Chapter 2

NATURAL DEDUCTION FOR PROPOSITIONAL BI

1. Introduction

BI has a simple and elegant proof theory, the presentation of which we begin in this section. We start with a definition of **BI** as a system of natural deduction, formulated in the sequential, or linearized, style. Rather than establish the metatheory of this system in the manner of Prawitz [Prawitz, 1965], we first formulate a representation of **BI**'s proofs as a λ-calculus, $\alpha\lambda$, with types given by **BI**'s propositions. We then establish normalization for $\alpha\lambda$. **BI**'s natural deduction system was introduced in [O'Hearn and Pym, 1999, Pym, 1999].

2. A Natural Deduction Calculus

In this section, we give a presentation of **BI** in sequential natural deduction form, *i.e.*, a sequential presentation based on introduction and elimination rules. Let L denote a set of atomic propositional letters and let p, q, *etc.* range over L. The set of **BI** propositions over L, $\mathcal{P}(L)$, is given by the following inductive definition:

PROPOSITIONS

$$
\begin{array}{rcll}
\phi & ::= & \mathrm{p} & \text{atoms} \\
 & \mid & I & \text{multiplicative unit} \\
 & \mid & \phi * \phi & \text{multiplicative conjunction} \\
 & \mid & \phi \mathrel{-\!\!*} \phi & \text{multiplicative implication} \\
 & \mid & \top & \text{additive unit} \\
 & \mid & \phi \wedge \phi & \text{additive conjunction} \\
 & \mid & \phi \rightarrow \phi & \text{additive implication} \\
 & \mid & \bot & \text{additive disjunctive unit} \\
 & \mid & \phi \vee \phi & \text{additive disjunction.}
\end{array}
$$

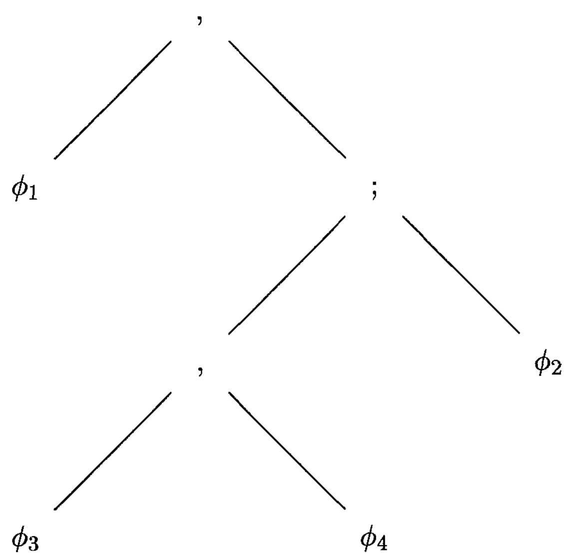
The additive connectives correspond to those of intuitionistic logic (**IL**), whereas the multiplicative connectives correspond to those of multiplicative, intuitionistic linear logic (**MILL**).

As we have seen, the presence of the additive and multiplicative implications as primitives forces to work with contexts which are not merely finite sequences but rather finite trees.

BUNCHES

$$\begin{array}{rcll} \Gamma & ::= & \phi & \text{propositional assumption} \\ & | & \emptyset_m & \text{multiplicative unit} \\ & | & \Gamma, \Gamma & \text{multiplicative combination} \\ & | & \emptyset_a & \text{additive unit} \\ & | & \Gamma ; \Gamma & \text{additive combination} \end{array}$$

For example, the bunch $\phi_1, ((\phi_3, \phi_4); \phi_2)$ may be drawn as



The main point of the definition of bunches is that ";" admits the structural properties of weakening and contraction, whereas "," does not: this distinction allows the correct formulation of the two implications. Bunches are structured as trees, with internal nodes labelled with either "," or ";" and leaves labelled with propositions. Bunches may be represented using lists of lists, *etc.* as described in [Read, 1988]. We write $\Gamma(\Delta)$, and refer to Δ as a *sub-bunch* of Γ, for a bunch Γ in which Δ appears as a sub-tree and write $\Gamma[\Delta'/\Delta]$ for Γ with Δ replaced by Δ'. We write $\Gamma(-)$ to denote a bunch Γ which is incomplete and which may be completed by placing a bunch in its hole, and will use this notation to refer to that part of $\Gamma(\Delta)$ which is not part of Δ. We require that "," and ";" be commutative monoids, giving rise to the coherent equivalence, $\Gamma \equiv \Gamma'$, as follows:

COHERENT EQUIVALENCE: $\Gamma \equiv \Gamma'$

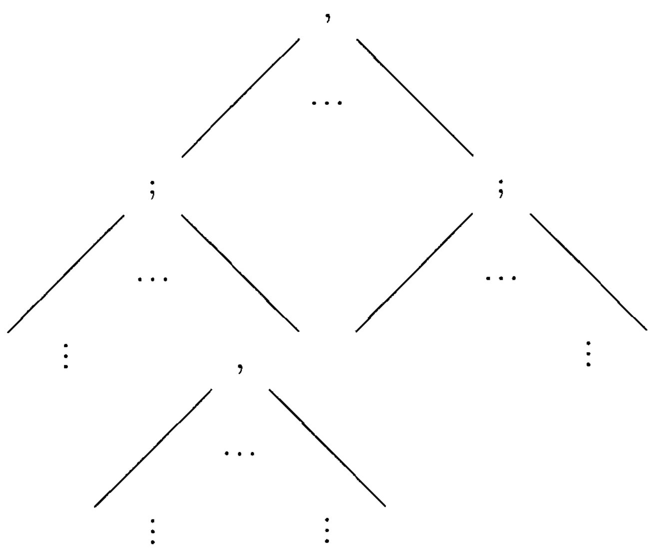
1 Commutative monoid equations for $\emptyset_a$ and ";".

2 Commutative monoid equations for $\emptyset_m$ and ",".

3 Congruence: if $\Delta \equiv \Delta'$ then $\Gamma(\Delta) \equiv \Gamma(\Delta')$.

Note that ";" and "," do not distribute over one another. We use $=$ for syntactic identity of bunches.

Although we have given the basic definition of bunches, a more structured presentation, stratified bunches, is possible. The idea is to stratify bunches into multiplicative and additive sub-bunches. So, if the top-most bunch-former is ",", then we get



and alternately if the top-most bunch-former is ";".

This presentation is discussed and exploited in Pablo Armelín's work on logic programming with **BI**, where it simplifies the definition of an operational semantics (*q.v.* Chapters 9 and 16).

We call the natural deduction system for propositional **BI**, given in Table 2.1, **NBI**.

IDENTITY AND STRUCTURE

$$\frac{}{\phi \vdash \phi}\ Axiom \qquad \frac{\Gamma \vdash \phi}{\Delta \vdash \phi} \equiv \text{ (where } \Delta \equiv \Gamma)\ E$$

$$\frac{\Gamma(\Delta) \vdash \phi}{\Gamma(\Delta; \Delta') \vdash \phi}\ W \qquad \frac{\Gamma(\Delta; \Delta) \vdash \phi}{\Gamma(\Delta) \vdash \phi}\ C$$

MULTIPLICATIVES

$$\frac{}{\emptyset_m \vdash I}\ I\,I \qquad \frac{\Gamma(\emptyset_m) \vdash \chi \quad \Delta \vdash I}{\Gamma(\Delta) \vdash \chi}\ I\,E$$

$$\frac{\Gamma \vdash \phi \quad \Delta \vdash \psi}{\Gamma, \Delta \vdash \phi * \psi}\ *I \qquad \frac{\Gamma(\phi, \psi) \vdash \chi \quad \Delta \vdash \phi * \psi}{\Gamma(\Delta) \vdash \chi}\ *E$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \mathrel{-\!\!*} \psi}\ {-\!\!*}\,I \qquad \frac{\Gamma \vdash \phi \mathrel{-\!\!*} \psi \quad \Delta \vdash \phi}{\Gamma, \Delta \vdash \psi}\ {-\!\!*}\,E$$

ADDITIVES

$$\frac{}{\emptyset_a \vdash \top}\ \top I \qquad \frac{\Gamma(\emptyset_a) \vdash \chi \quad \Delta \vdash \top}{\Gamma(\Delta) \vdash \chi}\ \top E$$

$$\frac{\Gamma \vdash \phi \quad \Delta \vdash \psi}{\Gamma; \Delta \vdash \phi \wedge \psi}\ \wedge I \qquad \frac{\Gamma(\phi; \psi) \vdash \chi \quad \Delta \vdash \phi \wedge \psi}{\Gamma(\Delta) \vdash \chi}\ \wedge E$$

$$\frac{\Gamma; \phi \vdash \psi}{\Gamma \vdash \phi \to \psi} \to I \qquad \frac{\Gamma \vdash \phi \to \psi \quad \Delta \vdash \phi}{\Gamma; \Delta \vdash \psi} \to E$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \phi}\ \bot E$$

$$\frac{\Gamma \vdash \phi_i}{\Gamma \vdash \phi_1 \vee \phi_2}\ (i = 1,\, 2) \quad \vee I \qquad \frac{\Gamma \vdash \phi \vee \psi \quad \Delta(\phi) \vdash \chi \quad \Delta(\psi) \vdash \chi}{\Delta(\Gamma) \vdash \chi}\ \vee E$$

Table 2.1. Propositional **NBI**

Notice that the introduction and elimination rules for additive and multiplicative implications, conjunctions and units are identical in form, following Prawitz's prescription [Prawitz, 1971]. The difference between them is the antecedent-combining operations they use. We can replace the $\wedge$E rule with the simpler, and perhaps more familiar, form

$$\frac{\Gamma \vdash \phi_1 \wedge \phi_2}{\Gamma \vdash \phi_i} \quad i = 1, 2.$$

As usual, we have the following easy lemma:

LEMMA 2.1 *1* $\Gamma(\phi_1, \phi_2) \vdash \psi$ *iff* $\Gamma(\phi_1 * \phi_2) \vdash \psi$*;*

2 $\Gamma(\phi_1; \phi_2) \vdash \psi$ *iff* $\Gamma(\phi_1 \wedge \phi_2) \vdash \psi$*.*

□

The additive maintenance of bunches may be made explicit by replacing each of the binary and ternary rules by their explicitly additive counterparts, as follows:

$$\frac{}{\phi; \Gamma \vdash \phi} \ \mathit{Axiom}$$

$$\frac{\Gamma \vdash \phi \to \psi \quad \Gamma \vdash \phi}{\Gamma \vdash \psi} \to E \qquad \frac{\Gamma \vdash \phi \quad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi} \wedge I$$

$$\frac{\Gamma \vdash \phi \vee \psi \quad \Gamma; \phi \vdash \chi \quad \Gamma; \psi \vdash \chi}{\Gamma \vdash \chi} \vee E.$$

Call the explicit system so obtained $\mathbf{NBI}^{\heartsuit}$. Then the following lemma follows by a familiar and straightforward induction on the structure proofs, using the admission by ";" of Weakening and Contraction:

LEMMA 2.2 $\Gamma \vdash \phi$ *is provable in* **NBI** *if and only if it is provable in* $\mathbf{NBI}^{\heartsuit}$*.*

In fact, Weakening and Contraction can be omitted from $\mathbf{NBI}^{\heartsuit}$ by modifying the way which is standard for intuitionistic logic [Troelstra and Schwichtenberg, 1996]. Weakening is pushed to the leaves of proofs via the axiom of the form

$$\frac{}{\Gamma; \phi \vdash \phi} \ \mathit{Axiom},$$

but must also be built into the binary multiplicative rules. For example, $-\!\!*\, E$ must be reformulated as

$$\frac{\Gamma \vdash \phi -\!\!* \psi \quad \Delta \vdash \phi}{(\Gamma, \Delta); \Theta \vdash \psi} -\!\!*\, E.$$

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