Chapter 5

TOPOLOGICAL KRIPKE SEMANTICS

1. Topological Kripke Models of Propositional BI with $\bot$

BI's Kripke semantics may be adapted to account for $\bot$ by moving from presheaves (or **Set**-valued functor categories) to sheaves on a topological space. Such a move permits a semantics in which we take an inconsistent world, at which $\bot$ is forced, together with a treatment of disjunction that exploits the structure of a topological space which admits a non-indecomposable[1] treatment of disjunction [Lambek and Scott, 1986]. Although this topological semantics is, perhaps, relatively obscure, our subsequent generalization, in § 3, to a semantics based on Grothendieck sheaves recovers the basis of the semantics in preordered monoids.

To understand the apparent need for non-indecomposability, consider a semantics based on a preordered monoid of worlds with an initial object, 0, used to interpret $\bot$, *i.e.*,

$$m \models \bot \quad \text{iff} \quad m = 0,$$

but with the indecomposable treatment of $\vee$,

$$m \models \phi \vee \psi \quad \text{iff} \quad m \models \phi \text{ or } m \models \psi,$$

i.e., rather than require (*cf.* [Lambek and Scott, 1986]) a cover of the world m by worlds n and n' and that both $n \models \phi$ and $n' \models \psi$, we simply

[1] An object C in a topos is ***indecomposable*** if, for all arrows $k : D \longrightarrow C$ and $l : E \longrightarrow C$ such that $[k + l] : D + E \longrightarrow C$ is an epimorphism, either k or l is an epimorphism.

require that m force ϕ or ψ. Then we find that

$$e \models ((\phi \multimap \perp) \multimap \perp) \vee (\phi \multimap \perp) \tag{5.1}$$

holds in this semantics even though

$$((\phi \multimap \perp) \multimap \perp) \vee (\phi \multimap \perp)$$

is not a theorem of **BI**. The failure of theoremhood may be seen from **BI**'s cut-free sequent calculus. To see that (5.1) holds, consider that $e \models \phi \multimap \perp$ either holds or not. If it does, then we are done immediately by the definition of $\models$. If not, then, by the definition of models, we can show that

$$\text{there exists a world } n \text{ such that } n \models \phi \text{ and } n \neq 0. \tag{5.2}$$

Now, suppose $e \models (\phi \multimap \perp) \multimap \perp$. By the definition of $\models$, this is the case if and only if, for all m, if $m \models \phi \multimap \perp$, then $m = 0$. A straightforward calculation reduces this to:

$$\text{for all } m \text{ there exists } n \text{ such that } n \models \phi \text{ and } n \neq 0, \text{ or } m = 0.$$

The result now follows from (5.2).

We remark that our topological interpretations of **BI** suggest a "spatial" interpretation of **BI**'s multiplicatives. We return to this remark in Chapters 9 and 16.

Recall, from Chapter 3, that a *topological monoid* $(\mathcal{X}, *, e)$ is a monoid in the category **Top** of topological spaces, *i.e.*, a topological space

$$(| \mathcal{X} |, \mathcal{O}(\mathcal{X}))$$

on which is defined a tensor product $* : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{X}$, with unit $e : 1 \longrightarrow \mathcal{X}$, such that usual monoidal diagrams commute [Mac Lane, 1971]. Recall also, from Lemma 3.1, that, in any topological monoid, the following distributive law holds: for all open sets U, V_i, for $i \in \mathcal{I}$, where $\mathcal{I}$ is some index set,

$$U * (\bigcup_i V_i) = \bigcup_i (U * V_i).$$

If $(| \mathcal{X} |, \Omega(\mathcal{X}))$ is a topological space and if $(*, e)$ are defined as above, then we refer to the *topological monoid* $(\mathcal{X}, *, e)$ *on* $(| \mathcal{X} |, \Omega(\mathcal{X}))$ *and* $(*, e)$. If $*$ and e are open, then we speak of the *open topological monoid* $(\mathcal{X}, *, e)$ *on* $(| \mathcal{X} |, \Omega(\mathcal{X}))$ *and* $(*, e)$.

LEMMA 5.1 *Let $(\mathcal{X}, *, e)$ be a topological monoid and let $U = \bigcup_i U_i$ and $V = \bigcup_j V_j$ be open covers. Then*

$$U * V \quad = \quad \bigcup_i \bigcup_j (U_i * V_j)\,.$$

PROOF An immediate consequence of the distributivity of $*$ over arbitrary unions. □

We must develop a few properties of sheaves. We follow the terminology and notation for sheaves used in [Lambek and Scott, 1986]. Similar structure is discussed in [Warner, 1983]. In particular, given a sheaf $F : \Omega(\mathcal{X})^{op} \longrightarrow \mathbf{Set}$, we have a mapping $F_{UV} : F(U) \longrightarrow F(V)$ just in case $V \subseteq U$. An $s \in F(U)$ is called *section over* U and we write $F_{UV}(s) = s\,|_V$ to denote the *restriction of* s *to* V.

LEMMA 5.2 (DAY'S PRODUCT FOR SHEAVES) *Let F and G be sheaves on a topological space $(|\mathcal{X}|, \Omega(\mathcal{X}))$. Let $(\mathcal{X}, *, e)$ be the open topological monoid on $(|\mathcal{X}|, \Omega(\mathcal{X}))$ and $(*, e)$. Then the functor $F \otimes G : \Omega(\mathcal{X})^{op} \longrightarrow$ **Set** defined by the co-end*

$$(F \otimes G)W = \int^{U,V} F(U) \times G(V) \times \Omega(\mathcal{X})^{op}(U * V, W),$$

i.e., *Day's tensor product, is a sheaf, as is the unit of $\otimes$, $\Omega(\mathcal{X})^{op}(-, I)$.*

PROOF Firstly, the product. Let $\bigcup_i U_i$ be an open cover of U and $\bigcup_j V_j$ be an open cover of V. By hypothesis, we have that there is a unique s in $F(U)$ such that $s\,|_{U_i} = s_i$, for all i, and that there is a unique t in $G(V)$ such that $t\,|_{V_j} = t_j$, for all j. Let $\bigcup_k W_k$ be an open cover of $W \subseteq U * V$ (so that $\bigcup_k W_k \subseteq (\bigcup_i U_i) * (\bigcup_j V_j)$). Since $\Omega(\mathcal{X})$ is a preorder, we require a unique r in $(F \otimes G)W$ such that

$$r\,|_{W_k} = r_k$$

for all i, j. Each $(F \otimes G)W_k$ is some set of pairs $[a, b]$ in which a is an element of some U_i and b is an element of some V_j. So we set $r = [s, t]$, Day's pairing of s and t.

Secondly, the unit. We must show that $\Omega(\mathcal{X})^{op}(-, e)$ is a sheaf. Let $\bigcup_i U_i$ be an open cover of U. We must show that there is a unique s in $\Omega(\mathcal{X})^{op}(U, e)$ such that $s\,|_{U_i} = s_i$, for all i. But $\Omega(\mathcal{X})^{op}$ is a preorder, so

$$\Omega(\mathcal{X})^{op}(U, I) \quad = \quad \begin{cases} \{*\} & \text{if } e \subseteq U \\ \emptyset & \text{otherwise,} \end{cases}$$

where $\{*\}$ denotes the one-element set. The result follows. $\square$

LEMMA 5.3 (DAY'S FUNCTION SPACE FOR SHEAVES) *Let F and G be presheaves on a topological space $(|\mathcal{X}|, \Omega(\mathcal{X}))$. Let $(\mathcal{X}, *, e)$ be the open topological monoid on $(|\mathcal{X}|, \Omega(\mathcal{X}))$ and $(*, e)$. Then the functor $F \multimap G : \Omega(\mathcal{X})^{op} \longrightarrow \mathbf{Set}$ defined by the end*

$$\begin{aligned}(F \multimap G)V &= \textstyle\int_U \mathbf{Set}[F(U), G(V * U)] \\ &\cong \mathbf{Set}^{\Omega(\mathcal{X})^{op}}[F, G(V * -)],\end{aligned}$$

i.e., *the right adjoint of $\otimes$, determines a sheaf.*

PROOF From the isomorphism given above, we can see that the proof must be similar to that for the intuitionistic function space.

We define

$$(F \multimap G)V = \mathbf{Set}^{\Omega(\mathcal{X})^{op}}[F(-), G(V * -)]$$

so that $(F \multimap G)V$ is the set of families $v(U) : F(U) \longrightarrow G(V * U)$ of functions, indexed by open sets U, such that, for all $U' \subseteq U$, the following naturality diagram commutes:

$$\begin{tikzcd} F(U) \arrow[r, "v(U)"] \arrow[d, "F(U' \subseteq U)"'] & G(V * U) \arrow[d, "G(1_V * (U' \subseteq U))"] \\ F(U') \arrow[r, "v(U')"'] & G(V * U') \end{tikzcd}$$

Let $\bigcup_j V_j$ be an open cover of V. We must show that there is a unique v in $(F \multimap G)(V)$ such that $v|_{V_j} = v_j$, for all j. This follows immediately from the following commuting naturality diagram, for each j,

$$\begin{tikzcd} U & F(U) \arrow[r, "v(U)"] \arrow[d] & G(V * U) \arrow[d, "G((V_j \subseteq V) * (U' \subseteq U))"'] & s \arrow[d, mapsto] & V \\ U' \arrow[u, "U' \subseteq U"] & F(U') \arrow[r, "v_j(U')"'] & G(V_j * U') & s_j & V_j \arrow[u, "V_j \subseteq V"'] \end{tikzcd}$$

which exists because $V_j \cdot U' \subseteq V * U$ and, since G is a sheaf, there is a unique $s \in G(V * U)$ such that $s|_{V_j * U} = s_j$ $(\in G(V_j * U))$.

The morphism part of the construction, as well as application, are defined in the usual way (see, for example, [O'Hearn et al., 1995]). □

Let $\mathbf{Sh}(\mathcal{X})$ denote the category of sheaves on the (open) topological monoid $(\mathcal{X}, *, e)$.

DEFINITION 5.4 (TOPOLOGICAL KRIPKE MODELS) *Let $(\mathcal{X}, *, e)$ be an open topological monoid and let $\mathcal{P}(L)$ denote the collection of* **BI** *propositions over a language L of propositional letters. A* topological Kripke model *is a triple*

$$\langle \mathbf{Sh}(\mathcal{X}), \models, [\![-]\!] \rangle,$$

where $\models \subseteq \mathcal{O}(X) \times \mathcal{P}(L)$, satisfying the conditions in Table 5.1 and $[\![-]\!] : \mathcal{P}(L) \rightharpoonup \mathbf{Sh}(\mathcal{X})$ is a partial function from the **BI** *propositions over L to the objects of $\mathbf{Sh}(\mathcal{X})$ such that:*

> *Kripke monotonicity: If $V \subseteq U$, then, for each $\phi \in \mathcal{P}(L)$, $U \models \phi$ implies $V \models \phi$.*

As before, wherever no confusion will arise, we shall refer to a model

$$\langle \mathbf{Sh}(\mathcal{X}), \models, [\![-]\!] \rangle$$

simply as $\mathcal{X}$. □

We define truth and validity for topological Kripke models just as in Chapter 4.

To see that this definition is consistent with the sub-object classifier semantics of intuitionistic logic [Lambek and Scott, 1986] consider the pullback diagram in $\mathbf{Sh}(\mathcal{X})$,

$$\begin{tikzcd} h^U \arrow[r, "\mu"] \arrow[d, "\bigcirc_{h^U}"'] & [\![\mathrm{p}]\!] \arrow[d, "\chi_U"] \\ 1 \arrow[r, "\top"'] & \Omega \end{tikzcd}$$

and note that an arrow $h^U \xrightarrow{\mu} [\![\mathrm{p}]\!]$ is, by the Yoneda lemma, determined uniquely by an element $\check{\mu} \in [\![\mathrm{p}]\!](U)$.

2. Soundness and Completeness for BI with $\perp$

THEOREM 5.5 (SOUNDNESS) *If $\Gamma \vdash \phi$ is provable in* **NBI**, *then $\Gamma \models \phi$.*

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