Chapter 1

Propositional Calculus

§10. The Language of $\mathcal{P}$

As discussed in the Introduction, we can study the nature of reasoning by studying logistic systems in which reasoning is represented in a precise way. We shall commence our study with a rather simple logistic system called $\mathcal{P}$, which is one formulation of propositional calculus. Many of the definitions and concepts which we shall introduce while studying $\mathcal{P}$ will also be applicable to richer logical systems. In addition, $\mathcal{P}$ will occur as a subsystem of some of these richer systems. Once we have discussed $\mathcal{P}$, which is an example of a logistic system, we will be in a good position to give a general explanation of what a logistic system is.

In order to make clear what aspects of reasoning will be represented in $\mathcal{P}$, we start by considering three examples of logical inferences.

EXAMPLE 1: Jack is asked whether he will go to the picnic on Saturday. He doesn't like to go to picnics in the rain, and he remembers that if it does not rain, he will be playing in a tennis match which conflicts with the picnic. Hence he replies,

(A_1) "If it rains, I will not go to the picnic."

(B_1) "If it does not rain, I will not go to the picnic."

(C_1) "Therefore, I will not go to the picnic."

EXAMPLE 2: We are given that w, z, and y are sets such that $z - y = \emptyset$ and $w \cap \widetilde{z} \subseteq y$ (where $\widetilde{z}$ is the complement of the set z), and we wish to show that $w \subseteq y$. So we assume that $z - y = \emptyset$ and $w \cap \tilde{z} \subseteq y$ and $x \in w$, and try to prove $x \in y$. Using the information that is given, it is not hard to show that

(A_2) If $x \in z$, then $x \in y$.

(B_2) If $x \notin z$, then $x \in y$.

(C_2) Therefore, $x \in y$.

EXAMPLE 3: Let us use $\star\star$ as a name for the sentence "The first sentence in this chapter which contains an occurrence of the symbol @ is not true." One may argue that:

(A_3) If $\star\star$ is true, then $\star\star$ is not true.

(B_3) If $\star\star$ is not true, then $\star\star$ is not true.

(C_3) Therefore, $\star\star$ is not true.

The reader need not be concerned with all the details of the arguments above, or even with whether the statements (A_i) and (B_i) were correctly inferred in each argument. The important point is that (C_i) is indeed a logical consequence of (A_i) and (B_i) in each argument.

Note that the main inference in each argument has the following form:

(A) If p, then q.

(B) If not p, then q.

(C) Therefore, q.

Obviously, the key words involved in this inference are "if", "then", and "not". These are connectives, words which can be used to construct more complex statements from simpler statements. Statements are sometimes called propositions (though some prefer to say that statements express propositions), and certain connectives which are important for logical purposes are represented in logistic systems as *propositional connectives.* A propositional calculus is a logistic system which formalizes the logical use of propositional connectives.

It is customary to use special symbols instead of English words for propositional connectives. This is very convenient, since the connectives are used very often and the symbols are quicker to write than the words (once one becomes accustomed to them). Also, the symbols allow one to exhibit very clearly the logical structure of a proposition which might be expressed in a variety of ways in English. Logistic systems which use symbols in this way are also referred to as systems of symbolic logic.

We shall express "If p, then q" symbolically as "$p \supset q$", and "not p" as "$\sim p$". In this notation, the inference above can be expressed as follows:

(A') $p \supset q$

(B') $\sim p \supset q$

(C') q

(The word "therefore" indicates that a conclusion is being reached, but has no logical content itself, so we have omitted it from (C').)

Before discussing the details of the logistic system $\mathcal{P}$, let us make a brief survey of the most important propositional connectives.

Negation ($\sim$)

We write the negation of the statement p as $\sim p$. In English this is generally expressed by inserting the word "not" into the sentence at an appropriate point. If w is the statement "everyone gets wet", $\sim w$ can be expressed by the statement "not everyone gets wet", but if r is the statement "it is raining", $\sim r$ is expressed by "it is not raining".

If p is true, then the statement $\sim p$ is false, while if p is a false statement, then $\sim p$ is a true statement. This is summarized in Figure 1.1. which we call the truth table for $\sim$. We write T for True, and F for False.

p	$\sim p$
T	F
F	T

Figure 1.1: Truth table for $\sim$

Conjunction ($\wedge$)

The conjunction of statements p and q is written as $[p \wedge q]$, and is generally expressed in English as "p and q" or "p, but q". Note that the statements "It is raining, and Jack is going to the picnic" and "It is raining, but Jack is going to the picnic" are *logically* equivalent, though they differ in emphasis. They both mean that the statement "It is raining" is true and

p	q	$p \wedge q$	$p \vee q$	$p \supset q$	$p \equiv q$	$p \not\equiv q$
T	T	T	T	T	T	F
T	F	F	T	F	F	T
F	T	F	T	T	F	T
F	F	F	F	T	T	F

Figure 1.2: Truth tables for $\wedge$, $\vee$, $\supset$, $\equiv$, $\not\equiv$

the statement "Jack is going to the picnic" is true. The statement $p \wedge q$ is true precisely when p and q are both true.

In Figure 1.2 we display the truth table for $\wedge$ and the other connectives which we will discuss below. Each horizontal line of the truth table corresponds to one of the logical possibilities with regard to the truth or falsity of the statements p and q. For example, in line 1 (below the bar) of the truth table, p is true and q is true and $[p \wedge q]$ is true, while in line 2, p is true and q is false and $[p \wedge q]$ is false.

Disjunction ($\vee$)

The disjunction of p and q is written as $[p \vee q]$, and generally expressed in English as "p or q". As can be seen from Figure 1.2, we regard $[p \vee q]$ as true if either or both of the statements p and q are true. Thus, we use $\vee$ for *inclusive* disjunction; we include the case where p and q are both true as one of the cases in which $[p \vee q]$ is true. For mathematical purposes, this is the generally accepted usage of the word "or". For example, 24 is regarded as a member of the set of integers which are multiples of 2 or multiples of 3. However, in ordinary usage the word "or" is sometimes used in the *exclusive sense*, in which "p or q" means p is true or q is true, but not both. For example, if Jack says, "I will go to the picnic or I will go to the tennis match", he may mean that he will go to one or the other, but not both. We will discuss exclusive disjunction below.

Implication ($\supset$)

$[p \supset q]$ means that the statement p implies the statement q, i.e., q is true whenever p is true. This can be translated into English in a variety of ways, including "if p, then q"; "q, if p"; "p only if q"; "p is sufficient for q"; "q is necessary for p" (i.e., "it is necessary for q to be true in order for p to be true").

We intend $\supset$ to represent *material implication*, so that the truth of $[p \supset q]$ depends only on the truth or falsity of the statements p and q.

Other notions of implication, in which the truth of "p implies q" may depend on such factors as the relevance of p to q, may be useful for certain purposes, but are much more complex than material implication, and seem not to be necessary for the formalization of mathematics, so we shall not consider them further here.

Note from Figure 1.2 that $[p \supset q]$ is regarded as true when p is false, without regard for q. The appropriateness of this way of defining material implication may be illuminated by supposing that p means "it rains", and q means "the street gets wet", so $[p \supset q]$ means "if it rains, then the street gets wet". If the street is protected from the rain in some way, so that on a certain day it rains, but the street stays dry, then on that day the statement "if it rains, then the street gets wet" is clearly false. However, if the street is not so protected, we would expect the statement "if it rains, then the street gets wet" to be true every day, whether or not it rains that day. In particular, if on a certain day it does not rain but the street gets wet because a pail of water is spilled on it, the statement "if it rains, then the street gets wet" is still true. Similarly, the statement is true on a day when it does not rain and the street remains dry. Thus the only case in which $[p \supset q]$ is false is that in which p is true and q is false.

Equivalence ($\equiv$)

$[p \equiv q]$ means that the statements p and q are equivalent with respect to truth; that is, they are both true or both false.

Figure 1.3 displays truth tables for $[[p \supset q] \wedge [q \supset p]]$ and $[p \equiv q]$. In each line of the truth table, the truth of each component statement is displayed under the main connective of that statement. It is easy to see that in every possible situation, $[[p \supset q] \wedge [q \supset p]]$ and $[p \equiv q]$ are both true or both false; that is, they are equivalent. Thus, p is equivalent to q means that p implies q and that q implies p. This leads to several alternative ways of expressing the assertion that p is equivalent to q which are encountered very frequently in mathematics.

p	q	$[[p \supset q]$	$\wedge$	$[q \supset p]]$	$[p \equiv q]$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	T	F	F	F
F	F	T	T	T	T

Figure 1.3: Truth tables for $[[p \supset q] \wedge [q \supset p]]$ and $[p \equiv q]$

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