

## Chapter 4

# HEYTING ALGEBRAS

We shall develop a duality for the category finitely presented Heyting algebras  $HA_{fp}$ . Using a combinatorial description  $M_H$  of  $HA_{fp}^{op}$  we shall prove that it is a Heyting category and hence, according to Theorem 3.8, the theory of Heyting algebras  $T_H$  admits a model completion  $T_H^*$ . Then we shall study some further properties of  $HA_{fp}^{op}$  and we shall derive some conclusions from these studies for intuitionistic propositional logic  $IpC$ . We introduce a sheaf semantics for second order logic and show how to use it to eliminate quantifiers in  $T_H^*$ .

### 1. Basic definitions

In this Chapter, we shall mainly deal with finite posets, to be indicated with the letters  $P, Q, \dots$ ; their elements will be usually written as  $p, q, \dots$  and their ordering (reflexive, transitive and antisymmetric) relations will be written simply as  $\leq$ , leaving the more explicit notation  $\leq_P, \leq_Q, \dots$  for contexts requiring such further specification. A poset  $P$  is called *rooted* iff it has a greatest element  $\rho(P)$  (sometimes we indicate it simply as  $\rho$  instead of  $\rho(P)$ ). If a basic finite poset  $L$  (the poset of ‘labels’) is fixed, we call an *L-evaluation* or simply an *evaluation* a pair  $\langle P, u \rangle$ , where  $P$  is a rooted finite poset and  $u : P \rightarrow L$  is an order-preserving map. This notion has a strict relation with finite Kripke models. In fact if  $\langle L, \leq \rangle$  is  $\langle \mathcal{P}(\vec{p}), \supseteq \rangle$  (where  $\vec{p}$  is a finite list of propositional letters), then an *L-evaluation*  $u : P \rightarrow L$  is the same as a Kripke model for the propositional intuitionistic language built up from  $\vec{p}$ .<sup>1</sup>

We define for every  $n \in \omega$  and for every pair of *L-evaluations*  $u : P \rightarrow L$  and  $v : Q \rightarrow L$ , the notions of being *n-equivalent* (written  $u \sim_n v$ ) and of

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<sup>1</sup> According to our conventions, we have that (for  $p, q \in P$ ) if  $p \leq q$  then  $u(p) \supseteq u(q)$ , that is we use  $\leq$  where standard literature uses  $\geq$ .

being *n-less than or equal to* (written  $u \leq_n v$ ). These notions are motivated by the fact (implicit in what is proved in the next section) that in the case of Kripke models being *n-equivalent* means exactly to satisfy the same formulas up to implicational degree *n*, see Exercise 1. Similarly,  $u \leq_n v$  means that *u* satisfies all formulas up to implicational degree *n* that *v* satisfies. We define also for two *L*-evaluations *u, v* the notions of being *infinitely equivalent* (written  $u \sim_\infty v$ ) and *infinitely less than or equal to* (written  $u \leq_\infty v$ ). All these notions are parallel to the analogous definitions introduced in [Fi1], [Fi2] for modal logic, see Chapter 5. We prefer to introduce them by means of Ehrenfeucht-Fraissé games.

Let  $u : P \rightarrow L$  and  $v : Q \rightarrow L$  be two *L*-evaluations. The *game* we are interested in has two players, player I and player II. Player I can choose either a point in *P* or a point in *Q* and player II must answer by choosing a point in the other poset and the only rule is that, if  $\langle p \in P, q \in Q \rangle$  is the last move played, then in the successive move the two players can only choose points  $\langle p', q' \rangle$  such that  $p' \leq p$  and  $q' \leq q$ . If  $\langle p_1, q_1 \rangle, \dots, \langle p_i, q_i \rangle, \dots$  are the points chosen at the end of the game, after infinitely many moves, player II wins iff for every  $i = 1, 2, \dots$ , we have that  $u(p_i) = v(q_i)$ . We say that

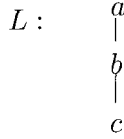
- $u \sim_\infty v$  iff *player II has a winning strategy*;
- $u \sim_n v$  (for  $n > 0$ ) iff *player II has a winning strategy for the first n-moves*, i.e. he has a winning strategy provided we stipulate that the game terminates after *n* moves;
- $u \sim_0 v$  iff  $u(\rho(P)) = v(\rho(Q))$ ;
- $u \leq_\infty v$  iff *player II has a winning strategy in the modified game*, where the word ‘modified’ refers to the fact that we have an additional rule forcing player I to play in the domain of *u* the first move;
- $u \leq_n v$  (for  $n > 0$ ) iff *player II has a winning strategy for the first n moves in the modified game*;
- $u \leq_0 v$  iff  $u(\rho(P)) \leq v(\rho(Q))$ .

The relations  $\sim_n$  and  $\sim_\infty$  are clearly equivalence relations, whereas the relations  $\leq_n$  and  $\leq_\infty$  are only reflexive and transitive. Notice also that for every *n*,  $u \sim_n v$  implies  $u \sim_0 v$  because  $\leq_L$  is antisymmetric. The straightforward proposition below (to be often used without explicit mention in the following) provides equivalent definitions. We fix a notation: if  $u : P \rightarrow L$  is an *L*-evaluation and  $p \in P$ ,  $u_p$  is *u* restricted in the domain to the downward closed subset  $\downarrow p = \{p' : p' \leq p\}$ .

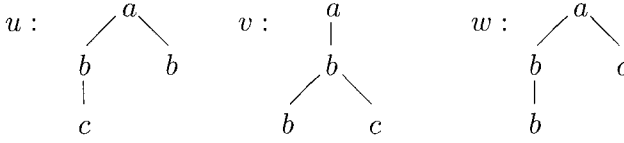
**PROPOSITION 4.1** *Given two L-evaluations  $u : P \rightarrow L, v : Q \rightarrow L$ , and  $n \in \omega$ , we have*

- (i)  $u \sim_{n+1} v$  iff  $\forall p \in P \exists q \in Q (u_p \sim_n v_q)$  and vice versa;
- (ii)  $u \leq_{n+1} v$  iff  $\forall p \in P \exists q \in Q (u_p \sim_n v_q)$ ;
- (iii)  $u \leq_\infty v$  iff  $(\forall p \in P \exists q \in Q \text{ such that } u_p \sim_\infty v_q) \text{ iff } (\exists q \in Q (u \sim_\infty v_q))$ ;
- (iv)  $u \sim_\infty v$  iff  $(u \leq_\infty v \text{ and } v \leq_\infty u)$ .  $\square$

**Example** Let  $L$  be a three element partial order with set of nodes  $\{a, b, c\}$  given by the following Hasse diagram



Consider the following three  $L$ -evaluations on three different four element posets:



In the above picture, we put in place of points of the poset the values of evaluations at these points.

Then  $u \sim_2 v$  but  $u \not\sim_3 v$ , and  $u \sim_1 w$  but  $v \not\sim_2 w$ .

Notice that saying that ' $u \sim_\infty v$ ' is equivalent to say that ' $u \sim_n v$  holds for every  $n$ '. In fact, on one side it is evident that if there exists an infinite strategy, then there are also strategies for  $n$  games for every  $n$ . Vice versa, suppose that there are all such finite strategies and suppose that player I chooses a point  $p \in P$ . Our posets are *finite* so that, as for every  $n \geq 0$  there exists  $q_n \in Q$  such that  $u_p \sim_n v_{q_n}$ , there is also  $q \in Q$  (independent of  $n$ ) such that  $u_p \sim_n v_q$  for every  $n$ . Player II answers this  $q$  and continuing this way, it is clear that we can define the desired winning infinite strategy.

With each  $L$ -evaluation  $u : P \rightarrow L$  we associate for every  $n \in \omega$  a set  $Type_n(u)$  of  $\sim_n$ -equivalence classes by:  $Type_n(u) = \{[u_p]_n : p \in P\}$  (where, of course,  $[u_p]_n$  is the  $\sim_n$ -equivalence class of  $u_p$ ). An important, although simple, fact is given by the following proposition:

**PROPOSITION 4.2** *Fix a finite poset  $L$  and  $n \in \omega$ ; then there are only finitely many equivalence classes of  $L$ -evaluations with respect to  $\sim_n$ .*

*Proof.* This is evident for  $n = 0$ . For  $n > 0$ , we argue by induction as follows. By Proposition 4.1(i), we have that  $u \sim_n v$  iff  $Type_{n-1}(u) = Type_{n-1}(v)$ ,

hence there cannot be more non  $\sim_n$ -equivalent  $L$ -evaluations than sets of  $\sim_{n-1}$  equivalence classes.<sup>2</sup>  $\square$

For infinite equivalence the result of the previous Proposition is not true, but on the other hand infinite equivalence can be characterized in terms of open maps (see Proposition 4.4 below). Recall from Chapter 2 that an order-preserving map  $h : P \rightarrow Q$  is called *open* iff for all  $p \in P, q \in Q$ , if  $q \leq h(p)$  then there exists  $p' \leq p$  such that  $h(p') = q$ . Notice that if  $Q$  is rooted and  $h$  open,  $h$  is surjective iff the inverse image of the root of  $Q$  is non-empty. As shown in Chapter 2, open maps are exactly Birkhoff duals of Heyting algebras homomorphisms.

LEMMA 4.3 *Given an  $L$ -evaluation  $u : P \rightarrow L$  and an open map  $h : Q \rightarrow P$ , we have that  $(u \circ h) \leq_\infty u$ . Moreover, if  $h$  is also surjective, then  $(u \circ h) \sim_\infty u$ .*

*Proof.* Clearly it is sufficient to prove the second part of the claim (for  $(u \circ h) \sim_\infty u_{h(\rho)}$ , hence  $(u \circ h) \leq_\infty u$  follows). So suppose that  $h$  is surjective. We have the following infinite strategy for player II: if player I plays  $q \in Q$ , the answer is  $h(q)$ , if he plays in  $P$  the answer is suggested by the openness of  $h$  (or by the surjectivity of  $h$  in the first move), so that we reach only positions of the kind  $\langle q, h(q) \rangle$ . It is evident that in this way player II wins.  $\square$

PROPOSITION 4.4 *For two  $L$ -evaluations  $u : P \rightarrow L, v : Q \rightarrow L$ , we have that  $u \sim_\infty v$  iff there is a commutative square*

$$\begin{array}{ccc} R & \xrightarrow{h} & P \\ k \downarrow & & \downarrow u \\ Q & \xrightarrow{v} & L \end{array}$$

*such that  $R$  is a finite rooted poset and  $h, k$  are open surjective maps. Moreover  $u \leq_\infty v$  iff there is a commutative square like the above one, with the only difference that now  $k$  is not required to be surjective.*

*Proof.* Suppose that  $u \sim_\infty v$ . Take  $R = \{\langle p, q \rangle : u_p \sim_\infty v_q\}$ . Order is the restriction of the product order on  $P \times Q$  and  $h, k$  are the two projections, restricted in their domains.  $R$  is clearly rooted, the square commutes and  $h, k$  are surjective and open. The latter can be shown as follows (e.g. for  $h$ ).

<sup>2</sup>This is only an exponential strict upper bound, because not all sets of  $\sim_{n-1}$ -equivalence classes are legal, i.e. are of the form  $Type_{n-1}(u)$  for some  $u$  (for instance, for  $n = 1$  only subsets of  $L$  having a greatest element are legal and the situation becomes more involved for larger  $n$ , see [Ur], [Gh1]).

Suppose that  $\langle p, q \rangle \in R$  and that  $p' \leq p$ ; we have that there exists  $q' \leq q$  such that  $u_{p'} \sim_\infty v_{q'}$ . For such  $q'$  we have that  $\langle p', q' \rangle \in R$ ,  $\langle p', q' \rangle \leq \langle p, q \rangle$  and  $h(p', q') = p'$ .

Conversely, suppose that there is a commutative square with the required properties. Now  $u \sim_\infty (u \circ h)$ , by Lemma 4.3 and similarly  $v \sim_\infty (v \circ k)$ , hence  $u \sim_\infty v$ .

The characterization of  $u \leq_\infty v$  follows from the above characterization of  $\sim_\infty$  and Proposition 4.1 (iii).  $\square$

It is interesting to know that, given an  $L$ -evaluation  $u$ , for sufficiently large  $n$  (strictly depending on  $u$ ) infinite equivalence to  $u$  is the same as  $n$ -equivalence to  $u$ . By  $ht(P)$ , we denote the height of the finite poset  $P$ , i.e. the length of a maximal chain in  $P$ . We have

**PROPOSITION 4.5** *Let  $u : P \rightarrow L$  be an  $L$ -evaluation. For  $n(P) = 2 ht(P) - 1$  we have that for every  $v : Q \rightarrow L$ ,  $u \sim_{n(P)} v$  iff  $u \sim_\infty v$ .*

*Proof.* The claim is clear for  $ht(P) = 1$ , because in this case  $P$  is a one-point poset and  $u \sim_1 v$  means that  $v$  is constant. Suppose that  $ht(P) > 1$  and that  $u \sim_{n(P)} v$ : we define an infinite strategy for player II. We recall that, as  $u \sim_{n(P)} v$ , for every  $q \in Q$  there must exist a point  $p \in P$  such that  $u_p \sim_{n(P)-1} v_q$ . Such  $p$  may or may not be the root of  $P$ . Player II behaves as follows: as long as player I plays a point  $q \in Q$ , such that  $u \sim_{n(P)-1} v_q$ , player II answers the root of  $u$ . After an initial (possibly empty) sequence of such moves we reach a position  $\langle \rho(P), q \rangle$  with  $u \sim_{n(P)-1} v_q$ . If now player I tries with  $p \in P$  (different from  $\rho(P)$ , otherwise answer is obviously  $q$ ), then there exists  $q' \leq q$  such that  $u_p \sim_{n(P)-2} v_{q'}$ : player II answers such  $q'$  and wins by induction hypothesis. If player I tries with  $q' \in Q$  such that there exists  $p \neq \rho(P)$  so that  $u_p \sim_{n(P)-1} v_{q'}$ , then player II answers such  $p$  and wins again by induction hypothesis.  $\square$

The next Lemma will complete the list of basic properties of the relations  $\sim_n$  and  $\leq_n$ . Before stating the Lemma, we introduce a useful construction on  $L$ -evaluations. Suppose that  $u : P \rightarrow L$ ,  $v : Q \rightarrow L$  are  $L$ -evaluations such that  $v \leq_0 u$ . The *grafting* of  $v$  below the root of  $P$  is an  $L$ -evaluation  $v \triangleleft u : P' \rightarrow L$  defined as follows.  $P'$  is  $P + Q$  (disjoint union as sets) with the following order  $\leq'$ :  $q \leq_{P'} p$  iff either  $(p = \rho(P))$  or  $(p, q \in P \text{ and } q \leq_P p)$  or  $(p, q \in Q \text{ and } q \leq_Q p)$ . Moreover,  $v \triangleleft u$  acts as  $u$  on  $P$  and as  $v$  on  $Q$  (this is order-preserving because  $v \leq_0 u$ ). It is immediately seen that:

**LEMMA 4.6** *( $v \triangleleft u$ )  $\circ i_Q = v$  (where  $i_Q$  is the open inclusion of  $Q$  into  $P'$ ), hence  $v \leq_\infty (v \triangleleft u)$ . Moreover, if for some  $n \in \omega$ ,  $v \leq_n u$ , then  $u \sim_n (v \triangleleft u)$*

$\square$

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