Chapter 1

INTRODUCTION TO PART I

Some of the content of Part I has appeared in [O'Hearn and Pym, 1999, Pym, 1999, Pym et al., 2000, Armelín and Pym, 2001]. References are given in the text as appropriate.

—DJP

1. A Proof-theoretic Introduction

One of the most important outcomes of the study of linear logic, much more than the formal system itself, is its revealing of the computational significance of the structural rules of Weakening and Contraction [Girard, 1987]. Logically, their absence leads to the decomposition of conjunction into *additive* (&) and *multiplicative* ($\otimes$) forms, which may be given a sequential natural deduction presentation as follows:

$$\frac{\Gamma \vdash \phi \quad \Gamma \vdash \psi}{\Gamma \vdash \phi \& \psi}\ \&I \qquad \frac{\Gamma \vdash \phi_1 \& \phi_2}{\Gamma \vdash \phi_i}\ (i = 1, 2) \quad \&E$$

$$\frac{\Gamma \vdash \phi \quad \Delta \vdash \psi}{\Gamma, \Delta \vdash \phi \otimes \psi}\ \otimes I \qquad \frac{\Gamma, \phi, \psi \vdash \chi \quad \Delta \vdash \phi \otimes \psi}{\Gamma, \Delta \vdash \chi}\ \otimes E.$$

If we have the rules of Weakening and Contraction

$$\frac{\Gamma \vdash \psi}{\Gamma, \phi \vdash \psi}\ W \qquad \frac{\Gamma, \phi, \phi \vdash \psi}{\Gamma, \phi \vdash \psi}\ C,$$

then these rules for $\otimes$ and & define the same connective, but without them the connectives are distinct. A similar decomposition obtains for disjunction, the multiplicative version arising in the classical (as opposed to intuitionistic) setting.

The decomposition of the connectives has a long history in the relevant logic tradition but the possibilities revealed by restricting structural rules were given a new perspective by the appealing "resource interpretations" of linear logic. The leading example is perhaps the *number-of-uses* reading in which a proof of a linear implication $\phi \multimap \psi$ determines a function that *uses* its argument exactly once. Like $\otimes$, the linear implication is multiplicative, which is to say that it uses separate contexts in its elimination rule.

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \multimap \psi} \multimap I \qquad \frac{\Gamma \vdash \phi \multimap \psi \quad \Delta \vdash \phi}{\Gamma, \Delta \vdash \psi} \multimap E.$$

However, an important message of linear logic is that, in order to obtain an expressive system, one cannot stay in the pretty-but-weak purely multiplicative system: it seems crucial to allow access to structurals in some manner. In linear logic, this is done via the "!" modality, which admits a recovery of intuitionistic (or additive) implication $\phi \to \psi$ as $!\phi \multimap \psi$. The number-of-uses reading of implication is extended to $!\phi$ as "as many ϕs as required".[1]

Access to the structurals may be recovered in another, rather different, way, not involving a modality. Just as we decomposed conjunction directly into multiplicative and additive parts, so we can decompose *implication* directly. The technical cost of this conceptual symmetry is that we must work with a more richly structured notion of sequent, entailing a more delicate analysis of the proof-theoretic relationship between implication and conjunction.

Implication is inextricably bound up with conjunction, or at least with antecedent-forming operations used to formulate sequents. This connection goes so far that it is sometimes said that an introduction rule

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \to \psi}$$

may be regarded, proof-theoretically, as defining the *meaning* of $\to$ [Sundholm, 1986]; it is clear that the character of the implication in a logic is, in a sense, determined by that of the comma or conjunction.

If, as is the case in **BI**, we have two forms of implication, then we are faced with the question of which of them to use in the introduction rule.

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi ? \psi}.$$

[1]The rôle of the modalities, or exponentials, is central to the development of linear logic.

That is, should the conclusion $\phi ? \psi$ in this rule be a multiplicative or additive implication ?

The connection between introduction rules and implications suggests a solution: If an antecedent-forming operation determines the behaviour of an implication, and we have two implications, then we should have two antecedent-forming operations. So, we postulate a new context-forming operation, ";", and stipulate that Contraction and Weakening are permitted for ";" but not for ",". The introduction rules then become

$$\frac{\Gamma , \phi \vdash \psi}{\Gamma \vdash \phi \mathrel{-\!\!*} \psi} \quad \text{and} \quad \frac{\Gamma ; \phi \vdash \psi}{\Gamma \vdash \phi \to \psi} .$$

The antecedents are no longer sequences; rather, they are trees with propositions as leaves and internal nodes labelled with "," or ";", or in short, *bunches* [Dunn, 1975, Belnap, 1982, Read, 1988].

Corresponding to **BI**'s natural deduction system is a simply-typed lambda calculus, $\alpha\lambda$, which gives a representation of **BI**'s natural deduction proof-objects. For example, the typing rules for the two kinds of lambda-abstraction, corresponding to the right rules for $\mathrel{-\!\!*}$ and $\to$ are, respectively

$$\frac{\Gamma , x : A \vdash M : B}{\Gamma \vdash \lambda x : A.M : A \mathrel{-\!\!*} B} \quad \text{and} \quad \frac{\Gamma ; x : A \vdash M : B}{\Gamma \vdash \alpha x : A.M : A \to B} .$$

Here we are working with bunches of typed variables, rather than bunches of formulæ, in which no variable may occur more than once. There are two combinators for application, one for each abstractor.

BI's proof theory may also be presented as a sequent calculus, in which, as explained in the introduction, elimination rules are replaced by rules which introduce connectives to the left-hand sides of sequents.

For example, the rules for eliminating $\to$ and $\mathrel{-\!\!*}$,

$$\frac{\Gamma \vdash \phi \to \psi \quad \Delta \vdash \phi}{\Gamma ; \Delta \vdash \psi} \to E \quad \text{and} \quad \frac{\Gamma \vdash \phi \mathrel{-\!\!*} \psi \quad \Delta \vdash \phi}{\Gamma , \Delta \vdash \psi} \mathrel{-\!\!*} E,$$

are replaced by

$$\frac{\Gamma \vdash \phi \quad \Delta(\psi) \vdash \chi}{\Delta(\Gamma ; \phi \to \psi) \vdash \chi} \to L \quad \text{and} \quad \frac{\Gamma \vdash \phi \quad \Delta(\psi) \vdash \chi}{\Delta(\Gamma , \phi \mathrel{-\!\!*} \psi) \vdash \chi} \to L,$$

respectively.

We establish the Cut-elimination theorem for **BI** and show that the natural deduction system and the sequent calculus are equivalent.

2. A Semantic Introduction

It is all very well to describe proof systems in this way and we can indeed argue that the proof-theoretic "meaning", in the sense of [Sundholm, 1986], is rather clear. However, we must ask what, if any, is the semantic significance of the resulting logic ? We argue herein that **BI** possesses three very natural semantics:

- Algebraic and topological models, in the tradition of Boole, Heyting and Tarski;
- Categorical models, in the tradition of Brouwer, Heyting, Kolmogorov, Dana Scott and Lambek;
- Kripke models, in the tradition of Kripke, Beth, Tarski and Joyal, as represented by [Lambek and Scott, 1986].

Of course, as we explain at the appropriate points in our development, these three semantics are intimately related to one another, being instances of the same abstract construction. However, their motivations and styles are sufficiently different to warrant separate presentations.

2.1 Algebraic and Topological Semantics

The first semantics, described in Chapter 3, follows, on the one hand, in the tradition of Heyting [Girard, 1989] and, on the other, in that of Tarski [Tarski, 1956]. We describe an algebraic structure corresponding to the logical structure of **BI**. Such a structure combines that required for the additive, intuitionistic, part of **BI**, *i.e.*, a Heyting algebra, with that required for the multiplicative part, *i.e.*, the $(\otimes, I, -\!\!\circ)$-fragment of intuitionistic linear logic. This latter structure is closely related to the algebraic structures described in [Restall, 1999, Trolestra, 1992].

We also give an associated syntactic calculus, which amounts to a Hilbert-type proof system for **BI**.

2.2 Categorical Semantics

The second semantics is a BHK-style semantics of the proof theory of propositional **BI** which arises from *doubly closed categories* (DCCs), in which a single category admits two closed structures or function spaces. It shows clearly the difference with linear logic, where two closed categories, connected by a monoidal co-monad [Barber, 1996], are usually used.

The purely multiplicative fragment of **BI**, sometimes called BCI logic, sometimes called multiplicative intuitionistic linear logic, has been studied in many different contexts but a categorical semantics is studied in some detail in Lambek's work on "Deductive Systems and Categories" [Lambek, 1968, Lambek, 1969, Lambek, 1972].

Categorical models of the proofs of predicate logics are a more delicate matter. Just as for intuitionistic predicate logic, for predicate **BI** we move to models with indexed, or fibred, structure [Seely, 1983].

2.3 Kripke Semantics

The third semantics is a Kripke-style semantics of formulæ, which combines Kripke's semantics of intuitionistic logic and Urquhart's semantics of multiplicative intuitionistic linear logic based on the idea of "pieces of information" [Urquhart, 1972]. This second semantics gives **BI** more of a genuine status as a logic: It gives us a way to read the formulæ as propositions that are true (or not) relative to a given world. It is motivated by the notion of resource, as discussed in the Introduction.

Recall from the Introduction that we require a resource to be a preordered commutative monoid, informally, a "Kripke resource monoid",

$$\mathcal{M} = (M, e, \cdot, \sqsubseteq),$$

in which we require the following bifunctoriality condition:

$$\text{if} \quad m \sqsubseteq m' \quad \text{and} \quad n \sqsubseteq n', \quad \text{then} \quad m \cdot n \sqsubseteq m' \cdot n'.$$

Whenever we refer to a preordered commutative monoid we assume that this condition holds.

Preordered commutative monoids may be used to provide a truth-functional, Kripke-style semantics for **BI**. The basic idea of **BI** is to allow the connectives of multiplicative intuitionistic linear logic, **MILL**, and intuitionistic logic, **IL**, to exist side-by-side, without recourse to the modalities used to recover intuitionistic strength in linear logic. This may be done with an inductively defined forcing relation $\models$. The clauses for the additives follow the intuitionistic pattern:

$$m \models \phi \wedge \psi \quad \text{iff} \quad m \models \phi \quad \text{and} \quad m \models \psi$$

$$m \models \phi \vee \psi \quad \text{iff} \quad \text{either} \quad m \models \phi \quad \text{or} \quad m \models \psi$$

$$m \models \phi \to \psi \quad \text{iff} \quad \text{for all} \quad n \sqsubseteq m, \quad n \models \phi \quad \text{implies} \quad n \models \psi.$$

The clauses for the multiplicatives are more interesting, following Urquhart's [Urquhart, 1972] semantics for **MILL**:

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