

BASIC TENSE LOGIC

1 WHAT IS TENSE LOGIC?

We approach this question through an example:

- (1) *Smith:* Have you heard? Jones is going to Albania!
Smythe: He won't get in without an extra-special visa.
Has he remembered to apply for one?
Smith: Not yet, so far as I know.
Smythe: Then he'll have to do so soon.

In this bit of dialogue the argument, such as it is, turns on issues of temporal order. In English, as in all Indo-European and many other languages, such order is expressed in part through changes in verb-form, or tenses. How should the logician treat such tensed arguments?

A solution that comes naturally to mathematical logicians, and that has been forcefully advocated in [Quine, 1960], is to regiment ordinary tensed language to make it fit the patterns of classical logic. Thus Equation 1 might be reduced to the quasi-English Equation 1 below, and thence to the 'canonical notation' of Equation 3:

- (2) Jones/visits/Albania at some time later than the present.

At any time later than the present, if Jones/visits/Albania then, then at some earlier time Jones/applies/for a visa.

At no time earlier than or equal to the present it is the case that Jones/applies/for a visa.

Therefore, Jones/applies/for a visa at some time later than the present.

- (3) $\exists t(c < t \wedge P(t))$
 $\forall t(c < t \wedge P(t) \rightarrow \exists u(u < t \wedge Q(u)))$
 $\neg \exists t((t < c \vee t = c) \wedge Q(t))$
 $\therefore \exists t(c < t \wedge Q(t)).$

Regimentation involves introducing quantification over instants $t, u, \dots$ of time, plus symbols of the present instant c and the earlier- later relation $<$. Above all, it involves treating such a linguistic item as 'Jones is visiting Albania' *not* as a complete sentence expressing a proposition and having a truth-value, to be symbolised by a sentential variable $p, q, \dots$, but rather as a predicate expressing a property on instants, to be symbolised by a one-place predicate variable $P, Q, \dots$. Regimentation has been called *detensing*

since the verb in, say, ‘Jones/visits/Albania at time t ’, written here in the grammatical present tense, ought really to be regarded as *tenseless*; for it states not a present fact but a timeless or ‘eternal’ property of the instant t . Bracketing is one convention for indicating such tenselessness. The knack for regimenting or detensing, for reducing something like Equation 1 to something like Equation 3, is easily acquired. The analysis, however, cannot stop there. For a tensed argument like that above must surely be regarded as an *enthymeme*, having as unstated premises certain assumptions about the structure of Time. Smith and Smythe, for instance, probably take it for granted that of any two distinct instants, one is earlier than the other. And if this assumption is formalised and added as an extra premise, then Equation 3, invalid as it stands, becomes valid.

Of course, it is the job of the cosmologist, not the logician, to judge whether such an assumption is physically or metaphysically correct. What *is* the logician’s job is to formalise such assumptions, correct or not, in logical symbolism. Fortunately, most assumptions people make about the structure of Time go over readily into first- or, at worst, second-order formulas.

1.1 Postulates for Earlier-Later

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| (B0) | Antisymmetry | $\forall x \forall y \neg (x < y \wedge y < x)$ |
| (B1) | Transitivity | $\forall x \forall y \forall z (x < y \wedge y < z \rightarrow x < z)$ |
| (B2) | Comparability | $\forall x \forall y (x < y \vee x = y \vee y < x)$ |
| (B3) | (a) Maximum | $\exists x \forall y (y < x \vee y = x)$ |
| | (b) Minimum | $\exists x \forall y (x < y \vee x = y)$ |
| (B4) | (a) No Maximals | $\forall x \exists y (x < y)$ |
| | (b) No Minimals | $\forall x \exists y (y < x)$ |
| (B5) | Density | $\forall x \forall y (x < y \rightarrow \exists z (x < z \wedge z < y))$ |
| (B6) | (a) Successors | $\forall x \exists y (x < y \wedge \neg \exists z (x < z \wedge z < y))$ |
| | (b) Predecessors | $\forall x \exists y (y < x \wedge \neg \exists z (y < z \wedge z < x))$ |
| (B7) | Completeness | $\forall U ((\exists x U(x) \wedge \exists x \neg U(x) \wedge$
$\forall x \forall y (U(x) \wedge$
$\wedge \neg U(y) \rightarrow x < y)) \rightarrow$
$(\exists x (U(x) \wedge$
$\wedge \forall y (x < y \rightarrow \neg U(y))) \vee$
$\exists x (\neg U(x) \wedge$
$\wedge \forall y (y < x \rightarrow U(y))))$ |
| (B8) | Wellfoundedness | $\forall U (\exists x U(x) \rightarrow \exists x (U(x) \rightarrow$
$\wedge \forall y (y < x \rightarrow \neg U(y)))$ |
| (B9) | (a) Upper Bounds | $\forall x \forall y \exists z (x < z \wedge y < z)$ |
| | (b) Lower Bounds | $\forall x \forall y \exists z (z < x \wedge z < y).$ |

For more on the development of the logic of time as a branch of applied first- and second-order logic, see [van Benthem, 1978].

The alternative to regimentation is the development of an autonomous *tense logic* (also called *temporal* logic or *chronological* logic), first undertaken in [Prior, 1957] (though several precursors are cited in [Prior, 1967]). Tense logic takes seriously the idea that items like ‘Jones is visiting Albania’ are already complete sentences expressing propositions and having truth-values, and that they should therefore be symbolised by sentential variables $p, q, \dots$. Of course, the truth-value of a sentence in the present tense may well differ from that of the corresponding sentence in the past or future tense. Hence, tense logic will need some way of symbolising the relations between sentences that differ only in the tense of the main verb. At its simplest, tense logic adds for this purpose to classical truth-functional sentential logic just two one-place connectives: the future-tense or ‘will’ operator F and the past-tense or ‘was’ operator P . Thus, if p symbolises ‘Jones is visiting Albania’, then Fp and Pp respectively symbolise something like ‘Jones is sooner or later going to visit Albania’ and ‘Jones has at least once visited Albania’. In reading tense-logical symbolism aloud. F and P may be read respectively as ‘it will be the case that’ and ‘it was the case that’. Then $\neg F\neg$, usually abbreviated G , and $\neg P\neg$, usually abbreviated H , may be read respectively as ‘it is always going to be the case that’ and ‘it has always been the case that’. Actually, for many purposes it is preferable to take G and H as primitive, defining F and P as $\neg G\neg$ and $\neg H\neg$ respectively. Armed with this notation, the tense-logician will reduce Equation 1 above to the stylised Equation 1.1 and then to the tense-logical Equation 5:

(4) Future-tense (Jones visits Albania)

Not future-tense (Jones visits Albania and not past-tense (Jones applies for a visa)).

Not past-tense (Jones applies for a visa) and not Jones applies for a visa.

Therefore, future-tense (Jones applies for a visa)

$$\begin{aligned}
 (5) \quad & Fp \\
 & \neg F(p \wedge \neg Pq) \\
 & \neg Pq \wedge \neg q \\
 & \therefore Fq.
 \end{aligned}$$

Of course, we will want some axioms and rules for the new temporal operators F, P, G, H . All the axiomatic systems considered in this survey will share the same standard format.

1.2 Standard Format

We start from a stock of sentential *variables* $p_0, p_2, p_2, \dots$, usually writing p for p_0 and q for p_1 . The (well-formed) *formulas* of tense logic are built

up from the variables using negation ($\neg$), and conjunction ($\wedge$), and the strong future (G) and strong past (H) operators. The *mirror image* of a formula is the result of replacing each occurrence of G by H and vice versa. Disjunction ($\vee$), material conditional ($\rightarrow$), material biconditional ($\leftrightarrow$), constant true ($\top$), constant false ($\perp$), weak future (F), and weak past (P) can be introduced as abbreviations.

As *axioms* we take all substitution instances of truth-functional tautologies. In addition, each particular system will take as axioms all substitution instances of some finite list of extra axioms, called the *characteristic axioms* of the system. As *rules of inference* we take Modus Ponens (MP) plus the specifically tense-logical:

Temporal Generalisation(TG): From α to infer $G\alpha$ and $H\alpha$

The *theses* of a system are the formulas obtainable from its axioms by these rules. A formula is *consistent* if its negation is not a thesis; a set of formulas is *consistent* if the conjunction of any finite subset is. These notions are, of course, relative to a given system.

The systems considered in this survey will have characteristic axioms drawn from the following list:

1.3 Postulates for a Past-Present-Future

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|---|--|
| <p>(A0) (a) $G(p \rightarrow q) \rightarrow (Gp \rightarrow Gq)$</p> <p>(c) $p \rightarrow GPP$</p> <p>(A1) (a) $Gp \rightarrow GGp$</p> <p>(A2) (a) $Pp \wedge Fq \rightarrow F(p \wedge Fq) \vee F(p \wedge q) \vee F(Fp \wedge q)$</p> <p>(b) $Pp \wedge Pq \rightarrow P(p \wedge Pq) \vee P(p \wedge q) \vee P(Pp \wedge q)$</p> <p>(A3) (a) $G\perp \vee FG\perp$</p> <p>(A4) (a) $Gp \rightarrow Fp$</p> <p>(A5) (a) $Fp \rightarrow FFp$</p> <p>(A6) (a) $p \wedge Hp \rightarrow FHp$</p> <p>(A7) (a) $Fp \wedge FG\neg p \rightarrow F(HFp \wedge G\neg p)$</p> <p>(b) $Pp \wedge PH\neg p \rightarrow P(GPp \wedge H\neg p)$</p> <p>(A8) $H(Hp \rightarrow p) \rightarrow Hp$</p> <p>(A9) (a) $FGp \rightarrow GFp$</p> | <p>(b) $H(p \rightarrow q) \rightarrow (Hp \rightarrow Hq)$</p> <p>(d) $p \rightarrow HFP$</p> <p>(b) $Hp \rightarrow HHp$</p> <p>(b) $H\perp \vee PH\perp$</p> <p>(b) $Hp \rightarrow Pp$</p> <p>(b) $Pp \rightarrow PPp$</p> <p>(b) $p \wedge Gp \rightarrow PGp$</p> <p>(b) $PHp \rightarrow HPP$</p> |
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A few definitions are needed before we can state precisely the basic problem of tense logic, that of finding characteristic axioms that ‘correspond’ to various assumptions about Time.

1.4 Formal Semantics

A *frame* is a nonempty set C equipped with a binary relation R . A *valuation* in a frame (X, R) is a function V assigning each variable p_i a subset of X . Intuitively, X can be thought of as representing the set of instants of time, R

the earlier-later relation, V the function telling us *when* each p_i is the case. We extend V to a function defined on *all* formulas, by abuse of notation still called V , inductively as follows:

$$\begin{aligned} V(\neg\alpha) &= X - V(\alpha) \\ V(\alpha \wedge \beta) &= V(\alpha) \cap V(\beta) \\ V(G\alpha) &= \{x \in X : \forall y \in X (xRy \rightarrow y \in V(\alpha))\} \\ V(H\alpha) &= \{x \in X : \forall y \in X (yRx \rightarrow y \in V(\alpha))\}. \end{aligned}$$

(Some writers prefer a different notion. Thus, what we have expressed as $x \in V(\alpha)$ may appear as $\|\alpha\|_x^V = \text{TRUE}$ or as $(X, R, V) \models \alpha[x]$.) A formula α is *valid* in a frame (X, R) if $V(\alpha) = X$ for every valuation V in (X, R) , and is *satisfiable* in (X, R) if $V(\alpha) \neq \emptyset$ for some valuation V in (X, R) , or equivalently if $\neg\alpha$ is not valid in (X, R) . Further, α is *valid* over a class $\mathcal{K}$ of frames if it is valid in every $(X, R) \in \mathcal{K}$, and is *satisfiable* over $\mathcal{K}$ if it is satisfiable in some $(X, R) \in \mathcal{K}$, or equivalently if $\neg\alpha$ is not valid over $\mathcal{K}$. A system $\mathbf{L}$ in standard format is *sound* for $\mathcal{K}$ if every thesis of $\mathbf{L}$ is valid over $\mathcal{K}$, and a sound system $\mathbf{L}$ is *complete* for $\mathcal{K}$ if conversely every formula valid over $\mathcal{K}$ is a thesis of $\mathbf{L}$, or equivalently, if every formula consistent with $\mathbf{L}$ is satisfiable over $\mathcal{K}$. Any set (let us say, finite) Φ of first- or second-order axioms about the earlier-later relation $<$ determines a class $\mathcal{K}(\Phi)$ of frames, the class of its *models*. The basic correspondence problem of tense logic is, given Φ to find characteristic axioms for a system $\mathbf{L}$ that will be sound and complete for $\mathcal{K}(\Phi)$. The next two sections of this survey will be devoted to representing the solution to this problem for many important Φ .

1.5 Motivation

But first it may be well to ask, why bother? Several classes of motives for developing an autonomous tense logic may be cited:

(a) *Philosophical* motives were behind much of the pioneering work of A. N. Prior, to whom the following point seemed most important: whereas our ordinary language is tensed, the language of physics is mathematical and so untensed. Thus, there arise opportunities for confusions between different ‘terms of ideas’. Now working in tense logic, what we learn is precisely how to avoid confusing the tensed and the tenseless, and how to clarify their relations (e.g. we learn that essentially the same thought can be formulated tenselessly as, ‘Of any two distinct instants, one /is/ earlier and the other /is/ later’, and tensedly as, ‘Whatever is going to have been the case either already has been or now is or is sometime going to be the case’). Thus, the study of tense logic can have at least a ‘therapeutic’ value. Later writers have stressed other philosophical applications, and some of these are treated elsewhere in this *Handbook*.

Handbook of Philosophical Logic

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