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PARTIAL LOGIC

INTRODUCTION

When I was originally asked to write about 'partial logic' for the first edition of the *Handbook*, I was a little puzzled: I was taken to be an expert in an apparently well defined subject area that I didn't know existed. But it turned out to be the sort of thing I had written about in my D.Phil. thesis, so I had somewhere to start. Nowadays the label 'partial logic' is much more familiar, and a lot of work is being done in the area it covers. The bulk of my own work, though—most of it dating right back to thesis days—has not yet been published: I have been bewilderingly bad about this. In particular, the various promises made in the first edition about forthcoming work have still not been fulfilled. In spite of this, I have resisted the temptation just to shove in more material of my own for the second edition—except in small ways here and there. Additions are largely in response to what has newly appeared in print.

A wide range of work will be surveyed (much more now than in the first edition), but the backbone of this chapter is the development of what I call 'simple partial logic'. It is against this backbone that other more sophisticated projects are discussed. Simple partial logic results from the simple-minded following through of the idea that classical logic may be loosened up to cater for non-denoting singular terms and neither-true-nor-false sentences—to cater for them in a uniform way as semantically 'undefined' items—and at the same time to cater for 'partially defined' functors: term-forming functors, predicates, and sentence connectives. These functors have to accommodate undefined arguments, but they may also produce undefined compounds even when all their arguments are fully defined. In particular, we shouldn't ignore sentence connectives of this kind: once loosened up, classical propositional logic needs to be filled out with connectives such as *interjunction* and *transplication*. The uniformity behind all this comes from the idea of representing partial functions by monotonic functions—as explained in Section 1—and using monotonically representable partial functions to interpret functors of whatever logical category.

All sections have undergone some stylistic revision for the second edition, and most of them have been expanded. Note that Section 2 now has more subsections: there is a new introductory subsection, which means that subsections 2.1 to 2.5 have become subsections 2.2 to 2.6; and the old subsection 2.6 has split into three—2.7 to 2.9—so that the old 2.7 is now 2.10. Section 4 has been disrupted in a similar way: the old subsection 4.1 has split into 4.1 and 4.2; subsection 4.3 is new; and the old subsection 4.2 has

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split into 4.4 and 4.5. There has been a more straightforward reorganization to Sections 6 and 7: a new subsection has been introduced as 6.3, which means that the old subsections 6.3 and 6.4 become 6.4 and 6.5; and the old subsection 7.2 has split into two: 7.2 and 7.3. The other Sections retain their original structure.

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Notation for *interjunction*:– In the first edition an interjunction sign was formed by juxtaposing two '$\times$'s: $\times\!\times$. This was a pity, because it made the symbol a bit too flat. Interjunction is a squadging of conjunction and disjunction, and so the symbol for it should be a simultaneous occurrence of '$\wedge$' and '$\vee$': ⩙. Sadly, the notation '$\times\!\times$' has found its way into the literature, and—much worse—this has sometimes become just two 'x's: xx. I urge anyone who wants to write an interjunction sign in the future to avoid 'xx' at all costs: '$\times\!\times$' is tolerable, but I recommend '⩙'.

1 A SKETCH OF SIMPLE PARTIAL LOGIC

1.1 Classical Semantics as Partial Semantics

In classical logic sentences are either true ($\top$) or false ($\bot$) and the interpretation of the standard sentence connectives can be given in the following way:

$$\neg\phi \quad \text{is} \quad \begin{cases} \top & \text{iff} \quad \phi \text{ is } \bot \\ \bot & \text{iff} \quad \phi \text{ is } \top, \end{cases}$$

$$\phi \wedge \psi \quad \text{is} \quad \begin{cases} \top & \text{iff} \quad \phi \text{ is } \top \quad \text{and} \quad \psi \text{ is } \top \\ \bot & \text{iff} \quad \phi \text{ is } \bot \quad \text{or} \quad \psi \text{ is } \bot, \end{cases}$$

$$\phi \vee \psi \quad \text{is} \quad \begin{cases} \top & \text{iff} \quad \phi \text{ is } \top \quad \text{or} \quad \psi \text{ is } \top \\ \bot & \text{iff} \quad \phi \text{ is } \bot \quad \text{and} \quad \psi \text{ is } \bot, \end{cases}$$

$$\phi \rightarrow \psi \quad \text{is} \quad \begin{cases} \top & \text{iff} \quad \phi \text{ is } \bot \quad \text{or} \quad \psi \text{ is } \top \\ \bot & \text{iff} \quad \phi \text{ is } \top \quad \text{and} \quad \psi \text{ is } \bot, \end{cases}$$

$$\phi \leftrightarrow \psi \quad \text{is} \quad \begin{cases} \top & \text{iff} \quad (\phi \text{ is } \top \quad \text{and} \quad \psi \text{ is } \top) \quad \text{or} \quad (\phi \text{ is } \bot \quad \text{and} \quad \psi \text{ is } \bot) \\ \bot & \text{iff} \quad (\phi \text{ is } \top \quad \text{and} \quad \psi \text{ is } \bot) \quad \text{or} \quad (\phi \text{ is } \bot \quad \text{and} \quad \psi \text{ is } \top). \end{cases}$$

For simple partial logic we shall adopt precisely these classical $\top/\bot$ conditions; only we give up the assumption that all sentences have to be classified either as $\top$ or as $\bot$. This leaves room for the classification *neither-$\top$-nor-$\bot$*. At present we are concerned merely to highlight a parallel with classical semantics, and under the parallel we can think of the third classification as a 'truth-value gap'. This thought is taken a little further in Sections 1.2

and 3. But the point, if any, of seeing the third classification as different in philosophical kind from $\top$ and $\bot$ will of course depend on what particular motivation we consider for adopting the forms of partial logic. (See, especially, Sections 2 and 5.)

To interpret universal and existential quantifiers over a given domain D, we shall again exploit the fact that the classical interpretation leaves room for a gap between $\top$ and $\bot$ when we write out $\top$-conditions and $\bot$-conditions separately. Assuming that a language has—or can be extended so as to have—a name $\bar{a}$ for each object a in D,

$$\forall x\phi(x) \quad \text{is} \quad \begin{cases} \top & \text{iff} \quad \phi(\bar{a}) \text{ is } \top \quad \text{for every } a \text{ in } D \\ \bot & \text{iff} \quad \phi(\bar{a}) \text{ is } \bot \quad \text{for some } a \text{ in } D, \end{cases}$$

$$\exists x\phi(x) \quad \text{is} \quad \begin{cases} \top & \text{iff} \quad \phi(\bar{a}) \text{ is } \top \quad \text{for some } a \text{ in } D \\ \bot & \text{iff} \quad \phi(\bar{a}) \text{ is } \bot \quad \text{for every } a \text{ in } D. \end{cases}$$

Most treatments of classical logic stipulate that the domain be non-empty. We shall not be so restrictive: D may be empty.

These $\top/\bot$-conditions for $\forall x$ and $\exists x$ of course presuppose a semantic account of predicate/singular-term composition. And this mode of composition deserves some attention, since it is the most familiar place to locate the cause of a sentence's being neither 'true' nor 'false'. It has been considered to give rise to a truth-value gap in two different ways: either (i) because a term t may lack a denotation and may, for this reason, make a sentence $\phi(t)$ neither true nor false; or (ii) because a predicate $\phi(x)$ may be only 'partially defined'—not either true or false of some object or objects—so that, if t denoted such an object, $\phi(t)$ would be neither true nor false. We shall want to accommodate both these ideas in one uniform account of predicate/singular-term composition. Our approach will be sketched in Section 1.2, along with an approach to functors which form singular terms from singular terms.

But there is one particular atomic predicate to consider immediately: the identity predicate. Once again we can adopt classical $\top$-conditions and $\bot$-conditions *verbatim* for a sentence $t_1 = t_2$:

$$t_1 = t_2 \quad \text{is} \quad \begin{cases} \top & \text{iff} \quad t_1 \text{ and } t_2 \text{ denote the same thing} \\ \bot & \text{iff} \quad t_1 \text{ and } t_2 \text{ denote different things.} \end{cases}$$

This means that if either t_1 or t_2 is non-denoting, then $t_1 = t_2$ is neither $\top$ nor $\bot$. Identity is an untypically straightforward case. At least, so it is if we restrict attention to a determinate relation over a discrete domain of objects—as we shall.

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Whatever general framework we set up for predicate/singular-term composition, our logic has so far been revealed as 'partial' only in the weak sense that it accommodates value-gaps that might arise from the interpretation of non-logical terms or predicates. This is because the interpretation of classical logical vocabulary is classical. But there is a stronger sense of 'partial logic': a logic will be partial in the stronger sense if it provides the resources for explaining why a sentence may be neither $\top$ nor $\bot$ in terms of logical vocabulary—vocabulary, that is, with a fixed meaning in the logic. We should look for modes of logical composition whose interpretation can give rise to truth-value gaps, even when any classical sentence constructed out of the same non-logical vocabulary (with the same interpretation) would have to be either $\top$ or $\bot$.

Assuming that we have worked out the general account of how non-denoting terms can give rise to truth-value gaps, a term-forming descriptions operator would be an example of gap-introducing logical vocabulary. This is because a term $\imath x\phi(x)$ may turn out not to denote, even when $\phi(x)$ is totally defined. Assuming that $\phi(x)$ is in fact totally defined, then the denotation conditions for $\imath x\phi(x)$ must be that if a is an object in the domain, then:

$$\imath x\phi(x) \text{ denotes } a \quad \text{iff} \quad \forall x[x = \bar{a} \leftrightarrow \phi(x)] \text{ is } \top,$$

where, as before, $\bar{a}$ is a name—pre-existing or specially introduced—for a. In other words, $\imath x\phi(x)$ denotes an object if and only if that object uniquely satisfies $\phi(x)$ and is non-denoting if there is no such object. Of course, we also have to consider the case where $\phi(x)$ is not totally defined, but the denotation conditions stated will continue to make sense. Furthermore, given the general constraint to emerge in Section 1.2, they will turn out to be the only possible ones for a determinate relation of identity over a discrete domain of objects (see Section 6.4).

These $\imath$-terms involve a rather complicated route to neither-$\top$-nor-$\bot$ sentences. There is a much more straightforward, and no less interesting, kind of gap-introducing vocabulary: sentence connectives. Consider the following $\top/\bot$-conditions for the connectives $\bowtie$ and $/$, the first of which we shall call *interjunction* and the second *transplication*:

$$\phi \bowtie \psi \quad \text{is} \quad \begin{cases} \top & \text{iff} \quad \phi \text{ is } \top \quad \text{and} \quad \psi \text{ is } \top \\ \bot & \text{iff} \quad \phi \text{ is } \bot \quad \text{and} \quad \psi \text{ is } \bot, \end{cases}$$

$$\phi \,/\, \psi \quad \text{is} \quad \begin{cases} \top & \text{iff} \quad \phi \text{ is } \top \quad \text{and} \quad \psi \text{ is } \top \\ \bot & \text{iff} \quad \phi \text{ is } \top \quad \text{and} \quad \psi \text{ is } \bot. \end{cases}$$

Notice that $\bowtie$ has the $\top$-conditions of $\wedge$ and the $\bot$-conditions of $\vee$, while $/$ has the $\top$-conditions of $\wedge$ but the $\bot$-conditions of $\rightarrow$. And so these connectives clearly meet our desideratum of introducing value gaps: we do not necessarily have to look to predicate/singular-term composition to find a logical explanation why a sentence may be neither $\top$ nor $\bot$. The particular usefulness of $\bowtie$ and $/$ will be touched upon in Section 2.2 and several later sections.

Among our logical vocabulary we shall also include a constantly true sentence $\top$, and a constantly false one $\bot$. Thus we are using '$\top$' and '$\bot$' both as truth-value labels and to stand for logical constants; and, in a similar way, we shall use '$*$' both to label the classification 'neither-$\top$-nor-$\bot$' and to stand for a sentence which is logically neither $\top$ nor $\bot$. There will also be a logically non-denoting singular-term, denoted by '$\circledast$'—which will be used also to denote the classification 'non-denoting'. In the presence of the term $\circledast$, we shall then be able to abandon $\imath$-terms without any loss in expressive power: this is explained in Section 6.4.

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Finally, we must consider the relation of (logical) consequence. Our semantical definition of 'ψ is a consequence of ϕ' is, loosely stated, that

(i) whenever ϕ is $\top$, ψ is $\top$, and (ii) whenever ψ is $\bot$, ϕ is $\bot$.

And so, yet again, we are using a definition which conjoins two formulations of the classical definition, one involving $\top$ and the other $\bot$—formulations which are equivalent in total logic, but not in partial logic. To illustrate the idea, consider for the moment just a propositional calculus with formulae built up from atomic sentences using the connectives we have introduced. Then 'interpretations' will simply be partial assignments of $\top$ and $\bot$ to atomic sentences, and formulae may be evaluated according to our $\top/\bot$-clauses for the connectives. We shall use '$\models$' for the relation of logical consequence, and so $\phi \models \psi$ if only if (i) and (ii) above both hold when 'whenever' is understood to mean 'under any partial assignment under which'. (By 'partial assignment' I do not mean to exclude total assignments: here, as elsewhere, 'partial' means 'not necessarily total'.)

The tendency among authors on partial logics of one sort or another is to take condition (i) on its own to define logical consequence; and sometimes (i) and (ii) are used to frame two separate notions—for example, in [Dunn 1975], [Hayes 1975] and, in disguised form, in [Woodruff 1970]. In [Cleave 1974], on the other hand, there is a (rather algebraic) version of our double-barrelled definition. And across the literature of the last twenty years the picture has not greatly changed. But perhaps making a choice between these alternatives is not such a fundamental matter. After all, we can define the two halves of our single notion:

$$\phi \models^{\top} \psi \quad \text{iff} \quad \phi \models * \vee \psi,$$
$$\phi \models^{\bot} \psi \quad \text{iff} \quad \phi \wedge * \models \phi.$$

And, putting them back together again,

$$\phi \models \psi \quad \text{iff} \quad \phi \models^{\top} \psi \text{ and } \phi \models^{\bot} \psi.$$

Or, if we invoke negation, either one of the halves on its own would do:

$$\phi \models \psi \quad \text{iff} \quad \phi \models^{\top} \psi \text{ and } \neg\psi \models^{\top} \neg\phi \quad \text{iff} \quad \phi \models^{\bot} \psi \text{ and } \neg\psi \models^{\bot} \neg\phi.$$

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