# Chapter 4

# SOUNDNESS AND COMPLETENESS

This chapter contains a proof that the basic tableau rules are sound and complete with respect to generalized Henkin models. Soundness is by the "usual" argument, is straightforward, and is what I begin with. Completeness is something else altogether. For that I use the ideas developed simultaneously in [Tak67, Pra68], where they were applied to give a non-constructive proof of a cut elimination theorem.

## 1. Soundness

Soundness means that any sentence having a tableau proof must be valid. Tableau soundness arguments follow the same pattern for all logics: some notion of satisfiability is defined for tableaus; then satisfiability is shown to be preserved by each tableau rule application. Note that in the following, $L^+(C)$ is used rather than $L(C)$, because formulas of the larger language $L^+(C)$ can occur in tableaus.

DEFINITION 4.1 (TABLEAU SATISFIABILITY) *A tableau branch is* satisfiable *if the set of formulas on it is satisfiable in a generalized Henkin model for $L^+(C)$ (see Definition 2.29). A tableau is* satisfiable *if some branch is satisfiable.*

Now, two key facts about these notions easily give us soundness. For the first, a closed tableau branch contains some formula and its negation, hence cannot be satisfiable. Since a closed tableau has every branch closed, we immediately have the following.

LEMMA 4.2 *A closed tableau cannot be satisfiable.*

The second key fact takes more work to prove, but the work is spread over several cases, each of which is rather simple.

LEMMA 4.3 *If a branch extension rule is applied to a satisfiable tableau, the result is another satisfiable tableau.*

**Proof** Suppose $\mathcal{T}$ is a satisfiable tableau. Then it has some satisfiable branch, say $\mathcal{B}$. Also suppose some branch extension rule is applied to $\mathcal{T}$ to produce a new tableau, $\mathcal{T}'$. It must be shown that $\mathcal{T}'$ is satisfiable.

The rule that was applied to turn $\mathcal{T}$ into $\mathcal{T}'$ may have been applied on a branch other than $\mathcal{B}$. In this case $\mathcal{B}$ is still a branch of $\mathcal{T}'$, and of course is still satisfiable, so $\mathcal{T}'$ is satisfiable. Now, for the rest of the proof assume a branch extension rule has been applied to the satisfiable branch $\mathcal{B}$ itself. And to be specific, say all the grounded formulas on $\mathcal{B}$ are true in the generalized Henkin model $\langle \mathcal{M}, \mathcal{A} \rangle$ with respect to the valuation $v$, where $\mathcal{M} = \langle \mathcal{H}, \mathcal{I}, \mathcal{E} \rangle$.

There are several cases, depending on which branch extension rule was applied. I consider only a few of these cases and leave the rest to you.

**Disjunction** Suppose the grounded formula $X \vee Y$ occurred on $\mathcal{B}$ and a rule was applied to it. Then in $\mathcal{T}'$ the branch $\mathcal{B}$ has been replaced with two branches: $\mathcal{B}$ lengthened with $X$, and $\mathcal{B}$ lengthened with $Y$. All formulas on $\mathcal{B}$ are true in $\langle \mathcal{M}, \mathcal{A} \rangle$ with respect to valuation $v$, hence $\mathcal{M} \Vdash_{v,\mathcal{A}} X \vee Y$. Then either $\mathcal{M} \Vdash_{v,\mathcal{A}} X$ or $\mathcal{M} \Vdash_{v,\mathcal{A}} Y$. In the first case, all members of $\mathcal{B}$ lengthened with $X$, and in the second case, all members of $\mathcal{B}$ lengthened with $Y$, are true in $\langle \mathcal{M}, \mathcal{A} \rangle$ with respect to $v$. Either way, some branch of $\mathcal{T}'$ is satisfiable.

**Existential Quantifier** Suppose the grounded formula $(\exists\alpha)\Phi(\alpha)$ occurred on $\mathcal{B}$ and a rule was applied to it, so that in $\mathcal{T}'$ branch $\mathcal{B}$ has been lengthened with $\Phi(p)$ where $p$ is a parameter new to $\mathcal{B}$, of the same type as $\alpha$.

Since all formulas on $\mathcal{B}$ are true in $\langle \mathcal{M}, \mathcal{A} \rangle$ with respect to $v$, $\mathcal{M} \Vdash_{v,\mathcal{A}} (\exists\alpha)\Phi(\alpha)$. Then, by definition of truth in a model, there must be some $\alpha$-variant $w$ of $v$ such that $\mathcal{M} \Vdash_{w,\mathcal{A}} \Phi(\alpha)$. Let $\sigma = \{p/\alpha\}$—the substitution that replaces $p$ by $\alpha$—and consider the valuation $w^\sigma$ (Definition 2.26). I claim all formulas on $\mathcal{B}$ extended with $\Phi(p)$ are true in $\langle \mathcal{M}, \mathcal{A} \rangle$ with respect to $w^\sigma$, so the extended branch is satisfiable.

First of all, $v$ and $w$ agree on all variables except $\alpha$. It is easy to see that $w$ and $w^\sigma$ agree on all variables except $p$, so the only variables on which $v$ and $w^\sigma$ can differ are $\alpha$ and $p$. But $\alpha$ does not occur free in any formula on $\mathcal{B}$, since these formulas are all grounded. And $p$ does not occur either, since $p$ was new to the branch. Consequently all formulas on $\mathcal{B}$ are true in $\langle\mathcal{M},\mathcal{A}\rangle$ with respect to $w^\sigma$, by Proposition 2.30.

Finally, note that since $p$ did not occur in $(\exists\alpha)\Phi(\alpha)$, then $\Phi(\alpha) = \Phi(p)\sigma$. We have $\mathcal{M} \Vdash_{w,\mathcal{A}} \Phi(\alpha)$, and by Proposition 2.31

$$\begin{aligned}\mathcal{M} \Vdash_{w,\mathcal{A}} \Phi(\alpha) &\Leftrightarrow \mathcal{M} \Vdash_{w,\mathcal{A}} \Phi(p)\sigma \\ &\Leftrightarrow \mathcal{M} \Vdash_{w^\sigma,\mathcal{A}} \Phi(p).\end{aligned}$$

This completes the argument for the existential case.

**Abstraction** Suppose the grounded formula

$$\langle\lambda\alpha_1,\ldots,\alpha_n.\Phi(\alpha_1,\ldots,\alpha_n)\rangle(\tau_1,\ldots,\tau_n)$$

occurred on $\mathcal{B}$, and a rule was applied to it, so that in $\mathcal{T}'$ branch $\mathcal{B}$ has been lengthened with $\Phi(\tau_1,\ldots,\tau_n)$. We are assuming that the formulas on $\mathcal{B}$ are all true in $\langle\mathcal{M},\mathcal{A}\rangle$ with respect to valuation $v$. I will show that this extends to include $\Phi(\tau_1,\ldots,\tau_n)$ as well.

Let $\sigma = \{\alpha_1/\tau_1, \ldots, \alpha_n/\tau_n\}$. This is free for $\Phi(\alpha_1,\ldots,\alpha_n)$ because $\tau_1, \ldots, \tau_n$ must be grounded, and parameters are never quantified or $\lambda$-bound. Now consider the valuation $v^\sigma$. Note the following useful items.

1 $v^\sigma(\alpha_i) = (v * \mathcal{I} * \mathcal{A})(\alpha_i\sigma) = (v * \mathcal{I} * \mathcal{A})(\tau_i)$

2 If $\beta$ is different from $\alpha_1, \ldots, \alpha_n$, $v^\sigma(\beta) = (v * \mathcal{I} * \mathcal{A})(\beta\sigma) = (v * \mathcal{I} * \mathcal{A})(\beta) = v(\beta)$.

Since $\langle\lambda\alpha_1,\ldots,\alpha_n.\Phi(\alpha_1,\ldots,\alpha_n)\rangle(\tau_1,\ldots,\tau_n)$ is on $\mathcal{B}$, we have

$$\mathcal{M} \Vdash_{v,\mathcal{A}} \langle\lambda\alpha_1,\ldots,\alpha_n.\Phi(\alpha_1,\ldots,\alpha_n)\rangle(\tau_1,\ldots,\tau_n).$$

For this to be the case

$$\begin{aligned}&\langle(v * \mathcal{I} * \mathcal{A})(\tau_1),\ldots,(v * \mathcal{I} * \mathcal{A})(\tau_n)\rangle \in \\ &\mathcal{E}((v * \mathcal{I} * \mathcal{A})(\langle\lambda\alpha_1,\ldots,\alpha_n.\Phi(\alpha_1,\ldots,\alpha_n)\rangle)).\end{aligned}$$

Since we have a generalized Henkin model, $\mathcal{A}$ is proper, so

$$\begin{aligned}&\mathcal{E}((v * \mathcal{I} * \mathcal{A})(\langle\lambda\alpha_1,\ldots,\alpha_n.\Phi(\alpha_1,\ldots,\alpha_n)\rangle)) = \\ &\{\langle w(\alpha_1),\ldots,w(\alpha_n)\rangle \mid w \text{ is an } \alpha_1,\ldots,\alpha_n\text{-variant of } v \\ &\ \text{and } \mathcal{M} \Vdash_{w,\mathcal{A}} \Phi(\alpha_1,\ldots,\alpha_n)\}\end{aligned}$$

and consequently $\mathcal{M} \Vdash_{w,\mathcal{A}} \Phi(\alpha_1,\ldots,\alpha_n)$ where $w$ is the $\alpha_1, \ldots,$ $\alpha_n$-variant of $v$ such that $w(\alpha_1) = (v * \mathcal{I} * \mathcal{A})(\tau_1), \ldots, w(\alpha_n) =$

$(v * \mathcal{I} * \mathcal{A})(\tau_n)$. But, by items 1 and 2 above, $v^\sigma$ itself is this $\alpha_1, \ldots,$ $\alpha_n$-variant of $v$. We thus have

$$\mathcal{M} \Vdash_{v^\sigma,\mathcal{A}} \Phi(\alpha_1, \ldots, \alpha_n).$$

Now, by Proposition 2.31,

$$\mathcal{M} \Vdash_{v,\mathcal{A}} \Phi(\alpha_1, \ldots, \alpha_n)\sigma,$$

that is,

$$\mathcal{M} \Vdash_{v,\mathcal{A}} \Phi(\tau_1, \ldots, \tau_n).$$

There are other cases—I leave them to you. ■

THEOREM 4.4 (SOUNDNESS) *If a sentence $\Phi$ of $L(C)$ has a tableau proof, $\Phi$ must be true in all generalized Henkin models with respect to $L(C)$.*

**Proof** Suppose $\Phi$ has a tableau proof, but is not true in every generalized Henkin model with respect to $L(C)$—I derive a contradiction. Since $\Phi$ is not true in every generalized Henkin model with respect to $L(C)$, $\{\neg\Phi\}$ is satisfiable, and by Proposition 2.33, is so in a generalized Henkin model with respect to $L^+(C)$. A tableau proof of $\Phi$ begins with a tableau consisting of a single branch, containing the single formula $\neg\Phi$, so this must be a satisfiable tableau. As we apply branch extension rules, we continue to get satisfiable tableaus, by Lemma 4.3. Since $\Phi$ is provable, we can get a closed tableau. Hence there must be a closed, satisfiable tableau, which is impossible according to Lemma 4.2. ■

Essentially the same argument also establishes the following.

THEOREM 4.5 *Let $S$ be a set of sentences and $\Phi$ be a single sentence of $L(C)$. If $\Phi$ has a tableau derivation from $S$, then $\Phi$ is a generalized Henkin consequence of $S$.*

## 2. Completeness

The proof of completeness, for basic tableaus, with respect to generalized Henkin models, is of considerable intricacy. It is spread over several subsections, each devoted to a single aspect of it. All the basic ideas go back to [Tak67, Pra68], where they were used to establish non-constructively a cut-elimination theorem for higher-order Gentzen systems. I also use aspects of the (second-order) presentation of [Tol75], in particular the central goal, for us, is to prove that something called a Hintikka set is satisfiable. This contains the essence of the proofs of [Tak67, Pra68]. [And71] abstracted the Takahashi, Prawitz ideas to prove a higher-order Model Existence Theorem which could simply have been cited, but the ideas of the completeness proof are pretty and deserve to be better known, hence the full presentation.

In outline, the completeness proof is as follows. In Section 2.1 the notion of a *Hintikka set* is defined: it is a set of grounded formulas of $L^+(C)$ meeting certain closure conditions bearing an obvious relationship to the tableau rules. In Section 2.2 *pseudo-models* are introduced. These are the closest we come, in higher-order logic, to the Herbrand models familiar in the first-order setting. Unfortunately, they will not look like proper models in the higher-order sense, because objects assigned as meanings for predicate abstracts might lie outside the range allowed for quantifiers. In Section 2.3 some rather technical (but important) results about the behavior of substitution in pseudo-models are shown. In Section 2.4 it is established that each Hintikka set is satisfiable in some pseudo-model. Section 2.5 shows that pseudo-models, in fact, are proper generalized Henkin models after all, and so each Hintikka set is satisfiable in such a model. Finally in Section 2.6 it is shown how to extract a Hintikka set from a failed tableau proof attempt, and this puts the last step in place for the completeness proof.

## 2.1 Hintikka Sets

Hintikka sets are fairly familiar from propositional and first-order logics—see [Fit96] and [Smu68] for instance. They play a similar role in the higher-order case, though arguments about them are much more complex. You should note that the basic tableau rules all correspond directly to Hintikka set conditions (I omit the connective $\equiv$ as a small convenience).

DEFINITION 4.6 (HINTIKKA SET) *A non-empty set $H$ of grounded formulas of $L^+(C)$ is a* Hintikka set *if it meets the following conditions.*

*1 Atomic Case. If $\Phi$ is atomic, not both $\Phi \in H$ and $\neg\Phi \in H$.*

*2 Conjunctive Cases.*

*(a) If $(\Phi \wedge \Psi) \in H$ then $\Phi \in H$ and $\Psi \in H$.*

*(b) If $\neg(\Phi \vee \Psi) \in H$ then $\neg\Phi \in H$ and $\neg\Psi \in H$.*

*(c) If $\neg(\Phi \supset \Psi) \in H$ then $\Phi \in H$ and $\neg\Psi \in H$.*

*3 Disjunctive Cases.*

*(a) If $(\Phi \vee \Psi) \in H$ then either $\Phi \in H$ or $\Psi \in H$.*

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