Chapter 4

KRIPKE SEMANTICS

1. Kripke Models of Propositional BI

The elementary account of a truth-functional semantics for **BI** that we gave in the Introduction, using "Kripke resource monoids", is, in its conceptual simplicity, rather appealing. However, its simplicity somewhat obscures an important aspect of the semantics of **BI**, namely the structure of the multiplicatives, $*$ and $-\!\!*$, which we must understand in the same (set-theoretic or, rather, categorical) terms as we understand the structure of the additives. In this chapter, we develop the elementary Kripke semantics in a categorical setting and establish a soundness theorem and a completeness theorem (restricted to exclude inconsistency, $\bot$).

Whilst the structure of $\wedge$ and $\rightarrow$ may be seen very clearly from the definition of the forcing relation, $\models$ (they are simply the usual set-theoretic product and implication, respectively) the structure of $*$ and $-\!\!*$ is, perhaps, less clear. To see their structure, we shall need to move away from the category of sets to the category (topos) of presheaves over a symmetric monoidal category and exploit Day's tensor product construction [Day, 1970], which will form the basis of the multiplicative part of our definition of Kripke models.

Let $(\mathcal{C}, I, \cdot)$ be a small monoidal category. Following [Day, 1970], we consider $\mathbf{Set}^{\mathcal{C}^{op}}$. The monoidal structure on $\mathcal{C}$ induces the following monoidal closed structure on $\mathbf{Set}^{\mathcal{C}^{op}}$: for $A, B : \mathcal{C}^{op} \longrightarrow \mathbf{Set}$, the co-end

$$(A \otimes B)m = \int^{n,n'} A(n) \times B(n') \times \mathcal{C}[m, n \cdot n'] \qquad (4.1)$$

defines the multiplicative product, $\otimes$, with the unit given by $\mathcal{C}[-, I]$ and the end

$$\begin{aligned} (A \multimap B)m &= \int_n \mathbf{Set}[A(n), B(m \cdot n)] \\ &\cong \mathbf{Set}^{\mathcal{C}^{op}}[A(-), B(m \cdot -)] \end{aligned} \tag{4.2}$$

defines the multiplicative function space, $\multimap$. The formulæ for

$$(A \otimes B)m \text{ and } (A \multimap B)m$$

are both contravariant in m, giving the morphism parts of the functors; see [Day, 1970, O'Hearn et al., 1995]. Moreover, if the monoidal structure on $\mathcal{C}$ is symmetric, then so is the induced one on $\mathbf{Set}^{\mathcal{C}^{op}}$. These definitions are elegantly explained in Ambler's thesis [Ambler, 1992], which considers the semantics of first-order linear logic in symmetric monoidal closed categories.

Recall from Chapter 3 that two general observations are useful for working with the tensor product given by Day's construction. The first is that we have a form of pairing operation: given $a \in EY$ and $b \in FY'$ we can form an element $[a, b] \in (E * F)(Y * Y')$. To see how this element is defined, consider that the co-end $E * F(X)$ may be described as a quotient of quintuples $(Y, Y', f : X \to_{\mathcal{C}} Y * Y', a \in EY, b \in FY')$. The pair ("Day's pairing operation") $[a, b]$ is then the equivalence class of $(Y, Y', 1_{Y*Y'}, a, b)$.

The second is a representation result which characterizes maps out of a tensor products: natural transformations $E * F \longrightarrow G$ are in bijection with families of functions

$$EY \times FY' \longrightarrow G(Y * Y')$$

natural in Y and Y'. To see why this is true, consider the definition of $-\!\!*$, and the isomorphism $[E * F, G] \cong [E, F -\!\!* G]$: the multi-map characterization is, essentially, forced by $-\!\!*$.

The usual categorical product and exponential lift, respectively pointwise and via Yoneda's lemma, to $\mathbf{Set}^{\mathcal{C}^{op}}$, which also has co-products. So, $\mathbf{Set}^{\mathcal{C}^{op}}$ has, using Day's construction, enough structure to interpret all of the connectives of propositional **BI**.[1]

[1]In Chapter 13, we shall see that we can also interpret predicate **BI** in presheaf categories, $\mathbf{Set}^{\mathcal{C}^{op}}$, in which $\mathcal{C}$ is small monoidal.

We can now define *Kripke models* of propositional **BI**. Our definition lives in the category $\mathbf{Set}^{\mathcal{M}^{op}}$, where $\mathcal{M} = (M, e, \cdot, \sqsubseteq)$ is a preordered commutative monoid, considered as a preordered monoidal category with an arrow from m to n if and only if $m \sqsubseteq n$.

Our definition of satisfaction is consistent with the definitions of the structure of $\otimes$ and $\multimap$ provided by (4.1) and (4.2) but enforces the structure they define via a forcing relation, $\models$. This setting is consistent with the BHK semantics for **BI**, developed in Chapter 3, in which we require that $*$ and $-\!\!*$ *be interpreted* using the (4.1) and (4.2). In the sequel, we will see that the basic truth-functional semantics described in the Introduction may be recovered as a simple restriction of this semantics. For now, it is sufficient to require that propositions be interpreted as objects of $\mathbf{Set}^{\mathcal{M}^{op}}$, a framework which ensures that we have enough structure to define an interpretation function and a satisfaction relation for propositions containing $*$ and $-\!\!*$.

DEFINITION 4.1 (KRIPKE MODELS) *Let $\mathcal{M} = (M, e, \cdot, \sqsubseteq)$ be a preordered commutative monoid, considered as a preordered commutative monoidal category with an arrow from m to n if and only if $m \sqsubseteq n$, and let $\mathcal{P}(L)$ denote the collection of* **BI** *propositions over a language L of propositional letters. A* Kripke model *is a triple*

$$\langle [\mathcal{M}^{op}, \mathbf{Set}], \models, [\![-]\!] \rangle,$$

where $[\mathcal{M}^{op}, \mathbf{Set}]$ is the category of presheaves over the preorder category $\mathcal{M}$ to **Set**, *$\models \subseteq M \times \mathcal{P}(L)$ is a satisfaction relation satisfying the constraints in Table 4.1 and $[\![-]\!] : \mathcal{P}(L) \rightharpoonup obj([\mathcal{M}^{op}, \mathbf{Set}])$ is a partial function from the* **BI** *propositions over L to the objects of $[\mathcal{M}^{op}, \mathbf{Set}]$ such that:*[2]

Kripke monotonicity (or Hereditary): If $n \sqsubseteq m$, then, for each $\phi \in \mathcal{P}(L)$, $m \models \phi$ implies $n \models \phi$.

Wherever no confusion will arise, we shall refer to a model

$$\langle [\mathcal{M}^{op}, \mathbf{Set}], \models, [\![-]\!] \rangle$$

simply as $\mathcal{M}$. □

To see that this definition is consistent with the sub-object classifier semantics of intuitionistic logic [Lambek and Scott, 1986] consider the pullback diagram in $\mathbf{Set}^{\mathcal{M}^{op}}$,

[2] We use a partial function because the interpretation of any given proposition need not be defined in all models.

$m \models \mathrm{p}$	iff	$[\![\mathrm{p}]\!](m) \neq \emptyset$, for $\mathrm{p} \in L$
$m \models \phi * \psi$	iff	for some $n, n' \in M$, $m \sqsubseteq n \cdot n'$, $n \models \phi$ and $n' \models \psi$
$m \models \phi \mathrel{-\!\!*} \psi$	iff	for all $n \in M$ $n \models \phi$ implies $m \cdot n \models \psi$
$m \models \phi \wedge \psi$	iff	$m \models \phi$ and $m \models \psi$
$m \models \phi \vee \psi$	iff	$m \models \phi$ or $m \models \psi$
$m \models \phi \rightarrow \psi$	iff	for all $n \sqsubseteq m$, $n \models \phi$ implies $n \models \psi$
$m \models \top$	for all	$m \in M$
$m \models I$	iff	$m \sqsubseteq e$
$m \not\models \bot$	for any	m

Table 4.1. Kripke Semantics

$$\begin{tikzcd}
h^m \arrow[r, "\mu"] \arrow[d, "\bigcirc_{h^m}"'] & {[\![\mathrm{p}]\!]} \arrow[d, "\chi_m"] \\
1 \arrow[r, "\mathrm{true}"'] & \Omega
\end{tikzcd}$$

and note that an arrow $h^m \xrightarrow{\mu} [\![\mathrm{p}]\!]$ is, by the Yoneda lemma, determined uniquely by an element $\check{\mu} \in [\![\mathrm{p}]\!](m)$ and so $[\![\mathrm{p}]\!](m)$ is non-empty just in case the square commutes. Note that it is conceptually important, in our setting, that propositions ϕ are interpreted as $[\![\phi]\!] \in obj([\mathcal{M}^{op}, \mathbf{Set}])$, thereby giving access to Day's constructions.

Let $\langle [\mathcal{M}^{op}, \mathbf{Set}], \models, [\![-]\!] \rangle$ be a Kripke model. If $m \models \phi$, then ϕ *is true at* m *in* $\mathcal{M}$. Where necessary, the forcing relation for a model $\mathcal{M}$ may be distinguished as $\models_{\mathcal{M}}$. We write

$$m \models_{\mathcal{M}} \Gamma \qquad \text{iff} \qquad m \models_{\mathcal{M}} \phi_{\Gamma},$$

where ϕ_Γ is the formula obtained from Γ by replacing each semicolon with $\wedge$ and each comma with $*$, with association respecting the tree structure of Γ. Then $m, \Gamma \models_{\mathcal{M}} \phi$ if $m \models_{\mathcal{M}} \Gamma$ implies $m \models_{\mathcal{M}} \phi$. This defines the truth of ϕ relative to Γ at m . Similarly, if, for all m in all $\mathcal{M}$, $m \models \phi$, then ϕ *is valid*. We write $\Gamma \models \phi$, if and only if, for all m in all $\mathcal{M}$, $m \models \Gamma$ implies $m \models \phi$. This defines the validity of ϕ relative to Γ.

It is a straightforward matter to generalize Definition 4.1 by replacing the category $[\mathcal{M}^{op}, \mathbf{Set}]$ with the category $[\mathcal{C}^{op}, \mathbf{Set}]$, where $\mathcal{C}$ is any symmetric monoidal category. The form of the necessary changes to the definition of $\models$ may be seen in [Lambek and Scott, 1986].

2. Soundness and Completeness for BI without $\perp$

We establish the soundness of **BI** without $\perp$ for elementary Kripke models. It can be argued that soundness holds even in the presence of $\perp$.

THEOREM 4.2 (SOUNDNESS OF **BI** FOR KRIPKE MODELS) *If* $\Gamma \vdash \phi$ *is provable in* **NBI** *without* $\perp$*, then* $\Gamma \models \phi$*.*

PROOF We pick an arbitrary KRM, $\mathcal{M}$, and proceed by induction on the structure of the proof Φ of $\Gamma \vdash \phi$ in **NBI**. We do just a few illustrative cases.

Axiom: If the last inference in Φ is an axiom inference, then it is immediate that, for any world m in any model $\mathcal{M}$, $m \models \phi$ implies $m \models \phi$.

$-\!\!*$ I: By the induction hypothesis, we have that, for all m, $m \models \Gamma, \phi$ implies $m \models \psi$. We must show that if $m \models \Gamma$, then $m \models \phi -\!\!* \psi$. We have that $m \models \phi -\!\!* \psi$ iff, for all n, $n \models \phi$ implies $m \cdot n \models \psi$. If $n \models \phi$, then $m \cdot n \models \Gamma, \phi$. By the induction hypothesis, $m \cdot n \models \psi$. Therefore $m \models \phi -\!\!* \psi$.

$-\!\!*$ E: Follows easily from the definition of $\models$: we must show that if $m \models \Gamma, \Delta$, then $m \models \phi$. We have that $m \models \Gamma, \Delta$ iff there are n and n' such that $m \sqsubseteq n \cdot n'$ and $n \models \Gamma$ and $n' \models \Delta$. By the induction hypothesis, $n \models \phi -\!\!* \psi$ and $n' \models \phi$. Therefore $n \cdot n' \models \psi$ and so, by monotonicity, $m \models \psi$.

$\to$I: The usual intuitionistic argument, *q.v.* [van Dalen, 1983].

$\to$E: Immediate from the definition of $\models$.

The arguments for the other cases are variations on these. □

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