Chapter 3

ALGEBRAIC, TOPOLOGICAL, CATEGORICAL

1. An Algebraic Presentation

In this section, we are primarily concerned with truth and provability, rather than the structure of proofs and so present a simple-minded algebraic semantics and associated calculus for **BI**. This presentation of **BI** makes little or no explicit use of bunches, *i.e.*, **BI**'s tree-structured contexts.

In order to motivate the models, however, it is useful to sketch briefly the categorical interpretation which lies at the core of **BI** and which we describe in more detail in the sequel.

Suppose, recalling our proof-theoretic introduction in Chapter 1, that we are to have a logic with two implications. Then, categorically, the natural notion of consequence arises from doubly closed categories, which are categories that possess two closed structures or function spaces. That is, we have a single category with two adjunctions

$$[A * B, C] \cong [A, B \mathrel{-\!\!*} C] \qquad \text{and} \qquad [A \wedge B, C] \cong [A, B \to C]$$

which determine the properties of $\mathrel{-\!\!*}$ and $\to$.[1] The algebraic models are collapsed versions of these structures, where the additive implication $\to$ corresponds to that of intuitionistic logic and the multiplicative $\mathrel{-\!\!*}$ to that of a basic substructural logic.

To describe the algebraic models, we recall firstly that Heyting algebras are the algebraic models of intuitionistic propositional logic. A

[1]These ideas are developed in § 3.

Heyting algebra is a lattice with greatest and least elements in which the meet $a \wedge b$ is *residuated*, *i.e.*, there is an implication operator, $\rightarrow$, satisfying

$$a \wedge b \leq c \quad \text{iff} \quad a \leq b \rightarrow c.$$

Secondly, an algebraic model of a basic substructural logic containing conjunction $*$, unit I and implication $-\!\!*$ is similar, except that $*$ is not required to be idempotent, *i.e.*, have the properties of meet, and I is not required to be top. That is, we would require a preorder with a (monotone) commutative monoid structure that is residuated, so that

$$a * b \leq c \quad \text{iff} \quad a \leq b -\!\!* c.$$

Because we have all of the connectives of intuitionistic logic and the basic substructural logic at the same time, we ask for an algebra that has both kinds of structure simultaneously.

A **BI**-*algebra is a Heyting algebra equipped with an additional residuated commutative monoid structure.*

It is important to note that the same underlying order is used to describe the residuated structure in both cases. Categorically, this corresponds to the two closed structures being defined with respect to a single class of arrows.

From this notion of **BI**-algebra, it is straightforward to derive a collection of axioms and rules — a Hilbert-type system, equivalent to **NBI** — for proving judgements $\phi \vdash \psi$, where the formulæ ϕ and ψ are built from propositional variables, the additive connectives $\rightarrow$, $\wedge$, $\top$, $\vee$ and $\bot$, and the multiplicative connectives I, $*$ and $-\!\!*$. The axioms and rules of this system are those for (some presentation of) intuitionistic propositional logic, together with the ones for the multiplicatives given in Table 3.1.

We say that "$\psi \vdash \phi$ is derivable or provable" to indicate that $\psi \vdash \phi$ may be proven using this system or, equivalently, when $[\![\psi]\!] \leq [\![\phi]\!]$ for all interpretations $[\![-]\!]$ in **BI**-algebras: **BI**-algebras obviously give sound and complete models of the proof system just given.

2. A Topological Presentation

A (commutative) topological monoid is a (commutative) monoid in the category **Top** of topological spaces and continuous maps between them, *i.e.*, a topological space $\mathcal{X}$, with open sets $\mathcal{O}(\mathcal{X})$, together with two arrows, a tensor product $* : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{X}$ and its unit $e : 1 \longrightarrow \mathcal{X}$ such that the usual monoidal diagrams commute [Mac Lane, 1971].

We need to interpret a formula $\phi * \psi$ as the tensor product, $U * V$ of the interpretations, respectively U and V, of ϕ and ψ. The tensor product of

$$\phi * (\psi * \chi) \dashv\vdash (\phi * \psi) * \chi \qquad \phi * I \dashv\vdash \phi \dashv\vdash I * \phi$$

$$\frac{\chi \vdash \phi \quad \eta \vdash \psi}{\chi * \eta \vdash \phi * \psi} \qquad \phi * \psi \vdash \psi * \phi$$

$$\frac{\chi * \phi \vdash \psi}{\chi \vdash \phi \mathrel{-\!\!*} \psi} \qquad \frac{\chi \vdash \phi \mathrel{-\!\!*} \psi \quad \eta \vdash \phi}{\chi * \eta \vdash \psi}$$

Table 3.1. Hilbert-type **BI**

two open sets is not necessarily open, however. Consequently, we must require that the monoidal structure be defined by open maps, *i.e.*, which map open sets to open sets.

An *open* topological monoid is one in which the maps $*$ and e, which define the monoidal structure, are open.

LEMMA 3.1 (DISTRIBUTIVITY) *Let $(\mathcal{X}, *, e)$ be a topological monoid. For all open sets U, V_i, $i \in \mathcal{I}$, where $\mathcal{I}$ is some indexing set,*

$$U * (\bigcup_i V_i) = \bigcup_i (U * V_i).$$

PROOF We have that $z \in U * (\bigcup_i V_i)$ iff there exist $x \in U$ and $y_j \in V_j$, for some j, such that $z = x * y_j$ iff $z \in \bigcup_i (U * V_i)$. □

The interpretation of **BI** in an open commutative topological monoid now follows exactly as for the interpretation of intuitionistic logic in a topological space, *i.e.*, with $[\![\bot]\!] = \emptyset$, with the addition of the following:

$$\begin{array}{rcl} [\![\phi * \psi]\!] & = & [\![\phi]\!] * [\![\psi]\!] \\ [\![I]\!] & = & e(1) \end{array}$$

and

$$[\![\phi \mathrel{-\!\!*} \psi]\!] \quad = \quad \bigcup_{i \in \mathcal{I}} \{U_i \mid U_i \text{ is open and } U_i * [\![\phi]\!] \subseteq [\![\psi]\!]\},$$

where $\mathcal{I}$ is an indexing set. This interpretation is well-defined:

LEMMA 3.2 (MULTIPLICATIVE FUNCTION SPACE) $[\![\phi \mathrel{-\!\!*} \psi]\!] * [\![\phi]\!] \subseteq [\![\psi]\!]$.

PROOF We have that $\bigcup_{i \in \mathcal{I}} (U_i * [\![\phi]\!]) \subseteq [\![\psi]\!]$, so that $(\bigcup_{i \in \mathcal{I}} U_i) * [\![\phi]\!] \subseteq [\![\psi]\!]$, by distributivity. □

We can obtain soundness and completeness for these models just as for **BI**-algebras.

3. A Categorical Presentation

The BHK semantics of proofs of **BI** rests directly on a class of models based on *doubly closed categories*, or *DCCs*. We refer to [Mac Lane, 1971] for basic categorical notions, such as cartesian, monoidal and closed structure. See also Lambek's related work [Lambek, 1968, Lambek, 1969, Lambek, 1972, Lambek, 1993], arising from his early work in mathematical linguistics [Lambek, 1958].

DEFINITION 3.3 (DCC) *A* doubly closed category *is a category equipped with two monoidal closed structures. A DCC is* cartesian *if one of the closed structures is cartesian and the other is symmetric monoidal and* bi-cartesian *(sometimes written bi-CDDC) if it also has finite coproducts.* □

To see how (bi-C)DCC structure corresponds to **BI**, consider the two adjunctions[2]

$$[F \otimes D, E] \cong [F, D \multimap E] \qquad [F \times D, E] \cong [F, D \to E],$$

where $\otimes$ is a symmetric monoidal product and $\times$ a cartesian product. In the proof theory, these adjunctions correspond to having one context combination corresponding to $*$ and another to $\wedge$. This leads directly to the rules

$$\frac{\Gamma\,,\,\phi \vdash \psi}{\Gamma \vdash \phi \mathrel{-\!\!*} \psi} \qquad \frac{\Gamma\,;\,\phi \vdash \psi}{\Gamma \vdash \phi \to \psi}$$

and to the tree-like structure of antecedents.

A DCC alone does not constitute a definition of a *model* of **BI**, for which we must also have an *interpretation* of **BI**'s syntax. Such an interpretation is a function from **BI**'s language of propositions to the objects of a DCC, defined by induction on the structure of propositions.

The interpretation of **BI** in a bi-cartesian DCC, with the two closed structures $(\times, 1, \to)$ and $(\otimes, I, \multimap)$ and co-product $(+, 0)$, is given by a

[2] In the sequel, we drop "cartesian" wherever no confusion is likely.

function $[\![-]\!]$ such that:

$$
\begin{array}{rcl}
[\![\phi \vee \psi]\!] & = & [\![\phi]\!] + [\![\psi]\!] \\
[\![\bot]\!] & = & 0 \\
[\![\phi \wedge \psi]\!] & = & [\![\phi]\!] \times [\![\psi]\!] \\
[\![\top]\!] & = & 1 \\
[\![\phi \to \psi]\!] & = & [\![\psi]\!] \to [\![\phi]\!]
\end{array}
\qquad
\begin{array}{rcl}
[\![\phi * \psi]\!] & = & [\![\phi]\!] \otimes [\![\psi]\!] \\
[\![I]\!] & = & I \\
[\![\phi -\!\!* \psi]\!] & = & [\![\psi]\!] -\!\!\circ [\![\phi]\!]
\end{array}
\tag{3.1}
$$

We interpret a bunch Γ by replacing each "," with $*$ and each ";" with $\wedge$. We write $[\![-]\!]_{\mathcal{D}}$ when we want to indicate that the interpretation is in the (bi-C)DCC $\mathcal{D}$.

One point which deserves comment here concerns disjunction. To interpret the elimination rule for $\vee$ we need to use distributivity of $+$ over both $\otimes$ and $\times$. To see why we get the needed distributivities in bi-cartesian DCCs note first that, since $E \otimes (-)$ and $E \times (-)$ are both left adjoints, they both preserve all colimits. Second, $+$ is a co-product. It follows that we have the isomorphisms

$$[\![\phi]\!] \times ([\![\psi]\!] + [\![\chi]\!]) \cong ([\![\phi]\!] \times [\![\psi]\!]) + ([\![\phi]\!] \times [\![\psi]\!])$$

$$[\![\phi]\!] \otimes ([\![\psi]\!] + [\![\chi]\!]) \cong ([\![\phi]\!] \otimes [\![\psi]\!]) + ([\![\phi]\!] \otimes [\![\chi]\!]).$$

The first of these two laws shows that DCCs are not models of linear logic: distributivity fails for linear logic's additives.

For the remainder of this chapter, we confine our attention to $\alpha\lambda$ without $\vee$ (or $\bot$) and references to $\alpha\lambda$ should be taken to exclude these connectives.

DEFINITION 3.4 (MODELS OF **BI** IN DCCS) *A categorical model of* **BI** *is a pair $\langle \mathcal{D}, [\![-]\!]_{\mathcal{D}} \rangle$, where $\mathcal{D}$ is a DCC and $[\![-]\!]_{\mathcal{D}}$ is an interpretation satisfying (3.1).* □

We will often omit the subscript, writing just $[\![-]\!]$, when no confusion is likely.

PROPOSITION 3.5 (WEAK SOUNDNESS FOR DCCS) *If $\Gamma \vdash \phi$ is provable in* **NBI** *and $[\![\Gamma]\!]_{\mathcal{D}}$ and $[\![\phi]\!]_{\mathcal{D}}$ are defined, then $[[\![\Gamma]\!]_{\mathcal{D}}, [\![\phi]\!]_{\mathcal{D}}] \neq \emptyset$.*

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