Chapter 5

EQUALITY

The basic tableau rules of Chapter 3 do not give any special role to equality. It is time to bring it into the picture. This is done by adding axioms to the tableau system, which has the effect of narrowing things to *normal* generalized Henkin models. In addition, some useful derived tableau rules will be presented.

1. Adding Equality

Leibniz's principle is that objects are equal just in case they have the same properties. This principle is most easily embodied in axioms, rather than in tableau-style rules.

DEFINITION 5.1 (EQUALITY AXIOMS) *Each sentence of the following form is an* equality axiom*:*

$$(\forall\alpha)(\forall\beta)[(\alpha = \beta) \equiv (\forall\gamma)(\gamma(\alpha) \supset \gamma(\beta))]$$

In this, $=$ *is of type* $\langle t, t\rangle$*, for some* t*, then* α *and* β *are of type* t *and* γ *is of type* $\langle t\rangle$. EQ *denotes the set of equality axioms.*

I will show that a closed formula Φ of $L(C)$ is valid in *normal* generalized Henkin models if and only if Φ has a tableau derivation from EQ. But before that is done I give some handy derived tableau rules, and examples of their use.

2. Derived Rules and Tableau Examples

There are two derived rules involving equality that are more "tableau-like" in flavor, and are what I primarily use in constructing tableau proofs and derivations. I do not know if they can serve as full replacements for the official Equality Axioms, since I have been unable to prove a completeness theorem using them. Nonetheless, the derived rules below are the ones I generally use in practice.

DEFINITION 5.2 (DERIVED REFLEXIVITY RULE) *For a grounded term* τ *of* $L^+(C)$, *at any point in a proof* $(\tau = \tau)$ *may be added to the end of a tableau branch. Schematically,*

$$\frac{}{(\tau = \tau)}$$

Justification of Derived Reflexivity Rule Let τ be a grounded term of type t. $(\tau = \tau)$ can be added to the end of a branch via the following sequence of steps.

$$\begin{array}{ll}
(\forall\alpha)(\forall\beta)[(\alpha = \beta) \equiv (\forall\gamma)(\gamma(\alpha) \supset \gamma(\beta))] & 1. \\
(\forall\beta)[(\tau = \beta) \equiv (\forall\gamma)(\gamma(\tau) \supset \gamma(\beta))] & 2. \\
[(\tau = \tau) \equiv (\forall\gamma)(\gamma(\tau) \supset \gamma(\tau))] & 3. \\
[(\tau = \tau) \supset (\forall\gamma)(\gamma(\tau) \supset \gamma(\tau))] & 4. \\
[(\forall\gamma)(\gamma(\tau) \supset \gamma(\tau)) \supset (\tau = \tau)] & 5.
\end{array}$$

$$\begin{array}{cc}
/ & \backslash \\
\neg(\forall\gamma)(\gamma(\tau) \supset \gamma(\tau)) \quad 6. & (\tau = \tau) \quad 7.
\end{array}$$

In this, 1 is an equality axiom; 2 is from 1 and 3 is from 2 by universal rules; 4 and 5 are from 3 by a conjunction rule; 6 and 7 are from 5 by a disjunction rule. Clearly the left branch continues to closure. The remaining open branch, the right one, indeed, has $(\tau = \tau)$ on it.

The next rule embodies the familiar notion of substitutivity of equals for equals.

DEFINITION 5.3 (DERIVED SUBSTITUTIVITY RULE) *Suppose* $\Phi(\alpha)$ *is a formula of* $L^+(C)$ *in which the variable* α *may have free occurrences, but no other variables occur free. Also suppose* τ_1 *and* τ_2 *are grounded terms of the same type as* α. *As usual, let* $\Phi(\tau_1)$ *denote the result of replacing free occurrences of* α *in* $\Phi(\alpha)$ *with occurrences of* τ_1*; and similarly for* $\Phi(\tau_2)$. *Then, if both* $\Phi(\tau_1)$ *and* $(\tau_1 = \tau_2)$ *occur on a tableau branch,* $\Phi(\tau_2)$ *can be added to the branch end. Schematically,*

$$\frac{\begin{array}{c}\Phi(\tau_1)\\ (\tau_1 = \tau_2)\end{array}}{\Phi(\tau_2)}$$

Justification of Derived Substitutivity Rule Assume τ_1 and τ_2 are grounded terms of type t, and $\Phi(\tau_1)$ and $(\tau_1 = \tau_2)$ occur on a tableau branch. I show $\Phi(\tau_2)$ can be added to the end of the branch.

$$\begin{array}{l}
\Phi(\tau_1) \\
(\tau_1 = \tau_2) \\
\vdots \\
(\forall\alpha)(\forall\beta)[(\alpha = \beta) \equiv (\forall\gamma)(\gamma(\alpha) \supset \gamma(\beta))] \quad 1. \\
(\forall\beta)[(\tau_1 = \beta) \equiv (\forall\gamma)(\gamma(\tau_1) \supset \gamma(\beta))] \quad 2. \\
[(\tau_1 = \tau_2) \equiv (\forall\gamma)(\gamma(\tau_1) \supset \gamma(\tau_2))] \quad 3. \\
[(\tau_1 = \tau_2) \supset (\forall\gamma)(\gamma(\tau_1) \supset \gamma(\tau_2))] \quad 4. \\
[(\forall\gamma)(\gamma(\tau_1) \supset \gamma(\tau_2)) \supset (\tau_1 = \tau_2)] \quad 5.
\end{array}$$

$$\begin{array}{ll}
\neg(\tau_1 = \tau_2) \quad 6. & (\forall\gamma)(\gamma(\tau_1) \supset \gamma(\tau_2)) \quad 7. \\
 & \langle\lambda\alpha.\Phi(\alpha)\rangle(\tau_1) \supset \langle\lambda\alpha.\Phi(\alpha)\rangle(\tau_2) \quad 8. \\
 & \begin{array}{ll}
 \neg\langle\lambda\alpha.\Phi(\alpha)\rangle(\tau_1) \quad 9. & \langle\lambda\alpha.\Phi(\alpha)\rangle(\tau_2) \quad 10. \\
 \neg\Phi(\tau_1) \quad 11. & \Phi(\tau_2) \quad 12.
 \end{array}
\end{array}$$

In this, 1 is an equality axiom; 2 is from 1 and 3 is from 2 by universal rules; 4 and 5 are from 3 by a conjunction rule; 6 and 7 are from 4 by a disjunction rule; 8 is from 7 by a universal rule, using the term $\langle\lambda\alpha.\Phi(\alpha)\rangle$; 9 and 10 are from 8 by a disjunction rule; 11 is from 9 and 12 is from 10 by a predicate abstract rule. The two left branches are closed, leaving the right one which contains $\Phi(\tau_2)$.

Now I give several examples of tableau derivations using the derived rules. The first example is (intentionally) a simple one. It appeared earlier as Example 2.9, where an informal reading was given, and validity was shown directly.

EXAMPLE 5.4 Here is a proof of $\langle\lambda X.(\exists x)X(x)\rangle(\langle\lambda x.x = c\rangle)$.

$$\begin{array}{l}
\neg\langle\lambda X.(\exists x)X(x)\rangle(\langle\lambda x.x = c\rangle) \quad 1. \\
\neg(\exists x)\langle\lambda x.x = c\rangle(x) \quad 2. \\
\neg\langle\lambda x.x = c\rangle(c) \quad 3. \\
\neg(c = c) \quad 4. \\
\ \ (c = c) \quad 5.
\end{array}$$

In this, 2 is from 1 by an abstract rule; 3 is from 2 by a universal rule; 4 is from 3 by an abstract rule, and 5 is by the derived reflexivity rule.

The next example shows how, by using the derived rules, we can reverse things and prove a version of the equality axioms.

EXAMPLE 5.5 I give a tableau proof (using derived rules, not axioms) of

$$(\forall\alpha)(\forall\beta)[(\forall\gamma)(\gamma(\alpha)\supset\gamma(\beta))\supset(\alpha=\beta)].$$

$$\begin{array}{ll}
\multicolumn{2}{l}{\neg(\forall\alpha)(\forall\beta)[(\forall\gamma)(\gamma(\alpha)\supset\gamma(\beta))\supset(\alpha=\beta)]\quad 1.} \\
\multicolumn{2}{l}{\neg(\forall\beta)[(\forall\gamma)(\gamma(P)\supset\gamma(\beta))\supset(P=\beta)]\quad 2.} \\
\multicolumn{2}{l}{\neg[(\forall\gamma)(\gamma(P)\supset\gamma(Q))\supset(P=Q)]\quad 3.} \\
\multicolumn{2}{l}{(\forall\gamma)(\gamma(P)\supset\gamma(Q))\quad 4.} \\
\multicolumn{2}{l}{\neg(P=Q)\quad 5.} \\
\multicolumn{2}{l}{\langle\lambda X.\neg(X=Q)\rangle(P)\supset\langle\lambda X.\neg(X=Q)\rangle(Q)\quad 6.} \\
\neg\langle\lambda X.\neg(X=Q)\rangle(P)\quad 7. & \langle\lambda X.\neg(X=Q)\rangle(Q)\quad 8. \\
\neg\neg(P=Q)\quad 9. & \neg(Q=Q)\quad 10. \\
 & (Q=Q)\quad 11.
\end{array}$$

Here 2 is from 1, and 3 is from 2 by an existential rule (P and Q are new parameters of the appropriate type); 4 and 5 are from 3 by a conjunctive rule; 6 is from 4 by a universal rule, using the grounded term $\langle\lambda X.\neg(X=Q)\rangle$; 7 and 8 are from 6 by a disjunction rule; 9 and 10 are from 7 and 8 by abstract rules; 11 is the derived reflexivity rule.

Though the derived tableau rules for equality allow us to prove the axioms, it does not follow they are their equivalent. To establish that, we would need to have a cut elimination theorem for the tableau system with the equality rules. And the way to prove cut elimination is to first have a completeness proof. I conjecture that such a completeness result is provable, but I don't know how to do it.

Exercises

EXERCISE 2.1 Prove the following characterization of equality—it says it is the smallest reflexive relation.

$$(\forall x)(\forall y)\{(x=y)\equiv(\forall R)[(\forall z)R(z,z)\supset R(x,y)]\}$$

EXERCISE 2.2 Give a tableau derivation of the following from EQ.

$$(\forall\alpha)(\forall\beta)[(\alpha=\beta)\supset(\forall\gamma)(\alpha(\gamma)\equiv\beta(\gamma))]$$

More generally, one can do the same with the following.

$$(\forall\alpha)(\forall\beta)[(\alpha=\beta)\supset(\forall\gamma_1)\cdots(\forall\gamma_n)(\alpha(\gamma_1,\ldots,\gamma_n)\equiv\beta(\gamma_1,\ldots,\gamma_n))]$$

3. Soundness and Completeness

The results of this section combine to prove the following.

THEOREM 5.6 *Let Φ be a closed formula and let S be a set of closed formulas of $L(C)$.*

1 *Φ is valid in all normal generalized Henkin models if and only if Φ has a tableau derivation from* EQ.

2 *Φ is a consequence of S with respect to normal generalized Henkin models if and only if Φ has a tableau derivation from $S \cup \mathsf{EQ}$.*

The theorem above combines soundness and completeness. One direction, soundness, is almost immediate. Every equality axiom is true in every *normal* generalized Henkin model, so the implications from right to left in Theorem 5.6 follow immediately from Theorems 4.4 and 4.5. As usual, the completeness direction is more work. The key item is to prove the following Proposition. Once we have it, completeness follows immediately using part 2 of Theorem 4.30.

PROPOSITION 5.7 *Given a generalized Henkin model in which all members of* EQ *are true, there is a* normal *generalized Henkin model in which exactly the same closed formulas of $L(C)$ are true.*

The rest of this section is given over to a proof of Proposition 5.7—it is broken up into constructions and Lemmas. The ideas are the same as in Gödel's original completeness proof for first-order logic with equality—bring equivalence classes into the picture.

For the rest of this section, assume $\langle \mathcal{M}, \mathcal{A} \rangle$ is a generalized Henkin model, $\mathcal{M} = \langle \mathcal{H}, \mathcal{I}, \mathcal{E} \rangle$, and all members of EQ are true in this model.

For $O_1, O_2 \in \mathcal{H}(t)$, let us write $O_1 =_{\mathcal{I}} O_2$ as a more readable alternative notation for $\langle O_1, O_2 \rangle \in \mathcal{I}(=^{\langle t,t \rangle})$. Thus $=_{\mathcal{I}}$ is the interpretation of the equality constant symbol (of a particular type, which will be indicated only if needed). Since all equality axioms are true in $\langle \mathcal{M}, \mathcal{A} \rangle$, it is an easy consequence that $=_{\mathcal{I}}$ is an equivalence relation.

For each $O \in \mathcal{H}(t)$, let $\overline{O}$ be the equivalence class determined by O, that is, $\overline{O} = \{O' \mid O =_{\mathcal{I}} O'\}$. Define a new Henkin domain mapping by setting $\overline{\mathcal{H}}(t) = \{\overline{O} \mid O \in \mathcal{H}(t)\}$. Also, define a new interpretation by setting $\overline{\mathcal{I}}(A)$ to be the equivalence class containing $\mathcal{I}(A)$, that is, $\overline{\mathcal{I}}(A) = \overline{\mathcal{I}(A)}$.

LEMMA 5.8 *If $\overline{O_1} = \overline{O_2}$ then $\mathcal{E}(O_1) = \mathcal{E}(O_2)$.*

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