Chapter 1

INTRODUCTION

This book is an example of fruitful interaction between (non-classical) propositional logics and (classical) model theory which was made possible due to categorical logic. Its main aim consists in investigating the existence of model-completions for equational theories arising from propositional logics (such as the theory of Heyting algebras and various kinds of theories related to propositional modal logic). The existence of model-completions turns out to be related to proof-theoretic facts concerning interpretability of second order propositional logic into ordinary propositional logic through the so-called 'Pitts' quantifiers' or 'bisimulation quantifiers'. On the other hand, the book develops a large number of topics concerning the categorical structure of finitely presented algebras, with related applications to propositional logics, both standard (like Beth's theorems) and new (like effectiveness of internal equivalence relations, projectivity and definability of dual connectives such as difference). A special emphasis is put on sheaf representation, showing that much of the nice categorical structure of finitely presented algebras is in fact only a restriction of natural structure in sheaves. Applications to the theory of classifying toposes are also covered, yielding new examples.

The book has to be considered mainly as a research book, reporting recent and often completely new results in the field; we believe it can also be fruitfully used as a complementary book for graduate courses in categorical and algebraic logic, universal algebra, model theory, and non-classical logics.

1. Motivating example

The origin of this work goes back to a surprising Theorem of A.M. Pitts, cf. [Pi2], stating that the second order intuitionistic propositional calculus IpC^2 can be interpreted into ordinary intuitionistic propositional calculus IpC. More precisely, THEOREM 1.1 (A.M. PITTS) *For each propositional variable x and for each formula t of IpC, there exist formulas $\exists^x t$ and $\forall^x t$ of IpC (effectively computable from t) containing only variables not equal to x which occur in t, and such that for any formula u not involving x, we have*

$$\vdash_{IpC} \exists^x t \to u \qquad \textit{iff} \qquad \vdash_{IpC} t \to u$$

and

$$\vdash_{IpC} u \to \forall^x t \qquad \textit{iff} \qquad \vdash_{IpC} u \to t.$$

Although the above result looks like a purely proof-theoretical fact, it can be interpreted model-theoretically in a quite interesting way as a statement about the theory of Heyting algebras. We summarize the main point below. Using the identification of intuitionistic formulas with the terms in the first order theory of Heyting algebras we can characterize semantically the 'Pitts' quantifiers' $\exists^x$ and $\forall^x$, as follows. For a formula $t(\vec{y}, x)$ of IpC, and a tuple of elements $\vec{a}$ from a Heyting algebra H, we have that

$$H \models (\exists^x t)(\vec{a}) = 1 \quad \text{iff} \quad H[\mathbf{x}]/t(\vec{a}, \mathbf{x}) \text{ is an extension of } H$$

where $H[\mathbf{x}]/t(\vec{a}, \mathbf{x})$ is the Heyting algebra of polynomials $H[\mathbf{x}]$ divided by the congruence generated by the condition $t(\vec{a}, \mathbf{x}) = 1$. Moreover

$$H \models (\forall^x t)(\vec{a}) = 1 \quad \text{iff} \quad H[\mathbf{x}] \models t(\vec{a}, \mathbf{x}) = 1.$$

The proof of these characterizations easily follows from Pitts' Theorem using any presentation for H.

This explanation of Pitts' quantifiers in terms of Heyting algebras can be used in order to show that the first order theory of Heyting algebras admits a model completion. In fact, it turns out that the system of equations and inequations with parameters $\vec{a}$ from H

$$t(\vec{a}, x) = 1 \;\&\; u_1(\vec{a}, x) \neq 1 \;\&\; \ldots \;\&\; u_m(\vec{a}, x) \neq 1 \tag{1.1}$$

is solvable in an extension of H iff the quantifier-free formula

$$(\exists^x t)(\vec{a}) = 1 \;\&\; (\forall^x (t \to u_1))(\vec{a}) \neq 1 \;\&\; \ldots$$

$$\ldots \;\&\; (\forall^x (t \to u_m))(\vec{a}) \neq 1 \tag{1.2}$$

is true in H. If the formula (1.2) is true, we can take $H[\mathbf{x}]/t(\vec{a}, \mathbf{x})$ as an extension of H in which the system (1.1) has a solution. Conversely, if the system (1.1) is solvable in an extension H', then we have a factorization

$$\begin{tikzcd} & H \arrow[dl] \arrow[dr] & \\ H[\mathbf{x}]/t(\vec{a},\mathbf{x}) \arrow[rr] & & H' \end{tikzcd}$$

showing that $H[\mathbf{x}]/t(\vec{a},\mathbf{x})$ is an extension of H in which the system (1.1) has the solution $x = \mathbf{x}$. This, together with the above characterization of Pitts' quantifiers, shows that formula (1.2) is true in H. Thus the class of existentially closed Heyting algebras is an elementary class and, as the above quantifier-elimination procedure is effective, it can easily be shown that the related first order theory is decidable. In Section 4.7, we shall provide examples of this decision procedure, together with a list of some basic properties of existentially closed Heyting algebras.

In this way Pitts' Theorem implies that the first order theory of Heyting algebras admits a model completion. The interesting point is that the converse is also true, in a quite general setting. In order to explain what we mean by this, we need a category-theoretic formulation of Pitts' Theorem. In this equivalent formulation, Theorem 1.1 just says that the opposite of the category of finitely presented Heyting algebras is a *Heyting category*.

The notion of Heyting category ([MR1], [MR2] or logos in [Pi1]) is a quite standard notion in categorical logic: Heyting categories are just 'Lindenbaum categories' for many-sorted intuitionistic first-order theories. A Heyting category is a category with finite limits in which finite joins, images and dual images among subobjects exist and are pullback-stable. Such a structure is needed in order to interpret first-order intuitionistic logic: terms are interpreted as arrows, formulas as subobjects and images and dual images along projections correspond to quantifiers. With each first-order many sorted intuitionistic theory, a Heyting category, built up in a completely syntactic way, can be associated: objects are formulas, arrows are equivalence classes (with respect to provable equivalence) of formulas which are provably functional in the restricted domains given by the source and the target of the arrow they define. Conversely, with each Heyting category, a first-order many sorted intuitionistic theory can be associated: we have one sort for each object, one term for each arrow, no relation symbols, and, as axioms, all the formulas which are 'internally true' in the given Heyting category. The two inverse passages are bijective, modulo the standard notion of equivalence between categories and modulo some natural notion of equivalence between theories.

Thus, using this category-theoretic formulation of Pitts' Theorem, we can say that the fact that HA_{fp}^{op}, i.e. the opposite to the category of finitely presented Heyting algebras, has enough categorical structure to classify internally a first-order intuitionistic theory implies (and actually it is equivalent to, see below) the fact that the first-order theory of Heyting algebras admits a model completion.

This is a rather interesting kind of connection: it says that the existence of a *classical* theory (the model completion) is equivalent to the existence of a suitable *intuitionistic* theory. Notice that the connection is not completely trivial, in the sense that it can be shown that the first-order intuitionistic theory classified by HA_{fp}^{op} is a theory speaking about Heyting algebras, but it differs considerably from the model completion of the theory of Heyting algebras. The two theories are indeed almost contradictory, for instance the statement

$$\forall x \forall y \; (x \vee y = 1 \Rightarrow (x = 1 \text{ or } y = 1))$$

is false in any existentially closed (non degenerate) Heyting algebra, but it is true in the theory classified by the opposite to the category of finitely presented Heyting algebras.

2. An overview of the book

We describe here the main *strategy* of the book. In Chapter 3 there is the proof of a theorem which generalizes the above observations for Heyting algebras. We take into consideration an arbitrary equational theory T satisfying a certain assumption (see next Section) which is rather strong in general, but which is often satisfied in varieties of algebras arising from logic. Under this assumption, we prove (Theorem 3.11) that

T admits a model completion iff $\mathbf{T}$ is an r-Heyting category,

where $\mathbf{T}$ is the opposite of the category $Alg(T)_{fp}$ of finitely presented T-algebras. In other words T admits a model completion iff the category $\mathbf{T}$ derived from T has some nice categorical structure.

The notion of r-Heyting category is obtained from the notion of Heyting category by replacing 'subobject' by 'regular subobject' everywhere in the definition. This modification is due to the fact that we prefer not to assume that monos are all regular in $\mathbf{T}$ (i.e. that epis are quotients in $Alg(T)_{fp}$), an assumption which holds for Heyting algebras as a consequence of the Beth property (BP), cf. Theorem 2.14, but which may fail in other cases.

In the following three Chapters, we apply Theorem 3.11 to two kinds of varieties of algebras: Heyting algebras and modal algebras. In both cases we adopt a similar strategy. Theorem 3.11 says that, under suitable assumptions, the existence of a model completion for T is equivalent to the existence of a certain categorical structure in $\mathbf{T}$. Usually it is not easy to decide directly whether $\mathbf{T}$ is an r-Heyting category. But, as this is a purely categorical property, we can study it in any category equivalent to $\mathbf{T}$. The strategy we adopt for an equational theory T can be summarized in the following four steps:

1 *Embedding.* Find an r-Heyting category $\mathcal{E}$ and an embedding

$$\mathbf{\Phi_T} : Alg(T)^{op}_{fp} \longrightarrow \mathcal{E}$$

which is conservative, preserves finite limits and all the other r-Heyting category structure that exists in $Alg(T)^{op}_{fp}$.

Conservativity ensures that the operations that can be performed in $Alg(T)^{op}_{fp}$ and are preserved by $\mathbf{\Phi_T}$ satisfy automatically any exactness properties that these operations satisfy in $\mathcal{E}$. In particular the operations of left ($\exists_f$) and right ($\forall_f$) adjoint to the pullback functors f^* (operating on regular subobjects, see Section 2.3) in $Alg(T)^{op}_{fp}$, if they exist, they automatically satisfy the Beck-Chevalley condition.

In the applications the category $\mathcal{E}$ is (equivalent to) the category of sheaves on the opposite of the category of finite T-algebras with the canonical topology.

2 *Duality.* Identify the image of $\mathbf{\Phi_T}$ in $\mathcal{E}$, i.e. describe in a convenient way a subcategory $\mathbf{M_T}$ of $\mathcal{E}$ so that we have a factorization of $\mathbf{\Phi_T}$

$$\begin{tikzcd}
Alg(T)^{op}_{fp} \arrow[rr, "\mathbf{\Phi_T}"] \arrow[dr] & & \mathcal{E} \\
& \mathbf{M_T} \arrow[ur, "\mathbf{\Psi_T}"'] &
\end{tikzcd}$$

with the first component being an equivalence of categories and $\mathbf{\Psi_T}$ being an inclusion.

In the applications this point is slightly reversed. It is usually more natural to define a 'duality' functor in the opposite direction, i.e. $\mathbf{M_T} \longrightarrow Alg(T)^{op}_{fp}$.

3 *Combinatorial condition for existence of adjoints.* Now the existence of the adjoints is reduced to the verification whether the existing adjoints in $\mathcal{E}$ when applied to objects coming from $Alg(T)^{op}_{fp}$ give objects coming from $Alg(T)^{op}_{fp}$, as well.

In applications, with the help of an appropriate description of $\mathbf{M_T}$, this can be reduced to an equivalent condition of a combinatorial nature, expressed in terms of Ehrenfeucht-Fraissé games on finite Kripke models.

4 *Verification of combinatorial conditions.* Last, but not least, the combinatorial conditions should be verified to establish whether the adjoints do exist, if they do $Alg(T)^{op}_{fp}$ is an r-Heyting category.

We believe that this method is general and can be applied in other similar contexts.

http://www.springer.com/978-1-4020-0660-9

Sheaves, Games, and Model Completions
A Categorical Approach to Nonclassical Propositional Logics
Ghilardi, S.; Zawadowski, M.
2002, IX, 245 p., Hardcover
ISBN: 978-1-4020-0660-9