

Chapter 3

CUT ELIMINATION AND THE DECISION PROBLEM

In Chapter 1, we discussed at some length the importance of cut elimination, both from a philosophical and from a technical viewpoint. Hitherto, however, we did not *prove* the cut elimination theorem for any of the systems so far introduced. This will be exactly the task of the present chapter. For a start, we shall present Gentzen's proof of the *Haauptsatz* for **LK**; coming to know how such a proof works is essential also from our perspective, for it allows to appreciate the role that structural rules play in it. Subsequently, we shall assess how Gentzen's strategy should be modified in order to obtain the elimination of cuts for systems lacking some of the structural rules. We shall also show, with the aid of appropriate counterexamples, that not all of our sequent systems are cut-free.

In the second part of this chapter, we shall examine one of the main applications of cut elimination, seeing how to extract out of it a *decision procedure* for many of our sequent calculi. Again, such methods do not work invariably in all cases: there are, indeed, systems which are known to be undecidable (and we shall briefly discuss them).

1. CUT ELIMINATION

1.1 Cut elimination for **LK**

The next definition introduces a calculus which is equivalent to **LK**, as we defined it back in Chapter 1.

Definition 3.1 (postulates of \mathbf{LK}^M). The system \mathbf{LK}^M is exactly like \mathbf{LK} , except for the fact that: 1) its basic expressions are inferences of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite, possibly empty, *multisets* of formulae of \mathcal{L}_0 ; 2) it contains no exchange rule; 3) the cut rule is replaced by the following rule (called *mix*):

$$\frac{\Gamma \Rightarrow \Delta \quad \Pi \Rightarrow \Sigma}{\Gamma, \Pi_A^* \Rightarrow \Delta_A^*, \Sigma} (A)$$

Here, both Δ and Π contain A , and Π_A^* (Δ_A^*) is the same multiset as Π (Δ), except for containing no occurrence of A . As far as there is no danger of ambiguity, we shall drop the subscript " A ". The formula A is called *mixformula*.

Proposition 3.1 (equivalence of \mathbf{LK}^M and \mathbf{LK}). $\vdash_{\mathbf{LK}} \Gamma \Rightarrow \Delta$ iff $\vdash_{\mathbf{LK}^M} \Gamma \Rightarrow \Delta$.

Proof. We confine ourselves to showing that the cut rule is equivalent to the mix rule. In fact:

$$\frac{\begin{array}{c} \frac{\Gamma \Rightarrow \Delta^*, A, \dots A}{\Gamma \Rightarrow \Delta^*, A} \quad \frac{A, \dots A, \Pi^* \Rightarrow \Sigma}{A, \Pi^* \Rightarrow \Sigma} \\ \hline \Gamma, \Pi^* \Rightarrow \Delta^*, \Sigma \end{array} \quad \text{(several CR)} \quad \text{(several CL)} \quad \text{(Cut)}}{\frac{\frac{\Gamma \Rightarrow \Delta, A \quad A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi^* \Rightarrow \Delta^*, \Sigma} \quad (A)}{\Gamma, \Pi \Rightarrow \Delta, \Sigma} \quad \text{(several WL, WR)}} \quad \square$$

Why, a reader could ask, did we introduce such a complicated and convoluted inference pattern as the mix rule, in place of the more natural and intuitively appealing cut rule? There is a reason, indeed, and it has to do precisely with the presence of contraction in \mathbf{LK} . We shall explain our move in due course; thus, the curious reader is begged to wait patiently until § 1.2.

What we shall do, for the time being, will be to prove a cut elimination theorem for \mathbf{LK}^M . To achieve this goal, we need a number of auxiliary notions. First of all, the concept of "mixproof" will permit us to focus on a quite small subset of the set of all proofs in \mathbf{LK}^M which contain one or more applications of mix¹.

Definition 3.2 (mixproofs and mix-free proofs). A proof \mathcal{D} in \mathbf{LK}^M is called a *mixproof* iff it contains just one application of mix, whose conclusion

S is the endsequent of the proof; it is called a *mix-free proof* iff it contains no application of mix at all.

Proposition 3.2 (circumscription of cut elimination). In \mathbf{LK}^M , if any mixproof \mathcal{D} of $\Gamma \Rightarrow \Delta$ can be transformed into a mix-free proof of the same sequent, then any arbitrary proof \mathcal{D}' of $\Gamma \Rightarrow \Delta$ can be transformed into a mix-free proof of the same sequent.

Proof (sketch). Let \mathcal{D} be any proof of $\Gamma \Rightarrow \Delta$ in \mathbf{LK}^M . Take the leftmost and uppermost application of mix in \mathcal{D} , and let $\Pi \Rightarrow \Sigma$ be its conclusion. The subproof \mathcal{D}' of \mathcal{D} whose endsequent is $\Pi \Rightarrow \Sigma$ is a mixproof which can thus be turned into a mix-free proof \mathcal{D}'' of $\Pi \Rightarrow \Sigma$. Now consider the result of replacing \mathcal{D}' in \mathcal{D} by \mathcal{D}'' , call it \mathcal{E} , and take the leftmost and uppermost application of mix in \mathcal{E} . By repeating this procedure as many times as there are applications of mix in \mathcal{D} , we get the required transformation. The details are left to the reader. \square

In virtue of the preceding lemma, it will suffice to show that any mixproof \mathcal{D} of $\Gamma \Rightarrow \Delta$ in \mathbf{LK}^M can be turned into a mix-free proof of the same sequent in \mathbf{LK}^M . To do so, we shall argue by induction on a special parameter, to be specified presently.

Definition 3.3 (rank of a sequent in a mixproof). Let \mathcal{D} be a mixproof whose final inference is:

$$\frac{\Gamma \Rightarrow \Delta \quad \Pi \Rightarrow \Sigma}{\Gamma, \Pi^* \Rightarrow \Delta^*, \Sigma}(A)$$

The *rank of the sequent* S in \mathcal{D} is denoted by $r_{\mathcal{D}}(S)$ and is so defined:

- If S belongs to the subproof \mathcal{D}' of \mathcal{D} whose endsequent is $\Gamma \Rightarrow \Delta$, $r_{\mathcal{D}}(S)$ is the maximal length (diminished by one) of an upward path of sequents S_1, \dots, S_n s.t. $S_1 = S$ and S_i ($1 \leq i \leq n$) contains A in its succedent;
- If S belongs to the subproof \mathcal{D}' of \mathcal{D} whose endsequent is $\Pi \Rightarrow \Sigma$, $r_{\mathcal{D}}(S)$ is defined in the same way, except for replacing " $\Gamma \Rightarrow \Delta$ " by " $\Pi \Rightarrow \Sigma$ " and "succedent" by "antecedent";
- $r_{\mathcal{D}}(\Gamma, \Pi^* \Rightarrow \Delta^*, \Sigma) = r_{\mathcal{D}}(\Gamma \Rightarrow \Delta) + r_{\mathcal{D}}(\Pi \Rightarrow \Sigma)$.

Definition 3.4 (rank of a subproof in a mixproof)². Let \mathcal{D} be a mixproof and \mathcal{D}' be any of its subproofs (possibly \mathcal{D} itself). The *rank of \mathcal{D}' in \mathcal{D}* is denoted by $r_{\mathcal{D}}(\mathcal{D}')$ or simply by $r(\mathcal{D}')$ and coincides by definition with $r_{\mathcal{D}}(S)$, where S is the endsequent of \mathcal{D}' .

Definition 3.5 (grade of a subproof in a mixproof). Let \mathcal{D} be a mixproof and \mathcal{D}' be any of its subproofs (possibly \mathcal{D} itself). The *grade of \mathcal{D}' in \mathcal{D}* is denoted by $g_{\mathcal{D}}(\mathcal{D}')$ or simply by $g(\mathcal{D}')$ and is the number of logical symbols contained in the mixformula A .

Definition 3.6 (index of a subproof in a mixproof). Let \mathcal{D} be a mixproof and \mathcal{D}' be any of its subproofs (possibly \mathcal{D} itself). The *index of \mathcal{D}' in \mathcal{D}* is denoted by $i_{\mathcal{D}}(\mathcal{D}')$ or simply by $i(\mathcal{D}')$ and is the ordered pair $\langle g(\mathcal{D}'), r(\mathcal{D}') \rangle$. Indexes are ordered *lexicographically*: that is, $\langle i, n \rangle \leq \langle j, m \rangle$ iff either $i < j$ or else ($i = j$ and $n \leq m$).

Before proving our cut elimination theorem, we settle some notational matters. When drawing the proof tree of a mixproof \mathcal{D} , we shall sometimes write $\Gamma \Rightarrow^n \Delta$, meaning thereby that $r_{\mathcal{D}}(\Gamma \Rightarrow \Delta)$ is n . We shall also write $n \sqcup m$ to denote the *maximum* of the ranks n and m . Now, we are ready to start.

Proposition 3.3 (cut elimination theorem for \mathbf{LK}^M : Gentzen 1935). Any mixproof \mathcal{D} of $\Gamma \Rightarrow \Delta$ in \mathbf{LK}^M can be turned into a mix-free proof \mathcal{D}' of the same sequent in the same calculus.

Proof. Let \mathcal{D} be a mixproof whose final inference is:

$$\frac{\Gamma \Rightarrow \Delta \quad \Pi \Rightarrow \Sigma}{\Gamma, \Pi^* \Rightarrow \Delta^*, \Sigma}(A)$$

We proceed by induction on $i(\mathcal{D})$, the index of \mathcal{D} .

$[i(\mathcal{D}) = \langle 0, 0 \rangle]$. Since $g(\mathcal{D}) = 0$, A is a variable, say p . As $r(\mathcal{D}) = 0$, in particular $r_{\mathcal{D}}(\Gamma \Rightarrow \Delta) = 0$. Hence, either $\Gamma \Rightarrow \Delta$ is an axiom, or else p is the principal formula of the inference whose conclusion is $\Gamma \Rightarrow \Delta$, which must perforce be WR. We have the following transformations:

$$\begin{array}{ccc} \vdots & & \vdots \\ \frac{p \Rightarrow p \quad \Pi \Rightarrow \Sigma}{p, \Pi^* \Rightarrow \Sigma} \text{ (p)} & \overset{T_1}{\rightsquigarrow} & \frac{\Pi \Rightarrow \Sigma}{p, \Pi^* \Rightarrow \Sigma} \text{ (several CL)} \\ \\ \vdots & & \vdots \\ \frac{\frac{\Lambda \Rightarrow \Theta}{\Lambda \Rightarrow \Theta, p} \text{ (WR)} \quad \Pi \Rightarrow \Sigma}{\Lambda, \Pi^* \Rightarrow \Theta, \Sigma} \text{ (p)} & \overset{T_2}{\rightsquigarrow} & \frac{\Lambda \Rightarrow \Theta}{\Lambda, \Pi^* \Rightarrow \Theta, \Sigma} \text{ (several WL, WR)} \end{array}$$

$[i(\mathcal{D}) = \langle 0, k \rangle, 1 \leq k]$. If $r(\mathcal{D}) = k > 0$, then either $r_{\mathcal{D}}(\Gamma \Rightarrow \Delta) > 0$ or $r_{\mathcal{D}}(\Pi \Rightarrow \Sigma) > 0$. We distinguish the two subcases.

$[r_{\mathcal{D}}(\Gamma \Rightarrow \Delta) > 0]$. Thus $\Gamma \Rightarrow \Delta$ is the conclusion of an inference where p can be either a principal, or an auxiliary, or a side formula. If it is a side formula, our strategy consists in "pushing cuts upwards" in such a way as to construct new proofs of $\Gamma, \Pi^* \Rightarrow \Delta^*, \Sigma$ containing mixproofs of grade 0 and of lower rank (hence of lower index) as subproofs. This entitles us to exploit our inductive hypothesis. Some examples:

$$\frac{\frac{\vdots}{\Gamma \Rightarrow^n \Delta} \quad \frac{\vdots}{\Pi \Rightarrow^m \Sigma}}{\Gamma \Rightarrow^{n+1} \Delta, B} (WR) \quad \frac{\vdots}{\Gamma, \Pi^* \Rightarrow^{n+m+1} \Delta^*, \Sigma, B} (p) \quad T3 \quad \sim \quad \frac{\frac{\vdots}{\Gamma \Rightarrow^n \Delta} \quad \frac{\vdots}{\Pi \Rightarrow^m \Sigma}}{\Gamma, \Pi^* \Rightarrow^{n+m} \Delta^*, \Sigma} (p) \quad \frac{\vdots}{\Gamma, \Pi^* \Rightarrow \Delta^*, \Sigma, B} (WR)$$

$$\frac{\frac{\vdots}{\Gamma \Rightarrow^n \Delta, B} \quad \frac{\vdots}{\Gamma \Rightarrow^m \Delta, C}}{\Gamma \Rightarrow^{(n \sqcup m)+1} \Delta, B \wedge C} (\wedge R) \quad \frac{\vdots}{\Pi \Rightarrow^r \Sigma} \quad T4 \quad \sim \quad \frac{\vdots}{\Gamma, \Pi^* \Rightarrow^{(n+r+1) \sqcup (m+r+1)} \Delta^*, \Sigma, B \wedge C} (p)$$

$$\frac{\frac{\vdots}{\Gamma \Rightarrow^n \Delta, B} \quad \frac{\vdots}{\Pi \Rightarrow^r \Sigma}}{\Gamma, \Pi^* \Rightarrow^{n+r} \Delta^*, \Sigma, B} (p) \quad \frac{\frac{\vdots}{\Gamma \Rightarrow^m \Delta, C} \quad \frac{\vdots}{\Pi \Rightarrow^r \Sigma}}{\Gamma, \Pi^* \Rightarrow^{m+r} \Delta^*, \Sigma, C} (p) \quad \frac{\vdots}{\Gamma, \Pi^* \Rightarrow \Delta^*, \Sigma, B \wedge C} (\wedge R)$$

$$\frac{\frac{\vdots}{\Gamma \Rightarrow^n \Delta, B} \quad \frac{\vdots}{C, \Pi \Rightarrow^m \Sigma}}{B \rightarrow C, \Gamma, \Pi \Rightarrow^{(n \sqcup m)+1} \Delta, \Sigma} (\rightarrow L) \quad \frac{\vdots}{\Lambda \Rightarrow^r \Theta} \quad T5 \quad \sim \quad \frac{\vdots}{B \rightarrow C, \Gamma, \Pi, \Lambda^* \Rightarrow^{(n+r+1) \sqcup (m+r+1)} \Delta^*, \Sigma^*, \Theta} (p)$$

$$\frac{\frac{\vdots}{\Gamma \Rightarrow^n \Delta, B} \quad \frac{\vdots}{\Lambda \Rightarrow^r \Theta}}{\Gamma, \Lambda^* \Rightarrow^{n+r} \Delta^*, \Theta, B} (p) \quad \frac{\frac{\vdots}{C, \Pi \Rightarrow^m \Sigma} \quad \frac{\vdots}{\Lambda \Rightarrow^r \Theta}}{C, \Pi, \Lambda^* \Rightarrow^{m+r} \Sigma^*, \Theta} (p) \quad \frac{\vdots}{B \rightarrow C, \Gamma, \Pi, \Lambda^*, \Lambda^* \Rightarrow \Delta^*, \Sigma^*, \Theta, \Theta} (\rightarrow L) \quad \frac{\vdots}{B \rightarrow C, \Gamma, \Pi, \Lambda^* \Rightarrow \Delta^*, \Sigma^*, \Theta} (CL, CR)$$

If the mixformula p is auxiliary, the strategy is basically identical, except that we possibly need to perform some adjustments by means of structural



<http://www.springer.com/978-1-4020-0605-0>

Substructural Logics: A Primer

Paoli, F.

2002, XIII, 305 p., Hardcover

ISBN: 978-1-4020-0605-0