

Chapter 5

DUALITY FOR MODAL ALGEBRAS

In this Chapter we shall develop a duality for finitely presented modal algebras in a similar way we have developed a duality for finitely presented Heyting algebras in the previous Chapter. We point out below to some slight differences between these dualities.

The duality for modal algebras has a parameter \mathbf{S} , being an equational theory containing the equational theory of $\mathbf{K4}$ -algebras with the finite model property and (AP) for finite algebras. In varieties of $\mathbf{K4}$ -algebras, the principal congruences do not correspond to elements of algebras, but rather to elements of form $a \wedge \Box a$. This is a source of some additional technical complications. In order to define properly the dual category $\mathbf{M}_{\mathbf{S}}$ we need to use games to define two kinds of relations \sim_n, \approx_n which reflects the difference between arbitrary elements of algebras and those that are of form $a \wedge \Box a$. The first relation serves to define morphisms and the second to define objects in the dual categories. As one could see from an exercise at the end of the previous Chapter the index n in the relation \sim_n was reflecting the implicational degree of the intuitionistic formulas. This time the index is related to the nesting of the necessity connective \Box . The site on which the sheaves are defined in the dual category differ slightly, as well. The frames we consider are the duals of all finite \mathbf{S} -algebras not only of those which are subdirectly irreducible.

In the exercises the reader will find some hints on how one can develop a similar duality for a theory \mathbf{S} without (AP) for finite algebras.

1. Frames, evaluations and games

In this Section we define frames, evaluations and games on evaluations. Games give rise to some equivalence relations. We also study some basic properties of these notions that will be used later.

Recall from Section 2.1 that **K4** denote the first order equational theory in the language $\wedge, \vee, \rightarrow, \neg, 0, 1, \Box$ consisting of the following axioms:

- (i) axioms of Boolean algebras;
- (ii) $\Box 1 = 1$ and $\Box(p \wedge q) = \Box p \wedge \Box q$;
- (iii) $\Box p \leq \Box \Box p$.

The connective \Diamond is defined in the usual way as $\neg \Box \neg$. *Modal formulas* are terms in the above language.

If **S** is an equational theory containing **K4** then by $Alg(\mathbf{S})$ we denote the category of all **S**-algebras (i.e. algebras satisfying the axioms of **S**), and by $Alg(\mathbf{S})_{fin}$ and $Alg(\mathbf{S})_{fp}$ we denote the full subcategories of $Alg(\mathbf{S})$ of finite and finitely presented **S**-algebras, respectively.

If X, Y are sets and $f : X \rightarrow Y$ is a function then Y^X denotes the set of functions from X to Y , and $f^\circ : 2^Y \rightarrow 2^X$ is the dual map of 'composing with f '. The Boolean operations on 2^X are denoted by $\wedge_X, \vee_X, \rightarrow_X, \neg_X, 0_X, 1_X$.

We will use the same letter for Boolean algebras and their universes. If B is a boolean algebra, and together with the operation $\Box : B \rightarrow B$ satisfies the axioms of **K4**, then the corresponding **K4**-algebra will be denoted by (B, \Box) . Recall that by a *frame* we mean a pair (X, R) where X is a finite set and R is a binary relation on X . The relation R is called the *accessibility relation* of the frame (X, R) . If R is transitive, the frame is called *transitive* as well. From now on all frames are assumed to be transitive. The *frame algebra* of (X, R) is the **K4**-algebra $(2^X, \wedge_X, \vee_X, \rightarrow_X, \neg_X, 0_X, 1_X, \Box_R)$ (to be denoted $(2^X, \Box_R)$), where for $v : X \rightarrow 2 \in 2^X$ and $x \in X$, $\Box_R(v)(x) = 1$ iff for all $y \in X$, $x R y$ implies $v(y) = 1$.

By a *morphism of frames* $f : (X, R) \rightarrow (Y, S)$ we mean a function $f : X \rightarrow Y$ which is an open frame map, i.e.

- (i) $x R x'$ implies $f(x) S f(x')$, for $x, x' \in X$;
- (ii) for any $x \in X$ and $y \in Y$, if $f(x) S y$ then there is $x' \in X$ such that $x R x'$ and $f(x') = y$.

It can be easily checked that $f : (X, R) \rightarrow (Y, S)$ is a frame morphism iff the dual map $f^\circ : (2^Y, \Box_S) \rightarrow (2^X, \Box_R)$ is a homomorphism of **S**-algebras.

In this way we have defined a category **F** of finite frames and morphisms of frames. The category **F**, as remarked in Section 2.1 is equivalent to the dual of the category of finite **K4**-algebras. More precisely, we have a functor

$$Alg(\mathbf{K4})_{fin} \longrightarrow \mathbf{F}^{op} \quad (5.1)$$

$$(B, \Box) \longmapsto (at(B), R_\Box)$$

where $at(B)$ is the set of atoms of the boolean algebra B and R_\Box is the binary relation on $at(B)$ defined by $x R_\Box x'$ iff $x \leq \Diamond x'$. The functor acts on morphisms in the obvious way. Moreover, we have a functor

$$\mathbf{F}^{op} \longrightarrow Alg(\mathbf{K4})_{fin} \quad (5.2)$$

$$(X, R) \longmapsto (2^X, \Box_R)$$

associating with a finite frame its frame algebra and with every frame morphism its dual map. They are essential inverses one to the other establishing the mentioned duality $Alg(\mathbf{K4})_{fin} \simeq \mathbf{F}^{op}$. Note that via this duality the empty frame (\emptyset, \emptyset) correspond to the $\mathbf{K4}$ -algebra in which $0 = 1$.

This duality restricts to some subcategories. Let \mathbf{S} be an equational theory containing $\mathbf{K4}$, $\mathbf{F_S}$ be the full subcategory of the category \mathbf{F} corresponding via the above duality to the subcategory $Alg(\mathbf{S})_{fin}$, i.e. we have $Alg(\mathbf{S})_{fin} \simeq \mathbf{F_S}^{op}$. By an \mathbf{S} -frame we mean an object of $\mathbf{F_S}$. For some theories \mathbf{S} , the objects of the category $\mathbf{F_S}$ can be described in a simple way in terms of their accessibility relations. We list below some of them (notice that any equational theory \mathbf{S} containing $\mathbf{K4}$ can be axiomatized by equations of the form $t = 1$, where t is any term, hence it is possible to introduce \mathbf{S} simply by specifying a set of terms, which could be viewed as a set of axioms of a logic):

| Logic \mathbf{S} | Axioms | Description of the accessibility relation |
|--------------------|--|---|
| S4 | $\mathbf{K4} + \Box p \rightarrow p$ | reflexive |
| S4.2 | $\mathbf{S4} + \Diamond \Box p \rightarrow \Box \Diamond p$ | reflexive and locally confluent |
| S4.3 | $\mathbf{S4} + \Box(\Box p_1 \rightarrow p_2) \vee \Box(\Box p_2 \rightarrow p_1)$ | reflexive and locally linear |
| Grz | $\mathbf{S4} + (\Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow p)$ | partial order |
| G | $\mathbf{K4} + (\Box(\Box p \rightarrow p) \rightarrow \Box p)$ | irreflexive |
| S5 | $\mathbf{S4} + \Diamond p \rightarrow \Box \Diamond p$ | equivalence relation |

A frame (X, R) is *locally confluent* iff for any $x, y, z \in X$, if $(xRy$ and $xRz)$ then there is $t \in X$ such that $(yRt$ and $zRt)$. A frame (X, R) is *locally linear* iff for any $x, y, z \in X$, if $(xRy$ and $xRz)$ then $(yRz$ or $zRy)$.

In all the above cases, the class of frame algebras of the finite frames indicated in the third column generates the variety $Alg(\mathbf{S})$, where \mathbf{S} is axiomatized as shown in the second column. The fact that $Alg(\mathbf{S})_{fin}$ generates $Alg(\mathbf{S})$ means that \mathbf{S} has *the finite model property*. The finite model property can be equivalently stated by asking that finitely generated free \mathbf{S} -algebras embed into

products of finite \mathbf{S} -algebras. In view of Proposition 2.19, the same property, if true, extends in our case to finitely presented algebras. In order to establish the finite model property, specific techniques are needed: in Section 2.1 we saw only how to get it quickly for the basic $\mathbf{K4}$ -case; we won't even sketch in the sequel the proofs for other cases (including the cases mentioned in the above table), the reader is referred to [CZ] as an excellent textbook on such questions. In fact, there is no need for our purposes to enter in such (interesting but non trivial) field. For instance, we never need to know that \mathbf{G} -algebras (also called *diagonalizable* algebras) can be axiomatized by the single equation $\Box(\Box p \rightarrow p) \rightarrow \Box p = 1$, we could simply define \mathbf{G} -algebras as the algebras belonging to the variety generated by the class of finite irreflexive frame algebras. Only proof-theoretic (or at least decidability) questions would be sensitive to the existence of nice axiomatizations, but such questions are outside the scope of this book. On the other hand, we shall almost exclusively deal with systems \mathbf{S} having the finite model property, so that all such systems are fully specified once the class of finite \mathbf{S} -frame algebras is given.

By an L -evaluation, or simply *evaluation*, $v : (X, R) \rightarrow L$ we mean a function $v : X \rightarrow L$, where (X, R) is a frame and L is a finite set. Note that if $L = \mathcal{P}(p_1, \dots, p_n)$ then an L -evaluation is nothing but a usual Kripke model which forces modal formulas in variables p_1, \dots, p_n (see the exercises for the description of the forcing relation).

We fix an arbitrary equational theory \mathbf{S} containing $\mathbf{K4}$ for the rest of the section. Consequently, from now on by a frame we mean an \mathbf{S} -frame. Let (X, R) be a frame, $Y \subseteq X$, $S = R \cap (Y \times Y)$. Then (Y, S) is a *generated subframe* of (X, R) iff for any $y \in Y$ and $x \in X$, if yRx then $x \in Y$. We have the following easy Lemma (for further information concerning classes of finite frames see the exercises):

- LEMMA 5.1 (i) *If (Y, S) is a generated subframe of an \mathbf{S} -frame (X, R) then (Y, S) is an \mathbf{S} -frame as well;*
- (ii) *If (X_i, R_i) are \mathbf{S} -frames for $i=1, \dots, n$ then the disjoint sum $\coprod_{i=1}^n (X_i, R_i)$ is an \mathbf{S} -frame;*
- (iii) *If $f : (X, R) \rightarrow (Y, S)$ is a surjective morphism of frames and (X, R) is an \mathbf{S} -frame then (Y, S) is an \mathbf{S} -frame as well.*

We shall define three relations on evaluations in terms of some games. They are variants of the Ehrenfeucht-Fraissé games adopted by K.Fine to the context of modal logic. In fact, we will describe only one game and the others will be obtained as slight modifications of its. Let $v : (X, R) \rightarrow L$ and $u : (Y, S) \rightarrow L$ be two L -evaluations, n be a natural number or ∞ . The n -game on v and u is played by two players, Player I and Player II. In the first move Player I chooses

one frame, either (X, R) or (Y, S) and a node in it. Player II answers by choosing a node in the other frame. After k moves the players already constructed sequences $\langle x_1, \dots, x_k \rangle, \langle y_1, \dots, y_k \rangle$ with $x_i R x_{i+1}, y_i S y_{i+1}$ for every $i < k$. Then in the $(k+1)$ -st move Player I chooses one of the frames, say (X, R) , and a node $x_{k+1} \in X$ such that $x_k R x_{k+1}$. Player II chooses a node in the other frame, say $y_{k+1} \in Y$, such that $y_k S y_{k+1}$. If Player II can't make a move¹ at his turn then he immediately loses. Otherwise, the game terminates after m moves if either Player I can't make a move or $m = n$ (if $n = \infty$ then $m = n$ means that both players plays infinitely many moves). Then Player II wins iff for all natural $k \leq m$, $v(x_k) = u(y_k)$.

We define for $n \geq 1$

- $v \approx_n u$ iff Player II has a winning strategy in n -game on v and u .
- $v \leq_n u$ iff Player II has a winning strategy in n -game on v and u modified so that Player I must play the first move in X .

For $n \geq 0$ and $x \in X, y \in Y$ we define

- $(v, x) \sim_n (u, y)$ iff Player II has a winning strategy in $n+1$ -game on v and u modified so that the first moves of the players must be x and y .

Notice that the relations \leq_n and \approx_n are defined on L -evaluations whereas the relation \sim_n is defined on L -evaluations with a distinguished nodes. The relations \sim_n and \approx_n are clearly equivalence relations, whereas the relation \leq_n is only reflexive and transitive. The equivalence class of a pair (v, x) with respect to the equivalence relation \sim_n will be denoted by $[(v, x)]_n$. The straightforward proposition below provides equivalent definitions of these relations.

PROPOSITION 5.2 *Given two L -evaluations $v : (X, R) \rightarrow L, u : (Y, S) \rightarrow L$, nodes $x \in X, y \in Y$, and $n \in \omega$. We have:*

- (i) $(v, x) \sim_0 (u, y)$ iff $v(x) = u(y)$;
- (ii) $(v, x) \sim_{n+1} (u, y)$ iff $v(x) = u(y)$, and $\forall x' \in X \exists y' \in Y$ (if $x R x'$ then $y S y'$ and $(v, x') \sim_n (u, y')$), and vice versa;
- (iii) $v \leq_{n+1} u$ iff $\forall x \in X \exists y \in Y (v, x) \sim_n (u, y)$;
- (iv) $v \approx_{n+1} u$ iff $v \leq_{n+1} u$ and $v \geq_{n+1} u$.
- (v) $(v, x) \sim_\infty (u, y)$ iff $v(x) = u(y)$, and $\forall x' \in X \exists y' \in Y$ (if $x R x'$ then $y S y'$ and $(v, x') \sim_\infty (u, y')$), and vice versa;

¹Since accessibility relations of our frames need not to be reflexive it may happen that there is no point accessible from a given point.

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