimisation of information gain in measurements can under certain conditions only be achieved with POVMs that are no PVMs. A comprehensive introduction to the topic of POVMs and their application in quantum foundations and experiments can be found in the monograph [13].

We will only be concerned with POVMs whose domains are finite Boolean algebras, which can be represented as power sets of finite value spaces $\Omega = \{1,2,\ldots,N\}$, $\Sigma = 2^{\Omega}$. Thus the definition of the full POVM follows from the additivity if only the map $i \mapsto E_i := E(\{i\})$ is given. Hence in the sequel we will simply refer to the POVM $E : X\left(\in 2^{\Omega}\right) \mapsto E(X)$ in terms of set $\{E_1,E_2,\ldots,E_N\}$.

The set of POVMs is known to contain noncommuting subsets that can be measured jointly, that is, their ranges can be contained in the range of one common POVM. Such families of POVMs are called *coexistent*. It has been shown that pairs or triples of unsharp spin observables are coexistent if their degree of unsharpness is large enough [12]. Let us consider spin 1/2 POVMs generated by effects of the form

$$E(n,\lambda) := \frac{1}{2}(I + \lambda n \cdot \sigma),$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ denotes the vector of Pauli spin matrices, n is a unit vector in $\mathbb{R}^3$ denoting a point on the unit sphere S^2, and $\lambda \in [0,1]$. The eigenvalues are $\frac{1}{2}(1 \pm \lambda)$, and the spectral projections are $P_n := \frac{1}{2}(I \pm n \cdot \sigma)$. Thus,

$$E(n,\lambda) = \frac{1}{2}(1+\lambda)P_n + \frac{1}{2}(1-\lambda)P_{-n}.$$

From this representation it is evident that the POVM $\{E(n,\lambda), E(-n,\lambda)\}$ is a smeared version of the PVM $\{P_n, P_{-n}\}$. This is the formal sense in which the former represents an unsharp spin.

A pair of sharp spin observables is noncommutative if their respective vectors n_1, n_2 are not collinear. Such pairs have no joint observable. But two unsharp spin observables can be coexistent. Necessary and sufficient conditions for this to happen are as follows [12]:

Theorem 2. *A pair of unsharp spin observables*
$a = \{E(n_1,\lambda), E(-n_1,\lambda)\}$, $a' = \{E(n_2,\lambda), E(-n_2,\lambda)\}$ *is coexistent if and only if*

$$\lambda \|n_1 + n_2\| + \lambda \|n_1 - n_2\| \leqslant 2. \tag{11}$$

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