

## CHAPTER 2

### FROM HILBERT TO KRONECKER

#### 1. INTRODUCTION. METAMATHEMATICS

##### 1.1. Introduction

Hilbert uses the expression «das inhaltliche logische Schliessen»<sup>1</sup> which I translate by "internal logic", rather than logic of content. Brouwer and H. Weyl<sup>2</sup> use also the expression to designate an inner logic different from formal (external) logic which mirrors only the superficial structure of mathematics. For Hilbert, internal logic is not ordinary or formal logic, the rôle of which is only ancillary, that is the demonstration of theorems in a given mathematical theory. But internal logic, often identified with metamathematics<sup>3</sup>, should be considered as an "intramathematics" in the sense that the inner consistency of axioms is more important than the deduction of particular theorems. In other words, proof theory <*Beweistheorie*> or <*Metamathematik*> is an internal logic to the extent it describes the inner workings of a mathematical theory. Proof theory has been seen as the theory of formal systems and, by extension, as the very embodiment of formalism. The hypothesis that I want to defend goes the other way : internal logic is the opposite of formalism and Hilbert's endeavour or programme could be formulated in the following terms : internal finitary logic reduces infinitary formal logic in the same manner that a finitary mathematical theory (like arithmetic) reduces the infinite problems of the theory of forms or the theory of invariants to a finite calculus.

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<sup>1</sup> Cf. D. Hilbert (1930).

<sup>2</sup> Weyl uses the term "intrinsic" which is very close to our "internal":

Each field of knowledge, when it cristallizes into a formal theory, seems to carry with it its intrinsic logic which is part of the formalized symbolic system and this logic will, generally speaking, differ in different fields (Weyl, 1968, III, p. 705).

<*Inhaltlich*> has been translated sometimes by "contentual", it could also be rendered by "concrete" or "substantive". Beyond stylistic reasons, my use of "internal" is pointed and refers to a foundational approach which I have attempted to justify elsewhere (see Gauthier, 1991).

<sup>3</sup> Cf. D. Hilbert (1932, III p. 174). For Hilbert's program, see G. Kreisel (1958) who does not however mention Kronecker's influence.

That hypothesis relies heavily on the assumption that Hilbert has been inspired by Kronecker's mathematical practice, especially by his fundamental work *Grundzüge einer arithmetischen Theorie der algebraischen Grössen* («Foundations of an arithmetical theory of algebraic quantities»). My contention is, that despite his rare admission of a Kroneckerian influence (see below), Hilbert saw Kronecker's work as a model of mathematical practice, not as a categorical imperative of philosophical import. Hardly a constant adherent to Kronecker's finitism, he nevertheless stressed the importance of finiteness results and the constructive content of mathematical results. My hypothesis, in this attempted reconstruction of Hilbert's programme, is that despite his opposition to Kronecker's anti-Cantorism, he wanted to save ideal structures (in a dialectical retreat from Brouwer's exclusivist attitude) by granting them a kind of ideal existence, that is consistency.

Hilbert's most important results must be replaced in the mathematical tradition he has inherited, the tradition of Gauss and Kronecker and I want to put the emphasis mainly on Kronecker who has inspired much of Hilbert's mathematical work. It is worth noting at first that Hilbert puts at the very foundation of his enterprise, the theory of finite intuitive arithmetic (arithmetical sentences without quantifiers); then follow quantified arithmetic sentences (with  $\exists$  or  $\forall$ ) which introduce an infinite (denumerable) number of elements, *e.g.* Euclid's theorem of the infinity of primes, Fermat's last theorem, etc., all theorems which are not immediately subjected to negation since they refer to the entire sequence (the set) of natural numbers, and finally, the transfinite mathematical statements which are transarithmetical by definition and which one must consider as ideal structures, much alike Kummer's ideal numbers, or more appropriately, as we shall see, as Kronecker's indeterminates *<Unbestimmte>*. In order to save Aristotelian logic, that is ordinary formal logic, Hilbert introduces a formalised language preserving classical laws of quantification for infinite arithmetical statements and for transfinite or transarithmetical statements. What Hilbert had sooner seen as formal logic was only the usual logic of ordinary mathematics interpreted as formal (external) calculus. But one had to go further to account for the internal character of intuitive finite arithmetic; from there, it should be possible to conceive an extension to the internal logic of arithmetic, that is a transarithmetical logic which could encompass the whole of mathematics. But the extension had to be conservative, *i.e.* the laws of arithmetic must remain valid and for that reason a consistency proof of infinite arithmetic (and analysis) was necessary.

Since finite intuitive arithmetic is self-consistent — here Hilbert concurs with Kronecker as is evident from Hilbert's early independence results in geometry and later in his foundational work — and immediately justified in intuition *<Anschauung>*, extended consistency has a conceptual *<begriffliche>* character that can be secured only by means of logic. Once consistency is obtained, ideal existence is warranted. I contrast here effective existence (of constructions) with ideal existence (of structures); the passage between the two is achieved by logic alone (what Hilbert called Aristotelian logic). Of course, the logic is non-constructive, but it must have a finitary embodiment, and that will be the task of finitist metamathematics conceived as an instrument for a consistency proof of analysis and set theory. The concepts of justification or certification, *<Sicherung>*, surveyability *<Uebersehbarkeit>*, are supposed to guarantee the finiteness enjoyed by intuitive arithmetic. If this analysis is

right, it shows that Hilbert's strategy for the consistency problem had to be motivated by a foundational approach akin to Kronecker's theory of arithmetic.

## 2. ARITHMETIC

Hilbert admires Kronecker's work in arithmetic, but he disapproves of his contempt for Cantor, whom Kronecker condemned as «perverter of youth». As Hilbert declared : « nobody will drive use from the paradise Cantor has created for us »<sup>4</sup>, and despite what Kronecker has said about the integers as creations of God<sup>5</sup>, there is no doubt that Cantor's paradise is more populated than Kronecker's. However, it is not divine inspiration that one finds in Kronecker, but Gaussian ideas, when he says that number is a creation of our mind, while space and time have an independent reality that cannot be determined *a priori* or in an absolute<sup>6</sup> fashion. Kronecker here follows Gauss and Riemann against Kant. But mathematics is the work of a finite mind and constructive methods - explicit solutions - must replace existence theorems as in the fundamental theorem of algebra where an algebraic equation without roots (solutions) leads to a contradiction. Hilbert will listen to Kronecker in his arithmetical works, but he will turn a deaf ear when he is able to travel the transcendental royal road of existence theorems in invariant theory.

Already in his works on number theory, Hilbert shows some reluctance to Kummer's and Kronecker's arithmetical spirit. In his report on *The theory of algebraic number fields*, Hilbert says :

I have attempted to bypass Kummer's heavy apparatus of calculation in order to abide by Riemann's precept, that is to obtain results through concepts and not by calculation<sup>7</sup>.

Modern mathematics stands under the sign of number « *unter dem Zeichen der Zahl* »<sup>8</sup> and the arithmetization of function theory (analysis) is meant to show that the proof of a mathematical fact is ultimately reducible to relations among rational integers<sup>9</sup>. Kronecker would not have said differently and the indeterminate coefficients or simply indeterminates <Unbestimmte> which he introduces in 1881 are algebraic quantifies (independent variables) playing the role of ideal extensions<sup>10</sup>. The theory of algebraic number fields is based on finitary concepts : in contemporary idiom, one says that a subfield  $F$  of complex numbers is a field of algebraic numbers if it is restricted to the field of rational numbers  $\mathbb{Q}$  - a field is any set of numbers in which

<sup>4</sup> See Hilbert (1926, p. 170).

<sup>5</sup> Cf. D. Hilbert (1932, I, p. 64 and 1932, III, p. 161). Hilbert adds that Kronecker had rejected everything that transcended the integers.

<sup>6</sup> Cf. L. Kronecker (1968, II, p. 249-274).

<sup>7</sup> Cf. L. Kronecker (1968, I, p. 67). Helmut Hasse adds that Hilbert has given new proofs free of Kummer's detailed and opaque calculations, (*idem*, p. 259).

<sup>8</sup> Underlined by Hilbert (1932, I, p. 66).

<sup>9</sup> (*Idem*, p. 66).

<sup>10</sup> Cf. L. Kronecker « Grundzüge einer arithmetischen Theorie der algebraischen Grössen » (1968, II, p. 237-387).

for two arbitrary numbers  $a$  and  $b$ ,  $a+b$ ,  $a-b$ ,  $ab$  and  $\sqrt[n]{b}$  (for  $b \neq 0$ ) are also contained in the set. In the case that  $F$  is algebraic number field, the subset of  $F$  containing only the algebraic integers  $\omega$  is a ring (Dedekind ring);  $\omega$  is here a complex number which is the root of a polynomial

$$b_0 x^n + b_1 x^{n-1} + \dots + b_n = 0$$

where the  $b_i$  are integers. An ideal  $A$  is generated by the algebraic integers  $\alpha_1, \alpha_2, \dots, \alpha_n$  of an algebraic number field  $K$  if it is defined by the sums

$$\lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \dots + \lambda_n \alpha_n$$

where the  $\lambda_i$  are arbitrary integers. The principal result in that context has been demonstrated by Dedekind and Kronecker and bears on the finite number of equivalence classes of an algebraic number field, a result which leads to the fundamental theorem on the unique factorization of an ideal into prime ideals (divisible by themselves and by the unit ideal): the main point here is the divisibility of any ideal by a finite number of ideals.

It is the field theory of the equivalence classes of ideals which attracts Hilbert and his last works on number theory treat of the relative Abelian fields and are the source of what is now called the class field theory. Hilbert maintains that in every case<sup>11</sup> one must find the class field  $K_k$  for an arbitrary base field  $k$  by purely arithmetical means, although there are transcendental methods available, like Dirichlet's (Dirichlet series). The fact that it is still difficult nowadays<sup>12</sup> to calculate the class number for equivalence classes of ideals is a testimony to Hilbert's arithmetic "ideal".

As a matter of fact, the most important results in number theory, the quadratic reciprocity law (and its generalizations), unique factorization — the fundamental theorem of arithmetic says that every integer is representable in a unique way by a product of prime factors — and its generalization in finite fields (of algebraic numbers), the distribution of primes

$$\lim_{n \rightarrow \infty} \frac{\pi(x)}{x/\log x} = 1$$

and Dirichlet's theorem on the infinity of primes in any arithmetic progression  $a + nb$  — proven by purely arithmetic means (Selberg in 1949) — all of them make manifest the finite character of arithmetic and if proofs are often analytical (or transcendental), the object is essentially finite. The same could be said in algebraic, or better, arithmetic geometry about Weil's results on the finite number of rational solutions

<sup>11</sup> Cf. « Ueber die Theorie der relativ-Abelschen Zahlkörper » (1932, II, p. 483-509).

<sup>12</sup> See K. Ireland and M. Rosen (1980, chap. 12).

over finite fields and Faltings's results on the finite number of rational points on any elliptic curve of genus  $\geq 2$  where Fermat's method of infinite descent<sup>13</sup>, although of arithmetic ascendancy, is often employed in a non-constructive or non-effective way. On the other side, the stochastic behaviour of primes seems also to call for non-effectivity, but as the counterpart of the regularity of integers, it is the combinatorial complexity engendered by the local distribution of (large) primes which accounts for the probabilistic effects in an absolute natural order (*i.e.* of integers).

Hilbert conceives the finitary ideal of arithmetic, but he manages his access to it via non-finitary means. Arithmetization of analysis is a goal for him as much as it is for Kronecker and he will even say that arithmetization of geometry is achieved in non-Euclidean geometries through the direct introduction of the number concept<sup>14</sup>. Hilbert's work in invariant theory goes in the same direction, as we shall now see.

### 3. ALGEBRA

Algebraic invariant theory stems from number theory, but its history is closely linked to geometry, since algebraic invariants correspond to invariant properties of geometric figures. P. Gordan has been the first mathematician to define a complete system of binary forms

$$ax^n + 2bxy + cy^n$$

(binary, *i.e.* in two variables) of arbitrary degree  $n$ ; such a system is finite and computable. Hilbert establishes the more general existence theorem on the finite number of forms in a system of arbitrary forms<sup>15</sup> with

$$F = A_1 F_1 + A_2 F_2 + \dots + A_m F_m$$

for definite forms  $F_1, F_2, \dots, F_m$  of the system and arbitrary forms with variables belonging to a given field. Here the basis theorem (of the system of forms) is the heart of the matter and it is not difficult to show its kinship with class field theory. Kronecker again opens the way. In his paper *On the full systems of invariants*<sup>16</sup>, Hilbert acknowledges that invariant theory is but one example (a remarkable one, to say the least) of the field theory for algebraic functions of several variables. Kronecker defines an algebraic function as the root of an irreducible equation  $f(x) = 0$  (of degree  $n$ ) where  $f(x)$  is an irreducible (or prime) polynomial in a domain of rationality  $\langle \text{Rationalitäts-Bereich} \rangle$  (*i.e.* field). Kronecker had already shown that in

<sup>13</sup> André Weil (1979) recognizes the importance of Kronecker in number theory and in algebraic geometry, as he has emphasized Fermat's achievements.

<sup>14</sup> Cf. D. Hilbert (1932, I, p. 64).

<sup>15</sup> Cf. D. Hilbert (1932, II, p. 199-257).

<sup>16</sup> Cf. D. Hilbert (1932, II, p. 287-365).

Internal Logic

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