MARIA LUISA DALLA CHIARA AND ROBERTO GIUNTINI

QUANTUM LOGICS

1 INTRODUCTION

The official birth of quantum logic is represented by a famous article of Birkhoff and von Neumann "The logic of quantum mechanics" [Birkhoff and von Neumann, 1936]. At the very beginning of their paper, Birkhoff and von Neumann observe:

> One of the aspects of quantum theory which has attracted the most general attention, is the novelty of the logical notions which it presupposes .... The object of the present paper is to discover what logical structures one may hope to find in physical theories which, like quantum mechanics, do not conform to classical logic.

In order to understand the basic reason why a non classical logic arises from the mathematical formalism of quantum theory (QT), a comparison with classical physics will be useful.

> There is one concept which quantum theory shares alike with classical mechanics and classical electrodynamics. This is the concept of a mathematical "phase-space". According to this concept, any physical system $\mathcal{S}$ is at each instant hypothetically associated with a "point" in a fixed phase-space Σ; this point is supposed to represent mathematically, the "state" of $\mathcal{S}$, and the "state" of $\mathcal{S}$ is supposed to be ascertainable by "maximal" observations.

Maximal pieces of information about physical systems are called also *pure states*. For instance, in classical particle mechanics, a pure state of a single particle can be represented by a sequence of six real numbers $\langle r_1, \ldots, r_6 \rangle$ where the first three numbers correspond to the *position*-coordinates, whereas the last ones are the *momentum*-components.

As a consequence, the phase-space of a single particle system can be identified with the set $\mathbb{R}^6$, consisting of all sextuples of real numbers. Similarly for the case of compound systems, consisting of a finite number n of particles.

Let us now consider an *experimental proposition* **P** about our system, asserting that a given physical quantity has a certain value (for instance: "the value of position in the x-direction lies in a certain interval"). Such

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a proposition **P** will be naturally associated with a subset X of our phase-space, consisting of all the pure states for which **P** holds. In other words, the subsets of Σ seem to represent good mathematical representatives of experimental propositions. These subsets are called by Birkhoff and von Neumann *physical qualities* (we will say simply *events*). Needless to say, the correspondence between the set of all experimental propositions and the set of all events will be many-to-one. When a pure state p belongs to an event X, we will say that our system in state p *verifies* both X and the corresponding experimental proposition.

What about the structure of all events? As is well known, the power-set of any set is a *Boolean algebra*. And also the set $\mathcal{F}(\Sigma)$ of all measurable subsets of Σ (which is more tractable than the full power-set of Σ) turns out to have a Boolean structure. Hence, we may refer to the following Boolean algebra:

$$\mathcal{B} = \langle \mathcal{F}(\Sigma), \subseteq, \cap, \cup, -, \mathbf{1}, \mathbf{0} \rangle,$$

where:

1) $\subseteq, \cap, \cup, -$ are, respectively, the set-theoretic inclusion relation and the operations intersection, union, relative complement;

2) $\mathbf{1}$ is the total space Σ, while $\mathbf{0}$ is the empty set.

According to a standard interpretation, $\cap, \cup, -$ can be naturally regarded as a set-theoretic realization of the classical logical connectives *and*, *or*, *not*. As a consequence, we will obtain a classical semantic behaviour:

- a state p verifies a conjunction $X \cap Y$ iff $p \in X \cap Y$ iff p verifies both members;
- p verifies a disjunction $X \cup Y$ iff $p \in X \cup Y$ iff p verifies at least one member;
- p verifies a negation $-X$ iff $p \notin X$ iff p does not verify X.

To what extent can such a picture be adequately extended to QT? Birkhoff and von Neumann observe:

> In quantum theory the points of Σ correspond to the so called "wave-functions" and hence Σ is ... a function-space, usually assumed to be Hilbert space.

As a consequence, we immediately obtain a basic difference between the quantum and the classical case. The *excluded middle principle* holds in classical mechanics. In other words, pure states semantically decide any event: for any p and X,

$$p \in X \text{ or } p \in -X.$$

QT is, instead, essentially probabilistic. Generally, pure states assign only probability-values to quantum events. Let ψ represent a pure state (a wave function) of a quantum system and let $\mathbf{P}$ be an experimental proposition (for instance "the spin value in the x-direction is up"). The following cases are possible:

(i) ψ assigns to $\mathbf{P}$ probability-value 1 ($\psi(\mathbf{P}) = 1$);

(ii) ψ assigns to $\mathbf{P}$ probability-value 0 ($\psi(\mathbf{P}) = 0$);

(iii) ψ assigns to $\mathbf{P}$ a probability-value different from 1 and from 0 ($\psi(\mathbf{P}) \neq 0, 1$).

In the first two cases, we will say that $\mathbf{P}$ is *true* (*false*) for our system in state ψ. In the third case, $\mathbf{P}$ will be *semantically indeterminate*.

Now the question arises: what will be an adequate mathematical representative for the notion of quantum experimental proposition? The most important novelty of Birkhoff and von Neumann's proposal is based on the following answer: "The mathematical representative of any experimental proposition is a closed linear subspace of Hilbert space" (we will say simply a *closed subspace*).[1] Let $\mathcal{H}$ be a (separable) Hilbert space, whose *unitary vectors* correspond to possible wave functions of a quantum system. The closed subspaces of $\mathcal{H}$ are particular instances of subsets of $\mathcal{H}$ that are closed under linear combinations and Cauchy sequences. Why are mere subsets of the phase-space not interesting in QT? The reason depends on the *superposition principle*, which represents one of the basic dividing line between the quantum and the classical case. Differently from classical mechanics, in quantum mechanics, finite and even infinite linear combinations of pure states give rise to new pure states (provided only some formal conditions are satisfied). Suppose three pure states ψ, ψ_1, ψ_2 and let ψ be a linear combination of ψ_1, ψ_2:

$$\psi = c_1\psi_1 + c_2\psi_2.$$

[1] A *Hilbert space* is a vector space over a *division ring* whose elements are the real or the complex or the quaternionic numbers such that

(i) An *inner product* $(. , .)$ that transforms any pair of vectors into an element of the division ring is defined;

(ii) the space is *metrically complete* with respect to the metrics induced by the inner product $(. , .)$.

A Hilbert space $\mathcal{H}$ is called *separable* iff $\mathcal{H}$ admits a countable basis.

According to the standard interpretation of the formalism, roughly this means that a quantum system in state ψ might verify with probability $|c_1|^2$ those propositions that are certain for state ψ_1 (and are not certain for ψ) and might verify with probability $|c_2|^2$ those propositions that are certain for state ψ_2 (and are not certain for ψ). Suppose now some pure states $\psi_1, \psi_2, \ldots$ each assigning probability 1 to a given experimental proposition **P**, and suppose that the linear combination

$$\psi = \sum_i c_i \psi_i \quad (c_i \neq 0)$$

is a pure state. Then also ψ will assign probability 1 to our proposition **P**. As a consequence, the mathematical representatives of experimental propositions should be closed under finite and infinite linear combinations. The closed subspaces of $\mathcal{H}$ are just the mathematical objects that can realize such a role.

What about the algebraic structure that can be defined on the set $C(\mathcal{H})$ of all mathematical representatives of experimental propositions (let us call them *quantum events*)? For instance, what does it mean *negation*, *conjunction* and *disjunction* in the realm of quantum events? As to negation, Birkhoff and von Neumann's answer is the following:

> The mathematical representative of the *negative* of any experimental proposition is the *orthogonal complement* of the mathematical representative of the proposition itself.

The orthogonal complement X' of a subspace X is defined as the set of all vectors that are orthogonal to all elements of X. In other words, $\psi \in X'$ iff $\psi \perp X$ iff for any $\phi \in X$: $(\psi, \phi) = 0$ (where (ψ, ϕ) is the inner product of ψ and ϕ). From the point of view of the physical interpretation, the orthogonal complement (called also *orthocomplement*) is particularly interesting, since it satisfies the following property: for any event X and any pure state ψ,

$$\psi(X) = 1 \text{ iff } \psi(X') = 0;$$

$$\psi(X) = 0 \text{ iff } \psi(X') = 1;$$

In other words, ψ assigns to an event X probability 1 (0, respectively) iff ψ assigns to the orthocomplement of X probability 0 (1, respectively). As a consequence, one is dealing with an operation that *inverts* the two extreme probability-values, which naturally correspond to the truth-values *truth* and *falsity* (similarly to the classical truth-table of negation).

As to conjunction, Birkhoff and von Neumann notice that this can be still represented by the set-theoretic intersection (like in the classical case). For, the intersection $X \cap Y$ of two closed subspaces is again a closed subspace. Hence, we will obtain the usual truth-table for the connective *and*:

$$\psi \text{ verifies } X \cap Y \text{ iff } \psi \text{ verifies both members.}$$

Disjunction, however, cannot be represented here as a set-theoretic union. For, generally, the union $X \cup Y$ of two closed subspaces is not a closed subspace. In spite of this, we have at our disposal another good representative for the connective *or*: the *supremum* $X \sqcup Y$ of two closed subspaces, that is the smallest closed subspace including both X and Y. Of course, $X \sqcup Y$ will include $X \cup Y$.

As a consequence, we obtain the following structure

$$\mathcal{C}(\mathcal{H}) = \langle C(\mathcal{H}), \sqsubseteq, \sqcap, \sqcup, {}', \mathbf{1}, \mathbf{0} \rangle,$$

where $\sqsubseteq, \sqcap$ are the set-theoretic inclusion and intersection; $\sqcup, {}'$ are defined as above; while $\mathbf{1}$ and $\mathbf{0}$ represent, respectively, the total space $\mathcal{H}$ and the null subspace (the singleton of the null vector, representing the smallest possible subspace). An isomorphic structure can be obtained by using as a support, instead of $C(\mathcal{H})$, the set $P(\mathcal{H})$ of all *projections* P of $\mathcal{H}$. As is well known projections (i.e. *idempotent* and *self-adjoint linear operators*) and closed subspaces are in one-to-one correspondence, by the projection theorem. Our structure $\mathcal{C}(\mathcal{H})$ turns out to simulate a "quasi-Boolean behaviour"; however, it is not a Boolean algebra. Something very essential is missing. For instance, conjunction and disjunction are no more distributive. Generally,

$$X \sqcap (Y \sqcup Z) \neq (X \sqcap Y) \sqcup (X \sqcap Z).$$

It turns out that $\mathcal{C}(\mathcal{H})$ belongs to the variety of all *orthocomplemented orthomodular lattices*, that are not necessarily distributive.

The failure of distributivity is connected with a characteristic property of disjunction in QT. Differently from classical (bivalent) semantics, a quantum disjunction $X \sqcup Y$ may be true even if neither member is true. In fact, it may happen that a pure state ψ belongs to a subspace $X \sqcup Y$, even if ψ belongs neither to X nor to Y (see Figure 1).

Such a semantic behaviour, which may appear *prima facie* somewhat strange, seems to reflect pretty well a number of concrete quantum situations. In QT one is often dealing with alternatives that are semantically determined and true, while both members are, in principle, strongly indeterminate. For instance, suppose we are referring to some one-half spin particle (say an electron) whose spin may assume only two possible values: either *up* or *down*. Now, according to one of the *uncertainty principles*, the spin in the x direction (*spin*$_x$) and the spin in the y direction (*spin*$_y$) represent two strongly *incompatible* quantities that cannot be simultaneously

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