Chapter 3

MODEL COMPLETIONS

1. r-Heyting categories

In this section we introduce the notions of r-regular and r-Heyting categories and study some of their basic properties. Roughly speaking, these notions are obtained from the extensively studied notions of regular and Heyting category (see e.g. [MR1], [MR2]]) 'by replacing monos with regular monos and regular epis by epis'. In case all subobjects are regular, the two notions coincide (this is evident from Proposition 3.3 below), so, for instance, any topos is r-regular and also r-Heyting. In case not all monos are regular, the two concepts are quite distinct: posets and order-preserving maps, for instance, form an r-Heyting category which is not even regular. As we saw in Proposition 2.14, regularity of monos in the opposite of the category of finitely presented algebras follows from some appropriate version of Beth theorem, which is often true (e.g. it holds in all varieties of Heyting and of **K4**-algebras, see [Ma5] and Section 5.6 below). Up to some extent, the theory of r-regular and r-Heyting categories goes parallel to that of regular and Heyting categories: some of the properties established in this section, for instance, are obtained through adaptations of standard arguments.

The best way to introduce r-regular categories is probably through stable factorization systems. Given a category **C**, a pair of classes of arrows $\langle \mathcal{E}, \mathcal{M} \rangle$ is said to be a *stable factorization system* for **C** iff the following four conditions are satisfied (see [FK], but we follow the equivalent formulation of [CJKP]):

(i) both $\mathcal{E}$ and $\mathcal{M}$ contain identities and are closed under left and right composition with isomorphisms;

(ii) each map f in **C** can be written as $m \circ e$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$;

(iii) whenever we have a commutative square,

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "u"'] & B \arrow[d, "v"] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$

with $m \in \mathcal{M}$ and $e \in \mathcal{E}$, there is a unique $w : B \longrightarrow C$ such that $w \circ e = u$ and $m \circ w = v$;

(iv) whenever we have a pullback square

$$\begin{tikzcd} A \arrow[r, "u'"] \arrow[d, "e'"'] & B \arrow[d, "e"] \\ C \arrow[r, "u"'] & D \end{tikzcd}$$

the fact that $e \in \mathcal{E}$ implies that $e' \in \mathcal{E}$.

The decomposition in (ii) is said to be a *factorization* for f; this factorization is unique in the sense that if $f = m \circ e$ can be factored as well as $m' \circ e'$, for $m' \in \mathcal{M}$ and $e' \in \mathcal{E}$, then using (iii), it can be shown that there is an invertible map w such that $w \circ e = e'$ and $m' \circ w = m$. In a factorization system, both $\mathcal{E}$ and $\mathcal{M}$ are closed under composition [CJKP]: let us recall how to show it for $\mathcal{M}$ (for $\mathcal{E}$ the proof is analogous). It is sufficient to have a characterization of arrows in $\mathcal{M}$, from which the desired property easily follows. So let us prove that for all $f : C \to D$, f belongs to $\mathcal{M}$ iff f is *orthogonal* to $\mathcal{E}$, i.e. iff the following condition is satisfied:

- 'for every $e \in \mathcal{E}$ and for every commutative square

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "u"'] & B \arrow[d, "v"] \\ C \arrow[r, "f"'] & D \end{tikzcd}$$

there is a unique $w : B \longrightarrow C$ such that $w \circ e = u$ and $f \circ w = v$.'

One side is just (iii); for the other side, take a factorization $m \circ e$ of f and consider the square

$$\begin{tikzcd} C \arrow[r, "e"] \arrow[d, "id_C"'] & D' \arrow[d, "m"] \\ C \arrow[r, "f"'] & D \end{tikzcd}$$

to get w such that $w \circ e = id_C$. As $e \circ w$ and $id_{D'}$, both fills the diagonal of the square

$$\begin{tikzcd} C \arrow[r, "e"] \arrow[d, "e"'] & D' \arrow[d, "m"] \\ D' \arrow[r, "m"'] & D \end{tikzcd}$$

they are equal by (iii), so e is iso and $f \in \mathcal{M}$ by (i).

Having established that an arrow belongs to $\mathcal{M}$ iff it is orthogonal to $\mathcal{E}$, it is not difficult to see that if $C \xrightarrow{m_1} D \xrightarrow{m_2} E$ are both orthogonal to $\mathcal{E}$, so is $m_2 \circ m_1$.

We say that a category $\mathbf{C}$ is *r-regular* iff it has finite limits and moreover taking $\mathcal{E}$ =all epis and $\mathcal{M}$ =all regular monos, we get a stable factorization system for $\mathbf{C}$. As conditions (i) and (iii) are trivially true in this case (by the definition of epi and regular mono), $\mathbf{C}$ is r-regular iff it has finite limits, each arrow has an epi/regular mono factorization and epis are stable under pullbacks.

PROPOSITION 3.1 *If $\mathbf{C}$ is r-regular, then the pullback functors operating on regular subobjects have left adjoints satisfying the Beck-Chevalley condition.*

Proof. The statement of the Proposition says that for every arrow $f : B \longrightarrow A$ in $\mathbf{C}$, for every regular subobject $S \hookrightarrow B$, there is a regular subobject $\exists_f(S) \hookrightarrow A$ satisfying the condition

$$\exists_f(S) \leq T \quad \text{iff} \quad S \leq f^*(T)$$

for every regular subobject $T \hookrightarrow A$. The Beck-Chevalley condition says that for every pullback square

$$\begin{tikzcd} C \arrow[r, "p_1"] \arrow[d, "p_2"'] & B_1 \arrow[d, "f_1"] \\ B_2 \arrow[r, "f_2"'] & A \end{tikzcd} \tag{3.1}$$

and for every regular subobject $S \hookrightarrow B_1$, the equation

$$f_2^*(\exists_{f_1}(S)) = \exists_{p_2}(p_1^*(S)) \tag{3.2}$$

holds in $Sub_r(B_2)$.

Let us fix a morphism $f : B \to A$ and a regular mono $s : S \hookrightarrow B$ in $\mathbf{C}$. We take as $\exists_f(S)$ the second component of the factorization of $f \circ s$. We can form a commuting diagram

$$\begin{tikzcd}
 & & \exists_f(S) \arrow[drr, "m"] \arrow[ddr] & & \\
S \arrow[urr, "e"] \arrow[r, "s"'] \arrow[dr] & B \arrow[rrr, "f"'] & & & A \\
 & f^*(T) \arrow[u, "t'"'] \arrow[rr, "f'"'] & & T \arrow[ur, "t"'] &
\end{tikzcd} \tag{3.3}$$

of all the named arrows; the arrows without names might not exist, but if they do, they are unique making the obvious shapes to commute, ($f^*(T)$ is a pullback of t along f). Now if $\exists_f(S) \leq T$, i.e. if $\exists_f(S) \to T$ exists in (3.3), then the two arrows

$$S \xrightarrow{s} B \xrightarrow{f} A \qquad S \xrightarrow{e} \exists_f(S) \longrightarrow T \xrightarrow{t} A$$

are equal, hence by the universal property of the pullback, $S \to f^*(T)$ exists in (3.3), showing that $S \leq f^*(T)$.

On the other hand, if $S \leq f^*(T)$ i.e. $S \to f^*(T)$ exists in (3.3), then the outer penthagon in (3.3) commutes, and $\exists_f(S) \to T$ exists, by the property (iii) of the definition of a stable factorization system, as e is an epi, and t is a regular mono.

It remains to show the Beck-Chevalley condition. Let us consider a pullback square (3.1) and let $S \overset{s}{\hookrightarrow} B_1$ be a regular mono. Take the further pullback

$$\begin{tikzcd}
p_1^*(S) \arrow[r] \arrow[d, "s'"'] & S \arrow[d, "s"] \\
C \arrow[r, "p_1"'] & B_1
\end{tikzcd}$$

What we have to show is that the factorization of $p_2 \circ s'$ is just (up to an isomorphism) the factorization of $f_1 \circ s$ pulled back along f_2. But it is a general property of stable factorization systems that in a pullback square

$$\begin{tikzcd} Z \arrow[r, "q_1"] \arrow[d, "q_2"'] & Y_1 \arrow[d, "g_1"] \\ Y_2 \arrow[r, "g_2"'] & X \end{tikzcd}$$

the factorization of q_2 is obtained by taking the factorization $m \circ e$ of g_1 and by successively pulling back m and e: this property is essentially due to condition (iv) ensuring that members of $\mathcal{E}$ are pullback-stable, whereas pullback-stability of members of $\mathcal{M}$ follows from conditions (i)-(iii) [CJKP] (in our case, anyway, stability of regular monos under pullbacks is a general fact). □

We shall reverse Proposition 3.1 in order to get an alternative definition of r-regular category. First, we prove a Lemma:

LEMMA 3.2 *Let* **C** *a category with finite limits, and* $f : B \to A$ *a morphism in* **C**. *Then*

(i) f *is epi iff for every regular subobject* $S \hookrightarrow A$*, we have that* $id_B \leq f^*(S)$ *implies* $id_A \leq S$*;*

(ii) *if, moreover, the pullback functor* $f^* : Sub_r(A) \to Sub_r(B)$ *has a left adjoint* $\exists_f : Sub_r(B) \to Sub_r(A)$*, then* f *is epi iff* $id_A \leq \exists_f(id_B)$.

Proof. Ad (i). Let $f : B \to A$ be an epi and let $S \hookrightarrow A$ be a regular mono such that $id_B \leq f^*(S)$; this means that we have a pullback square

$$\begin{tikzcd} B \arrow[r] \arrow[d, "id_B"'] & S \arrow[d] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

so $S \hookrightarrow A$ is epi as a second component of an epic arrow. Being also a regular mono, it is an isomorphism. Vice versa, suppose that $id_B \leq f^*(S)$ implies $id_A \leq S$ for all $S \in Sub_r(A)$ and suppose that $g_1 \circ f = g_2 \circ f$ for some parallel arrows of domain A. Let $S \stackrel{s}{\hookrightarrow} A$ be the equalizer of g_1, g_2; by the universal property of equalizers, we have a unique map $f' : B \to S$ such that $s \circ f' = f$. As s is mono, it turns out that the diagram http://www.springer.com/978-1-4020-0660-9

Sheaves, Games, and Model Completions
A Categorical Approach to Nonclassical Propositional Logics
Ghilardi, S.; Zawadowski, M.
2002, IX, 245 p., Hardcover
ISBN: 978-1-4020-0660-9