Chapter 5

ALGEBRAIC STRUCTURES

When studying a logical calculus S of any kind, it is extremely important to be in a position to find a class of adequate models for it - i.e. a class of algebraic structures which verify exactly the provable formulae of S. Thus, for example, it turns out that the algebraic counterpart of classical propositional logic are Boolean algebras, while intuitionistic propositional logic corresponds to Heyting algebras. As a rule, these correspondences pave the way for a profitable interaction: the investigation of models may yield several fruitful insights on the structure of the given calculus, and, conversely, it may even happen that proof-theoretical techniques be of some avail in proving purely algebraic results (Grishin 1982; Kowalski and Ono 2000).

Ono (200+*a*) compares proof-theoretical and algebraic investigations into the field of substructural logics with these remarks:

> Proof-theoretic methods have shown their effectiveness for particular substructural logics, e.g. logics formalized in cut-free sequent systems [...]. On the other hand, the semantical study up to now is quite unsatisfactory, and therefore the general study of substructural logics is far behind e.g. that in modal logic in recent years.

In our opinion, the reasons of the difficulties experienced in the algebraic study of substructural logics are essentially two:

1) It is not difficult to identify suitable classes of algebraic models for the different logics encountered in the previous chapters. Such structures, however, seem to have a restricted mathematical interest (with some notable exceptions, nevertheless). On the one hand, they do not arise often in areas of mathematics

other than logic; on the other hand, they are usually rather *weak* structures, with few interesting properties, and thus their theories of ideals, congruences and representation are either scarcely developed or even lacking.

2) A second ground of such a disrepair has to do with the idiosyncratic research styles of the various substructural schools. Linear logicians, for one, seem to assign a limited value to model-theoretic semantics in general (to which they prefer, far and away, the proof-theoretic semantics developed *within* the logical calculi themselves) and to traditional algebraic semantics in particular (usually replaced by phase semantics, of which more in Chapter 7, and by a denotational semantics for proofs). Relevance logicians, in turn, tend to focus on the *lattice-theoretical* properties of the different ℓ-semigroups they study, often viewed as lattices with additional operators (Urquhart 1996; Hartonas 1997). For example, the notion of *filter* one can often find in the literature of relevance logic (see e.g. Dunn 1986; Restall 2000) coincides with the concept of filter of the underlying lattice. Such a tendency, however, may sometimes obscure some fundamental algebraic properties more directly related to the behaviour of the semigroup operation[1].

Beside researchers working either in the linear or in the relevance tradition, over the last few years there have been authors who have undertaken a serious and in-depth investigation of the algebra of substructural logics in a general framework, i.e. without committing themselves to the partial perspective of a given logic. In particular, we are greatly indebted to the contributions of Avron (1988; 1990; 1998), Ono (1985; 1993; 200+*a*; 200+*b*) and Kowalski and Ono (2001), both for the results they proved and for the *ésprit de système* which underlies their works on the subject.

In the present chapter, we shall introduce the algebraic structures for substructural logics according to a rather traditional expository pattern. To begin with, we shall define such structures and study their elementary arithmetical properties. Subsequently, we shall introduce and investigate suitable concepts of homomorphism and ideal. Finally, we shall lay down the fundamentals of a representation theory. In the next chapter, we shall see how to match the logical calculi of Part II with the structures of the present chapter.

With regard to the definitions of the concepts needed for our treatment, we chose - as already pointed out in the Preface - to adopt a particular strategy. If we had started from scratch, defining and illustrating even such notions as "lattice", "congruence", "subdirect product", the chapter would have reached exorbitant lengths and would have become virtually unusable for readers with some algebraic background. As a consequence, we took a few basic notions for granted. However, we did not want to scare away the other readers; therefore, the unexperienced student will find all that is needed for a thorough comprehension of the chapter in the glossary of Appendix A.

1. *-AUTONOMOUS LATTICES

1.1 Definitions and elementary properties

In Chapter 2, we remarked that whenever one wants to investigate various logical systems in the lump, it is often expedient to focus on a suitable basic system and then consider its extensions. And so we did, appointing **LL** as the starting point for our syntactical investigations. Here we shall do the same - we shall pick a basic class of structures and see what happens by adding further axioms to its defining conditions. In the next chapter, we shall verify that the resulting atlas of algebras presents an exact correspondence with the atlas of logics of Table 2.2.

Definition 5.1 (*-autonomous lattice). A **-autonomous lattice* (or, briefly, a **-lattice*) is an algebra $\mathcal{A} = \, < A, +, -, 0, \sqcap, \sqcup >$ of type $< 2, 1, 0, 2, 2 >$, such that:

(C1) $< A, +, 0 >$ is an Abelian monoid;
(C2) $< A, -, \sqcap, \sqcup >$ is an involutive lattice;
(C3) $-0 \leq -x + y$ iff $x \leq y$.

In C3, " $\leq$ " denotes the induced lattice ordering of $< A, \sqcap, \sqcup >$. Henceforth, we agree to denote the element " -0" by the symbol "1" and to abbreviate $-(-x + -y)$ by $x \cdot y$.

Remark 5.1 (denominations of *-lattices). The name "*-autonomous lattice" is not standard in the literature. Rosenthal (1990) uses the expression "*-autonomous poset" for partially ordered structures satisfying C1, C3 and

(C2') $< A, -, \leq >$ is an involutive poset.

He chooses such a label in view of the connection existing between these structures and *-autonomous categories (Barr 1979; Seely 1989). Avron (1988; 1994), on the other hand, employs the denomination "additive relevant disjunction monoids" in order to refer to our *-lattices.

Example 5.1 (Girard 1987). Let $\mathcal{M} = \, < M, \bullet, e >$ be an Abelian monoid, and let $\emptyset \neq D \subseteq M$. We define the following operations on subsets of M:

$$X^{\perp} = \{x : \text{for every } y, y \in X \text{ only if } x \bullet y \in D\}$$
$$XY = \{x \bullet y : x \in X \text{ and } y \in Y\}$$
$$X \oplus Y = (X^{\perp}Y^{\perp})^{\perp}$$
$$0 = D = \{e\}^{\perp}$$

Let moreover $\mathcal{C}(\mathcal{M}) = \{X \subseteq M : X = X^{\perp\perp}\}$ and let Π be a subset of $\mathcal{C}(\mathcal{M})$ which contains 0 and is closed w.r.t. the operations $^{\perp}$, $\oplus$ and set-theoretical intersection ($\mathcal{C}(\mathcal{M})$ itself will do, for example). Then $\mathcal{A} = \ <\Pi, \oplus, ^{\perp}, 0, \sqcap, \sqcup>$ is a *-lattice where $X \sqcap Y = X \cap Y$ and $X \sqcup Y = (X \cup Y)^{\perp\perp}$.

This example will play a key role in Chapter 7, in the context of the relational semantics for nondistributive logics.

Example 5.2. Any Abelian ℓ-group $\mathcal{A} = \ < A, +, -, 0, \sqcap, \sqcup >$ is a *-lattice where $x \cdot y = x + y$. Any Boolean algebra can be presented as a *-lattice $\mathcal{A} = \ < A, +, -, 0, \sqcap, \sqcup >$ where $x + y = x \sqcup y$ and $x \cdot y = x \sqcap y$.

We list below some elementary arithmetical properties of *-lattices, mainly discovered by Avron (1988; 1990) and Casari (1989).

Proposition 5.1 (arithmetical properties of *-lattices). (i) $1 \leq -(-x+y) + -(-y+z) + -x+z$; (ii) $1 \leq -(-x+(-y+z)) + (-y+(-x+z))$; (iii) if $1 \leq x$ and $1 \leq -x+y$, then $1 \leq y$; (iv) if $1 \leq -x+y$ and $1 \leq -y+z$, then $1 \leq -x+z$; (v) $1 \leq -x+x$; (vi) $x \leq -(-x+y)+y$; (vii) $x + -(-y+y) \leq x$; (viii) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$; (ix) $x \cdot y = y \cdot x$; (x) $x \cdot 1 = x$; (xi) $x \leq y+z$ iff $-y \leq -x+z$; (xii) if $x \leq y$, then $x+z \leq y+z$ and $x \cdot z \leq y \cdot z$; (xiii) if $x \leq y$ and $x' \leq y'$, then $x+x' \leq y+y'$ and $x \cdot x' \leq y \cdot y'$; (xiv) $-(x+y) \leq -x + -y + -(1+1)$; (xv) $-(x+y+1) \leq -(x+1) + -(y+1)$; (xvi) $x \cdot y \leq z$ iff $x \leq -y+z$; (xvii) $x + (y \sqcap z) = (x+y) \sqcap (x+z)$; (xviii) $x \cdot (y \sqcup z) = (x \cdot y) \sqcup (x \cdot z)$; (xix) $(x+y) \sqcup (x+z) \leq x + (y \sqcup z)$; (xx) $x \cdot (y \sqcap z) \leq (x \cdot y) \sqcap (x \cdot z)$.

Proof. We prove (ii), (xi), the first half of (xii), (xiv), (xv), (xvii), leaving the rest as an exercise.

(ii) By C1, C2 $-x+(-y+z) \leq -y+(-x+z)$, whence our conclusion follows by C3. (xi) Immediate, applying C3 twice.

(xii) By (v), $1 \leq -(x+z)+x+z$, whence by C2 and C3 $-x \leq -(x+z)+z$. Now, suppose $x \leq y$, which in virtue of C2 implies $-y \leq -x$; we immediately get $-y \leq -(x+z)+z$, whence by C2 and (xi) $x+z \leq y+z$.

(xiv) Since $1 \leq -x+x$ and $1 \leq -y+y$, by (xiii) $1+1 \leq -x+x+-y+y$, whence by (xi) we get our conclusion.

(xv) By (vi), $0 \leq -(y+1)+y$ and thus, *via* (xii), $x+1 \leq x+y+1+-(y+1)$, whence by (xi) the desired conclusion follows.

(xvii) From $y \sqcap z \leq y$, by (xii), $x+(y \sqcap z) \leq x+y$ and similarly $x+(y \sqcap z) \leq x+z$. Hence $x+(y \sqcap z) \leq (x+y) \sqcap (x+z)$. Conversely, since $(x+y) \sqcap (x+z) \leq x+y$, by C3 $1 \leq -((x+y) \sqcap (x+z))+x+y$, whence $-(-((x+y) \sqcap (x+z))+x) \leq y$. Similarly $-(-((x+y) \sqcap (x+z))+x) \leq z$, which together with the previous inequality yields $-(-((x+y) \sqcap (x+z))+x) \leq y \sqcap z$. Two more applications of C3 give first $1 \leq -((x+y) \sqcap (x+z))+x+(y \sqcap z)$ and then $(x+y) \sqcap (x+z) \leq x+(y \sqcap z)$. □

Proposition 5.2 (*-lattices form a variety: Minari 200+). The class of *-lattices can be presented as a variety in the signature $< +, -, 0, \sqcap >$, defined by the equations:

$(E1)\, x+(y+z)=(x+y)+z$	$(E2)\, x+y=y+x$
$(E3)\, x+0=x$	$(E4)\, --x=x$
$(E5)\, -0 \sqcap (-x+x) = -0$	$(E6)\, x \sqcap (y \sqcap z)=(x \sqcap y) \sqcap z$
$(E7)\, x \sqcap y = y \sqcap x$	$(E8)\, x+(y \sqcap z)=(x+y) \sqcap (x+z)$
$(E9)\, x = x \sqcap -(-x \sqcap y)$	$(E10)\, x = x \sqcap (-(-x+y)+y)$

Proof. The previous equations are directly implied either by C1-C3 or else by Proposition 5.1. Hence, they hold in any *-lattice. Conversely, it is sufficient to prove by means of E1-E10 that (A) if $x \leq y$, then $-y \leq -x$ and that (B) $1 \leq -x+y$ iff $x \leq y$.

(*Ad* A). Suppose $x \sqcap y = x$. It follows $-x = -(x \sqcap y)$, i.e. $-y \sqcap -x = -y \sqcap -(x \sqcap y) = -y$ by E9, E7 and E4.

(*Ad* B). Let $1 = 1 \sqcap (-x+y)$. It follows that $0 = -(1 \sqcap (-x+y))$ and thus $0 \sqcap -(-x+y) = -(1 \sqcap (-x+y)) \sqcap -(-x+y) = -(-x+y)$, by E4, E7 and E9. Adding y to both sides, $-(-x+y)+y = (0 \sqcap -(-x+y))+y = y \sqcap (-(-x+y)+y)$, by E2, E3 and E8. As a consequence, using E6, $x \sqcap (-(-x+y)+y) = x \sqcap y \sqcap (-(-x+y)+y)$, whence by E10 $x = x \sqcap y$. For the other

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