Chapter 4

# OTHER FORMALISMS

In Chapter 2, we examined two different kinds of formalisms whose role is undoubtedly central in the proof theory of substructural logics: sequent calculi, on the one hand, and Hilbert-style systems, on the other. On that occasion, we noticed that there are at least six well-motivated axiomatic calculi - **HRW**, **HR**, **HRMI**, **HRM**, **HLuk**, and $\mathbf{HLuk_3}$ - which do not have any sequential counterpart, in that they seem scarcely amenable to a treatment by means of traditional sequents. As we already remarked, Hilbert-style calculi are definitely not the best one could hope for when it comes to engaging in proof search and theorem proving tasks. As a consequence, it seems desirable to find efficient and manageable formalisms also for the above-mentioned logics.

In the present chapter, we shall briefly illustrate three types of formalisms. In the opening section, we shall delve into some generalizations of ordinary sequent calculi. Each one of them arises when we abstract from some peculiar features of traditional sequents as conceived of by Gentzen, and allows to deal with one or more of the logics we mentioned in the previous paragraph. Subsequently, we shall try to get the hang of *proofnets*, which according to Girard (1987) constitute a sort of natural deduction calculus for linear logic. Finally, we shall turn to *resolution* calculi, taking a rapid look at resolution systems for some substructural logics.

The content of this chapter, as the reader will guess, is of a more advanced level than the material so far presented. The topics we shall discuss are at the very centre of contemporary research into the proof theory of substructural logics. This does not imply that we request, on the reader's part, any previous knowledge of the subject or technical prerequisites of any kind; rather, this means that we shall often be forced to skip a number of details and to present many results in a "dogmatic" manner, omitting their proofs or replacing them by examples which can give the reader some clue about how these proofs go through, or about how the calculi at issue actually work.

# 1. GENERALIZATIONS OF SEQUENT CALCULI

As we repeatedly pointed out, Gentzen's sequents are meant to represent inferences in a fully general and abstract form: we have an antecedent, which contains the premisses of the inference; a succedent, which contains its conclusions; and the arrow, which stands for the relation of consequence between the former and the latter.

If we examine with care the structure of sequents, however, we can spot at least three somewhat accidental features, three expressive limitations due to which our calculi do not seem to stand up to the required level of abstraction and generality:

- The arrow denotes a *binary* relation of consequence between an antecedent and a succedent. Why should it be so? In other words, why should the number of "cedents" be just two, and not three or more?
- "$\Gamma \Rightarrow \Delta$" means, informally, "$\Delta$ follows from $\Gamma$". What if I want to say that "either $\Delta$ follows from $\Gamma$, or $\Sigma$ follows from $\Pi$"? There seems to be no way to express such a disjunction by means of ordinary sequents. Should not we enrich the expressive power of our calculi to allow also for such patterns?
- Comma is the only way of bunching formulae together inside sequents. Cannot we think of other ways of providing them with an internal structure? Should not we try to refine, when circumstances seem to require it, our analysis of the logical role of comma?

All of the above questions have been successfully addressed by logicians working in the proof theory of nonclassical logics, who came up with such flexible formalisms as *$n$-sided sequent calculi*, *hypersequent calculi*, *Dunn-Mints calculi* and *display calculi*. Let us examine them one by one.

## 1.1 $N$-sided sequents

Finite-valued Lukasiewicz logics, like many other logics whose semantics is smooth and attractive, have always proved very resistant to any proof-theoretical analysis, especially by means of sequent calculi. In 1967, however, Rousseau (foreshadowed to some extent by Schröter 1955) thought out a promising new idea. His strategy can be summarized with a slogan: if twosided sequents are good for two-valued logic, multiple-valued logics need multiple-sided sequents.

Let us now try to be more precise. The classical sequent $\Gamma \Rightarrow \Delta$ holds, informally speaking, iff at least one of the $\Gamma$'s is false or at least one of the $\Delta$'s is true. In other words, it holds iff at least one of the $\Gamma$'s assumes the value 0 ("false") or at least one of the $\Delta$'s assumes the value 1 ("true"). Two values, two multisets of formulae. Now, suppose to be faced with an $n$-valued logic; what you will need, then, is an ordered $n$-tuple of multisets of formulae $\Gamma_0, ..., \Gamma_{n-1}$ which holds true iff there is a $j < n$ such that at least one of the $\Gamma_j$'s assumes the value $j$. This seems a fair generalization of the two-valued case; and such, in fact, is the intuitive idea behind Rousseau's investigations.

After Rousseau introduced (or rediscovered, if we take into account Schröter's contribution) $n$-sided sequents, many authors followed in his footsteps. Similar techniques are employed in several interesting papers and monographs on the proof theory of finite-valued logics, such as e.g. Carnielli (1991), Zach (1993), Baaz et al. (1994), Baaz, Fermueller et al. (1998), Gil et al. (1997, 1999). To give a significant example of this area of investigations, we present hereby a calculus for three-valued Lukasiewicz logic which is substantially equivalent to the one to be found in Baaz et al. (1994) - and not very distant from the original system by Rousseau.

Definition 4.1 (3-sided sequent). A *3-sided sequent* is an expression of the form $\Gamma_0|\Gamma_1|\Gamma_2$, where each $\Gamma_i$ is a finite, possibly empty multiset of formulae of £$_1$. Intuitively, $\Gamma_0|\Gamma_1|\Gamma_2$ says that at least one of the $\Gamma_0$'s is false, or at least one of the $\Gamma_1$'s is intermediate, or at least one of the $\Gamma_2$'s is true.

Definition 4.2 (postulates of **LLuk$_3$**). **LLuk$_3$** is a calculus whose basic expressions are 3-sided sequents. Its postulates are:

*Axioms*

$$A|A|A$$

*Structural rules*

$$\frac{\Gamma_0|\Gamma_1|\Gamma_2}{A, \Gamma_0|\Gamma_1|\Gamma_2}(W0) \qquad \frac{\Gamma_0|\Gamma_1|\Gamma_2}{\Gamma_0|A, \Gamma_1|\Gamma_2}(W1) \qquad \frac{\Gamma_0|\Gamma_1|\Gamma_2}{\Gamma_0|\Gamma_1|A, \Gamma_2}(W2)$$

$$\frac{A, A, \Gamma_0|\Gamma_1|\Gamma_2}{A, \Gamma_0|\Gamma_1|\Gamma_2}(C0) \qquad \frac{\Gamma_0|A, A, \Gamma_1|\Gamma_2}{\Gamma_0|A, \Gamma_1|\Gamma_2}(C1) \qquad \frac{\Gamma_0|\Gamma_1|A, A, \Gamma_2}{\Gamma_0|\Gamma_1|A, \Gamma_2}(C2)$$

$$\frac{A, \Gamma_0|\Gamma_1|\Gamma_2 \quad \Gamma_0|A, \Gamma_1|\Gamma_2}{\Gamma_0|\Gamma_1|\Gamma_2} (Cut0, 1)$$

$$\frac{\Gamma_0|A, \Gamma_1|\Gamma_2 \quad \Gamma_0|\Gamma_1|A, \Gamma_2}{\Gamma_0|\Gamma_1|\Gamma_2} (Cut1, 2)$$

$$\frac{A, \Gamma_0|\Gamma_1|\Gamma_2 \quad \Gamma_0|\Gamma_1|A, \Gamma_2}{\Gamma_0|\Gamma_1|\Gamma_2} (Cut0, 2)$$

*Operational rules*

$$\frac{\Gamma_0|\Gamma_1|A, \Gamma_2}{\neg A, \Gamma_0|\Gamma_1|\Gamma_2} (\neg 0) \qquad \frac{\Gamma_0|A, \Gamma_1|\Gamma_2}{\Gamma_0|\neg A, \Gamma_1|\Gamma_2} (\neg 1) \qquad \frac{A, \Gamma_0|\Gamma_1|\Gamma_2}{\Gamma_0|\Gamma_1|\neg A, \Gamma_2} (\neg 2)$$

$$\frac{\Gamma_0|\Gamma_1|A, \Gamma_2 \quad B, \Gamma_0|\Gamma_1|\Gamma_2}{A \to B, \Gamma_0|\Gamma_1|\Gamma_2} (\to 0)$$

$$\frac{\Gamma_0|A, B, \Gamma_1|\Gamma_2 \quad B, \Gamma_0|\Gamma_1|A, \Gamma_2}{\Gamma_0|A \to B, \Gamma_1|\Gamma_2} (\to 1)$$

$$\frac{A, \Gamma_0|A, \Gamma_1|B, \Gamma_2 \quad A, \Gamma_0|B, \Gamma_1|B, \Gamma_2}{\Gamma_0|\Gamma_1|A \to B, \Gamma_2} (\to 2)$$

Remark 4.1 (definability of connectives in **LLuk$_3$**). The rules for the connectives $\oplus$, $\otimes$, $\wedge$, $\vee$, $\mathbf{1}$ and $\mathbf{0}$ can be easily derived from the above rules, taking into account the mutual relationships among different connectives that hold in all our substructural logics and recalling that in all of Lukasiewicz logics $A \vee B$ is equivalent to $(A \to B) \to B$ and $\mathbf{1}$ is equivalent to $A \to A$.

Remark 4.2 (on the rules of **LLuk$_3$**). Notice that in **LLuk$_3$** operational, weakening and contraction rules are not divided into left and right rules, as usual, but rather into *0-, 1- and 2-rules*. So, for example, there are as many weakening rules as there are sides in a 3-sequent, i.e. three. We also have three cut rules, but for a different reason - in fact, there are as many cut rules as there are pairs of *different* truth values in our logic. Operational rules are directly obtained from the three-valued truth tables for the corresponding connectives. For example, negation shifts formulae from the "false" side on to the "true" side and *vice versa,* while leaving where they stand formulae on the "intermediate" side.

Remark 4.3 (on the substructural character of **LLuk$_3$**). The reader will have noticed that **LLuk$_3$** contains both weakening and contraction rules; therefore it is not, strictly speaking, a substructural calculus. In the next section, however, three-valued Lukasiewicz logic will be given formulations in terms of hypersequents where (internal) contraction rules do not hold. So, how does the matter stand? This seeming contradiction can be explained away by remarking that contraction rules, as we shall presently see, express here nothing more than the idempotency of *lattice-theoretical* disjunction. Therefore, they are similar to *external,* not internal, contraction rules in hypersequent calculi[1], and external structural rules always hold in such calculi.

Now, let us prove that **LLuk$_3$** actually corresponds to **HLuk$_3$**.

Definition 4.3 (formula-translation of a 3-sided sequent). Let $\Gamma_0 = A_{01}, ..., A_{0n}$, $\Gamma_1 = A_{11}, ..., A_{1m}$, and $\Gamma_2 = A_{21}, ..., A_{2p}$. Let moreover $\phi(A_{ij})$ be the formula $(A_{ij} \to \neg A_{ij}) \wedge (\neg A_{ij} \to A_{ij})$. The *formula-translation* $t(\Gamma_0|\Gamma_1|\Gamma_2)$ of the three-sided sequent $\Gamma_0|\Gamma_1|\Gamma_2$ is defined as follows if either $n \neq 0$ or $m \neq 0$ or $p \neq 0$:

$$\neg A_{01} \vee ... \vee \neg A_{0n} \vee \phi(A_{11}) \vee ... \vee \phi(A_{1m}) \vee A_{21} \vee ... \vee A_{2p}$$

If $n = m = p = 0$, then $t(\Gamma_0|\Gamma_1|\Gamma_2)$ is $\mathbf{0}$.

Proposition 4.1 (equivalence of **LLuk$_3$** and **HLuk$_3$**). (i) If $\vdash_{\mathbf{HLuk_3}} A$, then $\vdash_{\mathbf{LLuk_3}} \emptyset|\emptyset|A$; (ii) if $\vdash_{\mathbf{LLuk_3}} \Gamma_0|\Gamma_1|\Gamma_2$, then $\vdash_{\mathbf{HLuk_3}} t(\Gamma_0|\Gamma_1|\Gamma_2)$.

*Proof.* (i) Induction on the length of the proof of $A$ in **HLuk$_3$**. By way of example, let us consider the axiom F22. Let $B = A \to \neg A$, and let $\mathcal{D}, \mathcal{E}, \mathcal{F}$ be the following proofs:

$$\mathcal{D}: \quad \dfrac{\dfrac{\dfrac{A|A|A}{A|B,A,A|\neg A,A} \qquad \dfrac{A|A|A}{A|A,\neg A,B|\neg A,A}}{\emptyset|B,A|B,A} \qquad \dfrac{A|A|A}{A|B,A|A}}{B \to A|B,A|A}$$

http://www.springer.com/978-1-4020-0605-0

Substructural Logics: A Primer
Paoli, F.
2002, XIII, 305 p., Hardcover
ISBN: 978-1-4020-0605-0