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# MORE FREE LOGIC

By a *free* logic is generally meant a variant of classical first-order logic in which constant terms may, under interpretation, fail to refer to individuals in the domain $D$ over which the bound variables range, either because they do not refer at all or because they refer to individuals outside $D$. If $D$ is identified with what is assumed by the given interpretation to exist, in accord with *Quine's dictum* that "to be is to be the value of a [bound] variable,"[1] then a free variation on classical semantics does not require that all constant terms refer to existents, and in this sense such terms lack existential import.

Classical semantics treats free variables like constants, at least in the quantifier clause of the valuation rules. When we stipulate that $\exists xA$ is true iff $A$ is true for some assignment of a value $\alpha(x)$ in $D$ to $x$, we are treating $x$ at its free occurrences in $A$ as a constant that refers to $\alpha(x)$. In free semantics, free variables are also generally treated like constants, which means that they need not be assigned values in $D$; thus free variables and variable terms (such as $x + y$ or $1/x$) constructed from them also lack existential import. However, when reckoning the truth of $\exists xA$ in terms of the truth of $A$ for assignments of values to $x$, we consider only assignments $\alpha$ for which $\alpha(x) \in D$. Thus, although neither constant nor variable terms need refer to individuals in $D$, free semantics honors Quine's dictum.[2]

In classical semantics, free variables have existential import because $D$ is non-empty: there is always something in $D$ for $x$ to be assigned by $\alpha$. Variants of classical semantics in which this requirement is relaxed so that $D$ may be empty are said to be *inclusive*. A semantics that is free and inclusive is said to be *universally* free: the range of the bound variables may be empty, and even if it is not, neither constant nor variable terms have existential import.

This survey of free logic will begin by considering its motivation, then move to reviewing various kinds of free semantics and the syntactic proof systems designed to capture the forthcoming notions of logical truth or logical consequence, and conclude by describing some applications of free logics, notably free description theory. As this summary may suggest, my emphasis throughout will be on semantics. The account is self-contained

[1] Quine [1948, p. 15]. That Quine means *bound* variables here is clear from his earlier statement [p. 13] that "a theory is committed to those and only those entitites to which the bound variables of the theory must be capable of referring in order for the affirmations made in the theory to be true."

[2] Compare Bencivenga's [1986, p. 375] characterization: "A free logic is a formal system of quantification theory, with or without identity, which allows for some singular terms in some circumstances to be thought of as denoting no existing object, and in which quantifiers are invariably thought of as having existential import."



*D.M. Gabbay and F. Guenthner (eds.),*
*Handbook of Philosophical Logic, 2nd Edition, Volume 5,* 197–259.

and does not presuppose familiarity with [Bencivenga, 1986], reproduced in this volume. There is new material on motivation, applications, and neutral free semantics, while areas of overlap differ in detail and emphasis. Semantic options are laid out in greater detail, as are free description theories built upon them, but I pay less attention to the history of ideas.

## 1 QUICK REVIEW OF CLASSICAL FIRST-ORDER LOGIC

Partly to settle notation and terminology, and partly because free logics are variants of it, let us first quickly review classical first-order logic with identity, which we may take to be framed in formal first-order languages $L$.

The logical vocabulary of $L$ includes the identity operator $=$, plus an adequate set of quantifiers and truth-functional operators. Let us assume they are the universal quantifier $\forall$, negation $\neg$, and material conditional $\rightarrow$. The non-logical vocabulary of $L$ includes (individual) variables, plus perhaps constants, ($k-$place) function-names , and ($k-$place) predicates. We need not specify these symbols, which will vary with $L$; its sentences are to represent the logical forms of certain sentences of natural language, and its non-logical vocabulary will be chosen accordingly. Many formulations of free logic employ a 1-place existence predicate $E$ or $E!$, but such a predicate can generally be defined in terms of identity,[3] so we need not include it in the non-logical vocabulary. The proof method of $L$ will require an unbounded list of variables or special constants.

After defining *terms* as (1) variables, (2) names, and (3) complex terms $ft_1 \ldots t_k$, where the $t_i$ are terms and $f$ is a $k-$place function-name, the formation rules of $L$ identify *formulae* as (4) subject-predicate formulae $Pt_1 \ldots t_k$, where the $t_i$ are terms and $P$ is a $k-$place predicate, (5) identities $s = t$, where $s$ and $t$ are terms, (6) negations $\neg A$, where $A$ is a formula, (7) conditionals $(A \rightarrow B)$, where $A$ and $B$ are formulae, and (8) universals $\forall xA$, where $x$ is a variable and $A$ is a formula.[4] Identities and subject-predicate formulae are *atomic;* atomic formulae and their negations are *elementary.*

Subsequently, I shall use the following syntactical variables, with or without subscript: for variables: $x$, $y$, and $z$; for constants: $a$, $b$, and $c$; for function-names: $f$; for predicates: $P$; for terms: $s$ and $t$; for formulae: $A$, $B$, and $C$; for sets of formulae: $X$. Conjunctions $(A\&B)$, disjunctions $(A \vee B)$, biconditionals $(A \leftrightarrow B)$, and existentials $\exists xA$ may be defined as usual in terms of $\neg$, $\rightarrow$, and $\forall$. $s \neq t$ abbreviates $\neg s = t$. The outermost parentheses in conditionals, conjunctions, disjunctions, and biconditionals standing alone will be omitted. $ft_1 \ldots t_k$ and $Pt_1 \ldots t_k$ will be used with

[3] For exceptions, see [Garson, 1991], discussed below in Section 5.3, and [Gumb, 1998].

[4] To avoid the notational clutter that attends the use of single- and quasi-quotation, I shall generally follow Church [1956] in using symbols of $L$ as names for themselves and juxtaposition for juxtaposition.

the assumption that $f$ and $P$ are $k-$place; where necessary, commas and parentheses will disambiguate expressions, as in $Pf(x,y)$, and may also be inserted to enhance readability, as in $\exists x(x = fx)$.

An occurrence of a term $t$ in a formula $A$ is *bound* in $A$ provided it is an occurrence in a part $\forall xB$ of $A$, where $x$ occurs in $t$; an occurrence of $t$ in $A$ is *free* if it is not bound. The *bound (free)* variables of $A$ are those with a bound (free) occurrence in $A$. A *sentence* is a formula without free variables. $A(x_1, \ldots, x_k/t_1, \ldots, t_k)$ is the result of simultaneously replacing the $x_i$ at each free occurrence in $A$ by $t_i$, having (if necessary) first made such occurrences *free for* $t_i$ in $A$: if a free occurrence of $x_i$ in $A$ is in a part $\forall yB$, where $y$ occurs in $t_i$, replace each occurrence of $y$ in $\forall yB$ by the first variable that occurs in neither $A$ nor any of the $t_j$; relabel the result '$A$' and repeat until there are no such occurrences. I shall write $A(x_1, \ldots, x_k)$ for $A$ and $A(t_1, \ldots, t_k)$ for $A(x_1, \ldots, x_k/t_1, \ldots, t_k)$. $\exists!xA$ or $\exists!xA(x)$ abbreviates $\exists x\forall y(A(x/y) \leftrightarrow y = x)$, where $y$ is not $x$. In writing $\exists x(x = t)$, I assume that $x$ does not occur in $t$. The *universal closure* $\forall A$ of $A$ is $\forall x_1 \ldots \forall x_k A$, where the free variables of $A$ are $x_1, \ldots, x_k$.

An *interpretation* $I$ of $L$ is a pair $\langle D, d\rangle$, where $D$ is a set and $d$ is a denotation function defined on the constants, function-names, and predicates of $L$, such that:

i1. $D$ is non-empty;

i2. $d(a) \in D$;

i3. If $f$ is $k-$place, $d(f)$ is a total $k - ary$ function $D \to D$.

i4. If $P$ is $k-$place, $d(P)$ is a $k - ary$ relation in $D$.

An *assignment* $\alpha$ is a function that assigns individuals $\alpha(x)$ in $D$ to the variables. An $x$-variant of $\alpha$ is an assignment that differs from $\alpha$ at most at $x$.

Under $I$ and $\alpha$, terms refer to individuals of $D$ according to the reference rules:

r1. $x$ refers to $\alpha(x)$.

r2. $a$ refers to $d(a)$

r3. $ft_1 \ldots t_k$ refers to $d(f)(\alpha_1, \ldots, \alpha_k)$, if $t_i$ refers to $\alpha_i$.

Under $I$ and $\alpha$, formulae are true or false (and false if not true) according to the valuation rules:

v1. $Pt_1 \ldots t_k$ is true iff $\langle\alpha_1, \ldots, \alpha_k\rangle \in I(P)$, if $t_i$ refers to $\alpha_i$.

v2. If $s$ refers to $\alpha$ and $t$ to $\beta$, then $s = t$ is true iff $\alpha$ is $\beta$.

v3. $\neg A$ is true iff $A$ is false.

v4. $A \rightarrow B$ is false iff $A$ is true and $B$ is false.

v5. $\forall x A$ is false iff $A$ is false for some $x$-variant of $\alpha$.[5]

Since the referents of terms without variables and the truth-values of sentences are independent of $\alpha$, I shall speak of referents and truth-values under $I$ in such cases.

Logical relations and properties are defined as usual in terms of the totality of interpretations: $A$ is a *logical consequence* of $X (X \models A)$ iff there is no interpretation and assignment under which all the $X$-formulae are true and $A$ is false; $X$ is *satisfiable* iff there is some interpretation and assignment under which all the $X$-formulae are true; $A$ is *logically true (false)* iff $A$ is true (false) under each interpretation and assignment; $A$ and $B$ are *logically equivalent* iff, under each interpretation and assignment, $A$ is true iff $B$ is true. $A_1, \ldots, A_k \models B$ means: $\{A_1, \ldots, A_k\} \models B$. $X, A \models B$ means: $X \cup \{A\} \models B$. $X \not\models A$ means: not $X \models A$.

These definitions embody what Kleene [1967, p. 103] terms the *conditional* reading of free variables: free variables are treated by r1 as names of $D$-individuals. By contrast, the *generality* reading treats free variables as if they were universally quantified. It may be captured by stipulating that $A$ is true (false) under $I$ iff $A$ is true (false) under $I$ and $\alpha$ *for each* $\alpha$. We can then drop "and assignment" from the above definitions. However, we end up with weaker notions of logical consequence and logical equivalence (and a stronger notion of satisfiability). For the logical consequence relation $\models_g$, we have $X \models_g A$ iff $\forall X \models \forall A$, where $\forall X = \{\forall B : B \in X\}$, so that $X \models_g A$ if $X \models A$ but not conversely (*e.g.*, $Px \models_g \forall x Px$, but $Px \not\models \forall x Px$). If $X$ is a set of sentences, the two consequence relations coincide, since $X \models A$ iff $X \models \forall A$ and here we have $\forall X = X$.

From the semantic perspective assumed here, the aim of proof theory is to provide syntactic characterizations of logical properties and relations, which are defined in semantic terms. In particular, we want a syntactic notion of *proof from hypotheses* that captures the logical consequence relation: $A$ is provable from hypotheses in $X (X \vdash A)$ iff $A$ is a logical consequence of $X (X \models A)$, at least if $X$ is a set of sentences. A proof system with this property is said to be *strongly complete*. A proof system in which $A$ is

[5]Most presentations of free logic give a substitutional account of quantification, on which v5 would read instead: $\forall x A(x)$ is false iff $A(a)$ is false for some constant $a$. If $\forall x$ is to have the force of 'for all individuals $x$', every individual in $D$ must be named by some constant or other. If $D$ is uncountable, the terms and formulae of $L$ will then be undecidable. This awkward result may be avoided by proving, *via* the Löwenheim-Skolem theorem, that interpretations may be restricted to countable universes without altering logical consequence relations, so that no more than a countable infinity of constants need be assumed. By contrast, the objectual account of quantification given in v5 does not require an elaborate justification.

provable (from no hypotheses) iff $A$ is logically true ($\vdash A$ iff $\models A$) is *weakly complete.* If detachment or *modus ponens*

MP $$A, A \to B \vdash B$$

holds, and the deduction theorem or conditional proof

CP $$X \vdash A \to B \text{ provided } X, A \vdash B$$

holds for sentences $A$, then weak completeness is equivalent to: $X \vdash A$ iff $X \models A$ for *finite* sets $X$ of sentences.

Many strongly complete systems in a variety of styles are known for classical first-order semantics. It will be useful to give one that can be modified in simple ways to capture logical consequence for at least some free variations on classical semantics. The simplest proof systems to describe are Hilbert-style systems, which specify logical axioms and rules of inference, and define a proof of $A$ from hypotheses $X$ as a finite sequence $\langle A_1, \ldots, A_k \rangle$ such that $A_k = A$ and each $A_i$ is either a member of $X$, or a logical axiom, or is derived from previous formulae in the sequence by a rule of inference. Unlike natural deduction systems, in which some inference rules (such as CP) are conditional, those of a Hilbert-style system are (like MP) categorical.

Since the propositional part of the system does not matter here, we may adopt the simple inference rule

T $$A_1, \ldots, A_k \vdash B$$

if $B$ is a tautological consequence of $\{A_1, \ldots, A_k\}$, that is, there is no assignment of truth-values to universals and atomic formulae for which each $A_i$ is true and $B$ is false in virtue of rules v3 and v4.

The quantifier rule and axioms are as in [Church, 1956, p. 172]; the rule is generalization:

UG $$A \vdash \forall x A$$

and the axiom schemas are distribution and specification:

A1 $$\forall x(A \to B) \to (A \to \forall x B), \text{ if } x \text{ is not free in } A$$

A2 $$\forall x A(x) \to A(t)$$

Finally, we have the identity axiom schemas:

A3 $$x = x$$

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