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INTUITIONISTIC LOGIC

INTRODUCTION

Among these logics that deal with the familiar connectives and quantifiers two stand out as having a solid philosophical–mathematical justification. On the one hand there is a classical logic with its ontological basis and on the other hand intuitionistic logic with its epistemic motivation. The case for other logics is considerably weaker; although one may consider intermediate logics with more or less plausible principles from certain viewpoints none of them is accompanied by a comparably compelling philosophy. For this reason we have mostly paid attention to pure intuitionistic theories.

Since Brouwer, and later Heyting, considered intuitionistic reasoning, intuitionistic logic has grown into a discipline with a considerable scope. The subject has connections with almost all foundational disciplines, and it has rapidly expanded.

The present survey is just a modest cross-section of the presently available material. We have concentrated on a more or less semantic approach at the cost of the proof theoretic features. Although the proof theoretical tradition may be closer to the spirit of intuitionism (with its stress on *proofs*), even a modest treatment of the proof theory of intuitionistic logic would be beyond the scope of this chapter. The reader will find ample information on this particular subject in the papers of, e.g. Prawitz and Troelstra.

For the same reason we have refrained from going into the connection between recursion theory and intuitionistic logic. Section 8 provides a brief introduction to realizability.

Intuitionistic logic is, technically speaking, just a subsystem of classical logic; the matter changes, however, in higher-order logic and in mathematical theories. In those cases specific intuitionistic principles come into play, e.g. in the theory of choice sequences the meaning of the prefix $\forall\xi\exists x$ derives from the nature of the mathematical objects concerned. Topics of the above kind are dealt with in Section 9.

The last sections touch on the recent developments in the area of categorical logic. We do not mention categories but consider a very special case. There has been an enormous proliferation in the semantics of intuitionistic second-order and higher-order theories. The philosophical relevance is quite often absent so that we have not paid attention to the extensive literature on independence results. For the same reason we have not incorporated the intuitionistic ZF-like systems.

D.M. Gabbay and F. Guenthner (eds.),
Handbook of Philosophical Logic, 2nd Edition, Volume 5, 1–114.

Intuitionistic logic can be arrived at in many ways—e.g. physicalistic or materialistic—we have chosen to stick to the intuitionistic tradition in considering mathematics and logic as based on *human mental* activities. Not surprisingly, intuitionistic logic plays a role in constructive theories that do not share the basic principles of intuitionism, e.g. Bishop's constructive mathematics. There was no room to go into the foundations of these alternatives to intuitionism. In particular we had to leave out Feferman's powerful and elegant formalisations of operations and classes. The reader is referred to Beeson [1985] and Troelstra and van Dalen [1988] for this and related topics.

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1 A SHORT HISTORY

Intuitionism was conceived by Brouwer in the early part of the twentieth century when logic was still in its infancy. Hence we must view Brouwer's attitude towards logic in the light of a rather crude form of theoretical logic. It is probably a sound conjecture that he never read Frege's fundamental expositions and that he even avoided Whitehead and Russell's *Principia Mathematica*. Frege was at the time mainly known in mathematical circles for his polemics with Hilbert and others, and one could do without the *Principia Mathematica* by reading the fundamental papers in the journals. Taking into account the limited amount of specialised knowledge Brouwer had of logic, one might well be surprised to find an astute appraisal of the role of logic in Brouwer's Dissertation [Brouwer, 1907]. Contrary to most traditional views, Brouwer claims that logic does not precede mathematics, but, conversely, that logic depends on mathematics. The apparent contradiction with the existing practice of establishing strings of 'logical' steps in mathematical reasoning, is explained by pointing out that each of these steps represents a sequence of mathematical constructions. The logic, so to speak, is what remains if on takes away the specific mathematical constructions that lead from one stage of insight to the next.

Here it is essential to make a short excursion into the mathematical and scientific views that Brouwer held and that are peculiar to intuitionism. Mathematics, according to Brouwer, is a mental activity, sometimes described by him as the exact part of human thought. In particular, mathematical objects are mental constructions, and properties of these objects are established by, again, mental constructions. Hence, in this view, something holds for a person if he has a construction (or proof) that establishes it. Language does not play a role in this process but may be (and in practice: is) introduced for reasons of communication. 'People try by means of sounds and symbols to originate in other copies of mathematical constructions and reasonings which they have made themselves; by the same means they try to aid their own memory. In this way *mathematical language* comes into being, and as its special case *the language of logical reasoning*'. The next step taken by man is to consider the language of logical reasoning mathematically, i.e. to study its mathematical properties. This is the birth of *theoretical logic.*

Brouwer's criticism of logic is two-fold. In the first place, logicians are blamed for giving logic precedence over mathematics, and in the second place, logic is said to be unreliable (Brouwer [1907; 1908]). In particular, Brouwer singled out the *principle of the excluded third* as incorrect and unjustified. The criticism of this principle is coupled to the criticism of Hilbert's famous dictum that 'each particular mathematical problem can be solved in the sense that the question under consideration can either be affirmed, or refuted' [Brouwer, 1975, pp. 101 and 109].

Let us, by way of example, consider Goldbach's Conjecture, G, which states that each even number is the sum of two odd primes. A quick check tells us that for small numbers the conjecture is borne out: $12 = 5 + 7$, $26 = 13 + 13$, $62 = 3 + 59$, $300 = 149 + 151$. Since we cannot perform an infinite search, this simple method of checking can at best provide, with luck, a counter example, but not a proof of the conjecture. At the present stage of mathematical knowledge no proof of Goldbach's conjecture, or of its negation, has been provided. So can we affirm $G \vee \neg G$? If so, we should have a construction that would decide which of the two alternatives holds and provide a proof for it. Clearly we are in no position to exhibit such a construction, hence we have no grounds for accepting $G \vee \neg G$ as correct.

The undue attention paid to the principle of the excluded third, had the unfortunate historical consequence that the issues of the foundational dispute between the Formalists and the Intuitionists were obscured. An outsider might easily think that the matter was a dispute of two schools-one with, and one without, the *principle of the excluded third* (or *middle*), PEM for short. Brouwer himself was in no small degree the originator of the misunderstanding by choosing the far too modest and misleading title of 'Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten' for his first fundamental paper on intuitionistic mathematics. For the philosophical-mystical background of Brouwer's views, see [van Dalen, 1999a]; a foundational exposition can be found in [van Dalen, 2000].

The logic of intuitionism was not elaborated by Brouwer, although he proved its first theorem: $\neg\varphi \leftrightarrow \neg\neg\neg\varphi$.

The first mathematicians to consider the logic of intuitionism in a more formal way were Glivenko and Kolmogorov.

The first presented a fragment of propositional logic and the second a fragment of predicate logic. In 1928 Heyting independently formalised intuitionist predicate logic and the fundamental theories of arithmetic and 'set theory' [Heyting, 1930]. For historical details, cf. Troelstra [1978; 1981]. Heyting's formalization opened up a new field to adventurous logicians, but it did not provide a 'standard' or 'intended' interpretation, thus lacking the inner coherence of a conceptual explanation. In a couple of papers (cf. [Heyting, 1934]), Heyting presented from 1931 on the interpretation that we have come to call the *proof-interpretation* (cf. [Heyting, 1956, Chapter VII]). The underlying idea traces back to Brouwer: the truth of a mathematical statement is established by a proof, hence the meaning of the logical connective has to be explained in terms of proofs and constructions (recall that a proof is a kind of construction). Let us consider one connective, by way of example: A proof of $\varphi \to \psi$ is a construction which converts any proof of φ into a proof of ψ.

Note that this definition is in accord with the conception of mathematics (and hence logic) as a mental constructive activity. Moreover it does not require statements to be bivalent, i.e. to be either true or false. For example, $\varphi \to \varphi$ is true independent of our knowledge of the truth of φ. The proof-interpretation provided at least an informal insight into the mysteries of intuitionistic truth, but it lacked the formal clarity of the notion of truth in classical logic with its completeness property.

An analogue of the classical notion of truth value was discovered by Tarski, Stone and others who had observed the similarities between intuitionistic logic and the closure operation of topology (cf. [Rasiowa and Sikorski, 1963]). This so-called *topological interpretation* of intuitionistic logic also covers a number of interpretations that at first sight might seem to be totally devoid of topological features. Among these are the lattice (like) interpretations of Jaskowski, Rieger and others, but also the more recent interpretations of Beth and Kripke. All these interpretations are grouped together as semantical interpretations, in contrast to interpretations that are based on algorithms, one way or another.

A breakthrough in intuitionistic logic was accomplished by Gentzen in 1934 in his system of *Natural Deduction* (and also his *calculus of sequents*), which embodied the meaning of the intuitionistic connectives far more accurately than the existing Hilbert-type formalizations. The eventual recognition of Gentzen's insights is to a large extent due to the efforts of Prawitz who reintroduced Natural Deduction, and considerably extended Gentzen's work [1965; 1971].

In the beginning of the thirties the first meta-logical results about intuitionistic logic and its relation to existing logics appeared. Gödel, and independently Gentzen, formulated a translation of classical predicate logic into a fragment of intuitionistic predicate logic, thus extending early work of Glivenko [Glivenko, 1929; Gentzen, 1933; Gödel, 1932].

Gödel also established the connection between the modal logic **S4** and intuitionistic logic [Gödel, 1932].

The period after the Second World War brought new researchers to intuitionistic logic and mathematics. In particular Kleene, who based an 'effective' interpretation of intuitionistic arithmetic on the notion of recursive function. His interpretation is known as *realizability* (Kleene [1952; 1973]). In 1956 Beth introduced a new semantic interpretation with a better foundational motivation than the earlier topological interpretations, and Kripke presented a similar, but more convenient interpretation in 1963 [Kripke, 1965]. These new semantics showed more flexibility than the earlier interpretations and lent themselves better to the model theory of concrete theories. General model theory in the lattice and topological tradition had already been undertaken by the Polish school (cf. [Rasiowa and Sikorski, 1963]).

In the meantime Gödel had presented his Dialectica Interpretation [1958], which like Kleene's realizability, belongs to the algorithmic type of interpretations. Both the realizability and the Dialectica Interpretation have shown to be extremely fruitful for the purpose of Proof Theory.

Another branch at the tree of semantic interpretations appeared fairly recently, when it was discovered that sheaves and topoi present a generalisation of the topological interpretations [Goldblatt, 1979; Troelstra and van Dalen, 1988].

The role of a formal semantics will be expounded in Section 3. Its most obvious and immediate use is the establishing of underivability results in a logical calculus. However, even before a satisfactory semantics was discovered, intuitionists used to show that certain classical theorems were not valid by straightforward intuitive methods. We will illustrate the naive approach for two reasons. In the first place it is direct and the first thing one would think of, in the second place it has its counterparts in formal semantics and can be useful as a heuristics.

The traditional counterexamples are usually formulated in terms of a particular unsolved problem. The problem in the following example goes back to Brouwer. Consider the decimal expansion of $\pi : 3,14\ldots$, hardly anything is known about regularities in this expansion, e.g. it is not known if it contains a sequence of 9 nines. Let $A(n)$ be the statement 'the nth decimal of π is a nine and it is preceded by 8 nines'.

1. The principle of the excluded third is not valid.
   Suppose $\exists x A(x) \vee \neg\exists x A(x)$, then we would have a proof that either provides us with a natural number n such that $A(n)$, or that shows us that no such n exists. Since there is no such evidence available we cannot accept the principle of the excluded third.

2. The double negation principle is not valid. Observe that $\neg\neg(\exists x A(x) \vee \neg\exists x A(x))$ holds. In general the double negation of the principle of the excluded third holds, since $\neg\neg(\varphi \vee \neg\varphi)$ is equivalent to $\neg(\neg\varphi \wedge \neg\neg\varphi)$ and the latter is correct on the intuitive interpretations.

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