Chapter 2

PRELIMINARY NOTIONS

1. Basic algebraic structures

In this section we recall the main algebraic structures which will be investigated within the book. They are structures that provides an algebraic semantics for propositional logics. They are usually obtained by enriching posets by some algebraic operations. We are mainly interested in Heyting and modal algebras i.e. those algebras that provide counterparts of superintuitionistic and modal logics.

A partially ordered set (*poset*, for short) is a set P equipped with a reflexive, transitive and antisymmetric binary relation $\leq$. For such a poset, the *infimum* (resp. *supremum*) of a family $\{a_i\}_{i\in I}$ of elements of P is an element (it may or may not exists, but if it exists it is unique) $\bigwedge_i a_i \in P$ (resp. $\bigvee_i a_i \in P$) such that for all $b \in P$, we have

$$(\forall i \in I\; b \leq a_i) \qquad \text{iff} \qquad b \leq \bigwedge_i a_i$$

(or

$$(\forall i \in I\; a_i \leq b) \qquad \text{iff} \qquad \bigvee_i a_i \leq b)$$

respectively). In case the index I is empty, the above conditions say that the infimum of the empty set is the maximum element of P and the supremum of the empty set is just the minimum.

We recall some facts about *adjoints among posets*, although they can be deduced from the general results about categories given in the Appendix, it is worth having a direct knowledge of what happens in this special case. The right adjoint f_* (resp. left adjoint f^*) to an order-preserving map $f : P \to Q$ among posets, is an order-preserving map in the opposite direction, satisfying

$$f(a) \leq b \qquad \text{iff} \qquad a \leq f_*(b)$$

(or

$$b \leq f(a) \qquad \text{iff} \qquad f^*(b) \leq a$$

respectively) for all $a \in P, b \in Q$. Such a right (left) adjoint may not exists, but if it exists it is unique. It is easily seen that left adjoints preserve existing suprema and right adjoints existing infima: the latter, for instance, is shown by an easy chain of equivalences as follows

$$\frac{\frac{\frac{\frac{a \leq f_*(\bigwedge_i b_i)}{f(a) \leq \bigwedge_i b_i}}{\forall i\ f(a) \leq b_i}}{\forall i\ a \leq f_*(b_i)}}{a \leq \bigwedge_i f_*(b_i)}$$

yieldying $f_*(\bigwedge_i b_i) = \bigwedge_i f_*(b_i)$ as a is arbitrary. If P is complete (i.e. iff all suprema -or equivalently all infima- exist), then any order-preserving map $f : P \to Q$ has a right adjoint iff it preserves suprema and has a left adjoint iff it preserves infima. Such adjoints are easily seen to be given by the following formulas:

$$f_*(b) = \bigvee_{f(a) \leq b} a \qquad f^*(b) = \bigwedge_{b \leq f(a)} a$$

for all $b \in Q$.

A (meet) *semilattice* is a commutative idempotent monoid, i.e. a structure $(M, \wedge, \top)$ satisfying the equations

$$a \wedge b = b \wedge a, \quad a \wedge \top = a, \quad a \wedge a = a, \quad a \wedge (b \wedge c) = (a \wedge b) \wedge c \quad (2.1)$$

for all $a, b, c \in M$. Putting

$$a \leq b \qquad \text{iff} \qquad a \wedge b = a$$

we can define a partial order in any semilattice; the operation $\wedge$ turns out to be the infimum (also called *meet*) of the pair $\{a, b\}$ and $\top$ turns out to be the maximum element. In fact, one can equivalently define a semilattice as a partially ordered set in which infima exist for all finite sets of elements (this includes the maximum element, seen as the infimum over the empty set).

Many important further operations can be characterized with respect to the partial order so introduced: in order to obtain the notion of a *lattice*[1] one simply has to require that also suprema (called *joins* as well) exist for all finite sets; equivalently, a lattice is a semilattice with another binary operation $\vee$

[1]Notice that we always require the presence of $\bot$ and $\top$ in a lattice (this is different from some common literature). Sometimes, we also use the notation $\mathbf{1}$ for $\top$ and $\mathbf{0}$ for $\bot$.

and another constant $\bot$ satisfying equations (2.1) (with $\wedge, \top$ replaced by $\vee, \bot$ respectively) and moreover the following absorption laws

$$a \wedge (a \vee b) = a, \quad a \vee (a \wedge b) = a.$$

A lattice is said to be *distributive* iff it satisfies one of the two (equivalent) equations

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

In a given semilattice M it may happens that for $a, b \in M$ the supremum of the set $\{c \,|\, a \wedge c \leq b\}$ exists; such an element is called the *relative pseudocomplement* of a relative to b (or the *implication* of a and b, using logical terminology) and is written as $a \to b$. Otherwise said, $a \to b$, if it exists, it is the unique element satisfying the condition

$$a \wedge c \leq b \qquad \text{iff} \qquad c \leq a \to b$$

for all c. A *Brouwerian semilattice* is a semilattice in which all implications among pairs of elements exist and a *Heyting algebra* is a Brouwerian semilattice which is also a distributive lattice. Brouwerian semilattices (hence also Heyting algebras) may be equivalently introduced for instance through the equations

$$\begin{array}{cc} a \wedge (a \to b) = a \wedge b & b \wedge (a \to b) = b \\ a \to (b \wedge c) = (a \to b) \wedge (a \to c) & a \to a = \top. \end{array}$$

This shows that Heyting algebras form an equational class, i.e. a *variety*. An important example of a Heyting algebra is given by the open sets of a topological space; here the partial order is inclusion, (finite) meets and joins are intersections and unions, whereas implication of the open subsets a and b is the interior of $a' \cup b$ (where a' is the complement of a). The most important example for us is given by the downward closed subsets $\mathcal{D}(P)$ of a poset P ($a \subseteq P$ is downward closed iff $p \in a$ and $q \leq p$ imply $q \in a$): here the partial order, joins and meets are again inclusion, intersections and unions, respectively, whereas the implication of a and b is

$$a \to b \;=\; \{p \in P \,|\, \forall q \leq p \; (q \in a \Rightarrow q \in b)\}.$$

A finite distributive lattice is always a Heyting algebra, because a finite distributive lattice is complete and, thanks to distributivity, for any element a, the order preserving map $a \wedge (-)$ preserves suprema, so that it has a right adjoint $a \to (-)$. For the same reason, a finite Brouwerian semilattice is always a Heyting algebra: in fact joins exists and are distributive as $a \wedge (-)$ preserves them (being a left adjoint).

In a Heyting algebra H, *negation* is introduced through

$$\neg a = a \to \bot;$$

such operation satisfies many usual laws, but not all the classical ones (for instance, only three of the four De Morgan identities hold). A *Boolean algebra* is a Heyting algebra in which we have $\neg\neg a = a$ (or, equivalently, $a \vee \neg a = \top$) for all a.

In a distributive lattice lattice D certain elements play a special role, they are the join-irreducible ones. We say that a is *join-irreducible* iff for all $n \geq 0, b_1, \dots, b_n \in D$, we have

$$\text{if } a \leq b_1 \vee \cdots \vee b_n \text{ then for some } 1 \leq i \leq n, \quad a \leq b_i$$

(notice that join-irreducible elements are non-zero, i.e. different from $\bot$, by taking $n = 0$ in the above definition). In distributive lattices of the kind $\mathcal{D}(P)$, where P is a finite poset, join-irreducible elements are those of the kind $\downarrow p$ for $p \in P$ (here $\downarrow p$ is $\{q \in P \mid q \leq p\}$). In a Boolean algebra B, join-irreducible elements are called *atoms* and turn out to be just the minimal non-zero elements. A Boolean algebra B may have no atoms (in this case we say that it is *atomless*), or, at the extreme opposite, it may happen that for any non-zero $b \in B$ there is an atom $a \leq b$ (in this case, we say that B is *atomic*).

Distributive lattices and Boolean algebras (also Brouwerian semilattices, but we shall not use this further result) are *locally finite varieties*, namely varieties in which finitely generated algebras are finite; this is easily seen, e.g. in the case of Boolean algebras, from the fact that if the set G generates the algebra B, then every element of B admits a representation of the kind $\bigwedge_i \bigvee_j x_{ij}$ where i, j range over finite sets of indices and where x_{ij} is either g or $\neg g$ for some $g \in G$. This is not true for Heyting algebras since the free Heyting algebra on one generator is infinite.

Modal algebras are just Boolean algebras endowed with a further unary 'necessity' operator $\Box$ satisfying the conditions:

$$\Box(a \wedge b) = \Box a \wedge \Box b \qquad \Box\top = \top;$$

the 'possibility' operator $\Diamond$ is introduced in any modal algebra through the definition $\Diamond a = \neg\Box\neg a$. We shall mainly deal with **K4**-*algebras*, i.e. modal algebras in which the operator $\Box$ satisfies the further axiom

$$\Box a \leq \Box\Box a.$$

We shall meet in the book many interesting varieties of $K4$-algebras; for the moment let us only mention **S4**-*algebras* (or interior algebras or topological Boolean algebras), which are characterized by the further axiom

$$\Box a \leq a.$$

The main examples of modal algebras we are interested in are obtained through frames. A *frame* is a pair (X, R), where X is a set endowed with a relation (the *accessibility* relation of the frame); a *transitive frame* is a frame in which R is assumed also to be transitive and a *preordered frame* is a transitive frame in which R is also reflexive. Given a frame (resp. a transitive frame, a preordered frame) (X, R), we can turn $\mathcal{P}(X)$ into a modal algebra (resp. into a $\mathbf{K4}$-algebra, into an $\mathbf{S4}$-algebra) by putting, for $a \subseteq X$

$$\Box_R a \;=\; \{p \in X \mid \forall q \in X(pRq \Rightarrow q \in a)\};$$

the corresponding definition of the possibility operator is

$$\Diamond_R a \;=\; \{p \in X \mid \exists q \in X(pRq \;\&\; q \in a)\}.$$

Let us mention how to describe quotients in Heyting and $K4$-algebras. The central notion to this respect is the notion of a *filter* F, which makes sense at the level of a semilattice R (although it becomes fully operative only when there are implications): this is a subset of R satisfying the following requirements

- $\top \in F$;
- if $a_1, a_2 \in F$, then $a_1 \wedge a_2 \in F$;
- if $a_1 \in F$ and $a_1 \leq a_2$, then $a_2 \in F$.

Given a subset $S \subseteq R$, there exists the minimum filter $[S]$ containing S, which is given by

$$[S] \;=\; \{b \in R \mid \exists n \geq 0, \exists a_1, \ldots, a_n \in S \text{ s.t. } a_1 \wedge \cdots \wedge a_n \leq b\}.$$

In particular, the minimum (or *principal*) filter containing an element a is just $[a] = \{b \mid a \leq b\}$. For the case of modal algebras, the relevant notion is the notion of *modal filter*, which is an ordinary filter satisfying the further condition

- if $a \in F$, then $\Box a \in F$.

A formula for the minimum modal filter $[S]_m$ containing a set S can be easily given; it simplifies considerably for the case of $K4$-algebras where we have

$$[S] \;=\; \{b \in R \mid \exists n \geq 0, \exists a_1, \ldots, a_n \in S \text{ s.t. } \Box^+ a_1 \wedge \cdots \wedge \Box^+ a_n \leq b\}$$

(here $\Box^+ a_i$ stands for $a_i \wedge \Box a_i$). Consequently, the principal modal filter corresponding to an element a is just $[a]_m = [\Box^+ a]$.

In Heyting algebras, the lattice of filters and the lattice of congruences are isomorphic; given a congruence $\simeq$, we can associate to it the filter $\{a \mid a \simeq \top\}$ and given a filter F we can associate to it the congruence $a \simeq b$ iff $a \leftrightarrow b \in F$

Sheaves, Games, and Model Completions
A Categorical Approach to Nonclassical Propositional Logics
Ghilardi, S.; Zawadowski, M.
2002, IX, 245 p., Hardcover
ISBN: 978-1-4020-0660-9