Chapter 2

BASIC PROOF SYSTEMS FOR SUBSTRUCTURAL LOGICS

In this chapter, we shall introduce sequent calculi and Hilbert-style calculi for several substructural logics. A standard way to accomplish such tasks, in handbooks whose scope comprises various logical systems, is to focus on a basic system and then consider its extensions. Such extensions can be either axiomatic (the language remains the same as in the basic calculus, but more postulates are added) or linguistic (the language is enriched by new logical constants and, possibly, some postulates governing these new symbols are introduced). The choice of such a basic system must perforce be, to some extent, arbitrary. However, a delicate tradeoff is involved: this system must be neither too weak, for it would lack any intrinsic interest, nor too strong, since its extensions would be too limited in number.

In view of these considerations, we choose to take as our starting point what is usually known as the "subexponential fragment of classical propositional linear logic without additive constants". This is a certainly interesting system and, as we shall see in this chapter and in the next, is general enough to admit a wide range of substructural logics as extensions. Before presenting our basic sequent calculus **LL**, however, we have to dispatch some tedious but unavoidable preliminaries.

1. SOME BASIC DEFINITIONS AND NOTATIONAL CONVENTIONS

Since our notation differs in part from other standard articles or textbooks on linear and substructural logics, we included a synoptic table of notations for the reader's convenience (Table 2.1):

Table 2.1. A synopsis of notations.

Our notation	Girard 1987	Troelstra 1992	Restall 2000
$\neg$	$\perp$	$\sim$	$\sim$
$\otimes$	$\otimes$	$\bigstar$	$\circ$
$\oplus$	$⅋$	$+$	$+$
$\wedge$	$\&$	$\sqcap$	$\wedge$
$\vee$	$\oplus$	$\sqcup$	$\vee$
$\rightarrow$	$\multimap$	$\multimap$	$\rightarrow$
!	!	!	!
?	?	?	?
$\mathbf{1}$	1	1	1
$\mathbf{0}$	$\perp$	$\mathbf{0}$	$\mathbf{0}$
$\boldsymbol{T}$	T	T	T
$\perp$	0	$\perp$	$\perp$

Definition 2.1 (some conventions about languages). Throughout this volume we shall be concerned with propositional languages, each containing a denumerable set of propositional variables and a given number of connectives, drawn from the set

$$\{\neg, \otimes, \oplus, \rightarrow, \wedge, \vee, \supset, \mathbf{1}, \mathbf{0}, \boldsymbol{T}, \perp, !, ?\}.$$

Of these, $\otimes$, $\oplus$, $\wedge$, $\vee$, $\rightarrow$, $\supset$ are *binary*; $\neg, !, ?$ are *unary*; $\mathbf{1}, \mathbf{0}, \boldsymbol{T}, \perp$ are *nullary*. Nullary connectives are sometimes referred to as *propositional constants*. The connectives $\neg$, $\otimes$, $\oplus$, $\rightarrow$, $\mathbf{1}, \mathbf{0}$ are called *group-theoretical*; the connectives $\wedge$, $\vee$, $\supset$, $\boldsymbol{T}$, $\perp$ are called *lattice-theoretical*; finally, $!, ?$ are the *exponentials*. We adopt the convention according to which unary connectives bind stronger than either $\otimes$, $\oplus$, $\wedge$ or $\vee$, which in turn bind stronger than either $\rightarrow$ or $\supset$.

Hereafter, we list the languages on which our calculi will be based, together with their respective sets of logical constants:

$£_1$: $\{\neg, \otimes, \oplus, \to, \wedge, \vee, \mathbf{1}, \mathbf{0}\}$;
$£_2$: $\{\neg, \otimes, \oplus, \to, \wedge, \vee, \mathbf{1}, \mathbf{0}, \boldsymbol{T}, \perp\}$;
$£_3$: $\{\neg, \otimes, \oplus, \to, \wedge, \vee, \mathbf{1}, \mathbf{0}, \boldsymbol{T}, \perp, !, ?\}$;
$£_4$: $\{\to\}$;
$£_5$: $\{\neg, \otimes, \oplus, \to, \mathbf{1}, \mathbf{0}\}$;
$£_6$: $\{\otimes, \to, \wedge, \vee, \mathbf{1}\}$;
$£_7$: $\{\neg, \otimes, \oplus, \to, \wedge, \vee, \supset, \mathbf{1}, \mathbf{0}\}$.

The letter "£" will refer to a generic language in the above list. By VAR(£) and FOR(£) we shall denote, respectively, the set of all the propositional variables and of all the well-formed formulae of the language £.

Definition 2.2 (some conventions about calculi). Formal calculi - whether axiomatic or sequent calculi - will be referred to by boldface capital letters. The letter "**S**" will stand for a generic calculus; the letter "**L**", followed by a specific letter, will be employed to refer to sequent calculi (a convention which should be reminiscent of Gentzen's usage of the same letter in "**LK**" and "**LJ**"); likewise, the letter "**H**", followed by a specific letter, will designate Hilbert-style calculi. If **S** is a calculus, then:

- $\mathbf{S_i}$ will denote its *purely implicational* fragment, based on the language $£_4$;
- $\mathbf{S_g}$ will denote its *group-theoretical* fragment, based on the language $£_5$;
- $\mathbf{S_+}$ will denote its *positive* fragment, based on the language $£_6$.

If **S** is any sequent calculus, by writing $\vdash_{\mathbf{S}} \Gamma \Rightarrow \Delta$ we shall mean that $\Gamma \Rightarrow \Delta$ is a theorem of **S**. Moreover, by writing $\vdash_{\mathbf{S}} \Gamma \Leftrightarrow \Delta$ (or by saying that $\Gamma \Leftrightarrow \Delta$ is a theorem of **S**) we shall mean that both $\vdash_{\mathbf{S}} \Gamma \Rightarrow \Delta$ and $\vdash_{\mathbf{S}} \Delta \Rightarrow \Gamma$. If **S** is any axiomatic calculus, by writing $\vdash_{\mathbf{S}} A$ we shall mean that A is a theorem of **S**. Moreover, by writing $\vdash_{\mathbf{S}} A \leftrightarrow B$ (or by saying that $A \leftrightarrow B$ is a theorem of **S**) we shall mean that both $\vdash_{\mathbf{S}} A \to B$ and $\vdash_{\mathbf{S}} B \to A$.

Definition 2.3 (some conventions about sequents). Throughout this chapter, we shall adopt the same definitions and conventions about sequents that we stated in Definitions 1.1-1.6. With one notable exception, however: capital Greek letters will not stand for *sequences* of formulae of the language at issue, but for *multisets* of formulae of such language. Multisets can be rigorously defined (see e.g. Troelstra 1992, p. 2), but this is not necessary in our context: suffice it to say that multisets are aggregates where the ordering of the elements does not matter (whereas it matters for sequences), but their multiplicity does (while it does not for sets). So, for example, $\{A, B\}$ is the same multiset as

$\{B, A\}$, but $\{A, A, A\}$ is not the same multiset as $\{A\}$. As a rule, outer brackets will be omitted: as it is customary to do, we shall write $A_1, ..., A_n \Rightarrow B_1, ..., B_m$ in place of the more correct $\{A_1, ..., A_n\} \Rightarrow \{B_1, ..., B_m\}$.

2. SEQUENT CALCULI

2.1 The calculus LL

It is now time to come to the heart of the matter, and present our basic sequent calculus.

Definition 2.4 (postulates of **LL**). The calculus **LL**, based on the language $£_1$, has the following postulates:

Axioms

$$A \Rightarrow A$$

Structural rules

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\,(Cut)$$

Operational rules

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}\,(\neg L) \qquad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}\,(\neg R)$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \otimes B, \Gamma \Rightarrow \Delta}\,(\otimes L) \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Pi \Rightarrow \Sigma, B}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, A \otimes B}\,(\otimes R)$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Pi \Rightarrow \Sigma}{A \oplus B, \Gamma, \Pi \Rightarrow \Delta, \Sigma}\,(\oplus L) \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \oplus B}\,(\oplus R)$$

$$\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \; \frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\,(\wedge L) \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\,(\wedge R)$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\,(\vee L) \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B} \; \frac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \vee B}\,(\vee R)$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Pi \Rightarrow \Sigma}{A \rightarrow B, \Gamma, \Pi \Rightarrow \Delta, \Sigma} (\rightarrow L) \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B} (\rightarrow R)$$

$$\frac{\Gamma \Rightarrow \Delta}{\mathbf{1}, \Gamma \Rightarrow \Delta} (\mathbf{1}L) \qquad \Rightarrow \mathbf{1} \ (\mathbf{1}R)$$

$$\mathbf{0} \Rightarrow \ (\mathbf{0}L) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \mathbf{0}} (\mathbf{0}R)$$

Notice that **LL** contains "covert" exchange rules: using multisets instead of sequences, we are allowed to perform arbitrary permutations either in the antecedent or in the succedent. Beside such rules, the only explicit structural rule of **LL** is the cut rule.

Proposition 2.1 (theorems of **LL**). The following sequents are provable in **LL**: (i) $\Rightarrow A \rightarrow A$; (ii) $A \rightarrow (B \rightarrow C) \Rightarrow B \rightarrow (A \rightarrow C)$; (iii) $A \rightarrow B, B \rightarrow C \Rightarrow A \rightarrow C$; (iv) $A \otimes (B \otimes C) \Leftrightarrow (A \otimes B) \otimes C$; (v) $A \oplus (B \oplus C) \Leftrightarrow (A \oplus B) \oplus C$; (vi) $A \otimes B \Leftrightarrow B \otimes A$; (vii) $A \oplus B \Leftrightarrow B \oplus A$; (viii) $A \rightarrow (B \rightarrow C) \Leftrightarrow A \otimes B \rightarrow C$; (ix) $A, B \Rightarrow A \otimes B$; (x) $A \Leftrightarrow \neg\neg A$; (xi) $A \rightarrow \neg B \Leftrightarrow B \rightarrow \neg A$; (xii) $A \rightarrow B \Leftrightarrow \neg A \oplus B$; (xiii) $\neg(A \otimes B) \Leftrightarrow \neg A \oplus \neg B$; (xiv) $\neg(A \oplus B) \Leftrightarrow \neg A \otimes \neg B$; (xv) $A \wedge B \Rightarrow A$; (xvi) $A \wedge B \Rightarrow B$; (xvii) $(A \rightarrow B) \wedge (A \rightarrow C) \Leftrightarrow A \rightarrow B \wedge C$; (xviii) $A \Rightarrow A \vee B$; (xix) $B \Rightarrow A \vee B$; (xx) $(A \rightarrow C) \wedge (B \rightarrow C) \Leftrightarrow A \vee B \rightarrow C$; (xxi) $A \vee (B \wedge C) \Rightarrow (A \vee B) \wedge (A \vee C)$; (xxii) $(A \wedge B) \vee (A \wedge C) \Rightarrow A \wedge (B \vee C)$; (xxiii) $A \oplus (B \wedge C) \Leftrightarrow (A \oplus B) \wedge (A \oplus C)$; (xxiv) $A \otimes (B \vee C) \Leftrightarrow (A \otimes B) \vee (A \otimes C)$; (xxv) $A \otimes (B \wedge C) \Rightarrow (A \otimes B) \wedge (A \otimes C)$; (xxvi) $(A \oplus B) \vee (A \oplus C) \Rightarrow A \oplus (B \vee C)$; (xxvii) $\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$; (xxviii) $\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B$; (xxix) $A \wedge (B \wedge C) \Leftrightarrow (A \wedge B) \wedge C$; (xxx) $A \wedge B \Leftrightarrow B \wedge A$; (xxxi) $A \wedge A \Leftrightarrow A$; (xxxii) $A \vee (B \vee C) \Leftrightarrow (A \vee B) \vee C$; (xxxiii) $A \vee B \Leftrightarrow B \vee A$; (xxxiv) $A \vee A \Leftrightarrow A$; (xxxv) $A \wedge (B \vee A) \Leftrightarrow A$; (xxxvi) $A \vee (B \wedge A) \Leftrightarrow A$; (xxxvii) $\mathbf{1} \Rightarrow A \rightarrow A$; (xxxviii) $\neg\mathbf{1} \Leftrightarrow \mathbf{0}$; (xxxix) $\neg\mathbf{0} \Leftrightarrow \mathbf{1}$; (xl) $A \otimes \mathbf{1} \Leftrightarrow A$; (xli) $A \oplus \mathbf{0} \Leftrightarrow A$.

Proof. For its most part, this lemma will be left as an exercise for the reader. We only present a couple of examples: the left-to-right part of (viii) and the right-to-left part of (xxiv).

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