

Chapter 1

THE ROLE OF STRUCTURAL RULES IN SEQUENT CALCULI

1. THE "INFERENTIAL APPROACH" TO LOGICAL CALCULUS

Substructural logics owe their name to the fact that an especially immediate and intuitive way to introduce them is by means of sequent calculi *à la* Gentzen where one or more of the structural rules (weakening, contraction, exchange, cut) are suitably restricted or even left out. We do not assume the reader to be familiar with the terminology of the preceding sentence, which will be subsequently explained in full detail - but if only she has some acquaintance with the history of twentieth century logic, at least the name of *Gerhard Gentzen* should not be completely foreign to her.

Gentzen, a German logician and mathematician who is justly celebrated as one of the most prominent figures of contemporary logic, introduced both natural deduction and sequent calculi in his doctoral thesis *Untersuchungen über das logische Schliessen* (translated into English as *Investigations into Logical Deduction*: Gentzen 1935). In a sense, as we shall see below, Gentzen can also be considered as the founding father of substructural logics (Došen 1993). Any investigation concerning this topic, therefore, cannot fail to take Gentzen's *Untersuchungen* as a starting point. And so shall we do.

Gentzen describes as follows the philosophical motivation that led him to set up his calculus of natural deduction (p. 68):

The formalization of logical deduction, especially as it has been developed by Frege, Russell, and Hilbert, is rather far removed

from the forms of deduction used in mathematical proofs [...]. In contrast, I intended first to set up a formal system which comes as close as possible to actual reasoning.

Natural deduction, according to Gentzen, has thus a decisive edge over Hilbert-style axiomatic calculi: its formal derivations reflect more closely some concrete structural features of informal mathematical proofs - most notably, the use of *assumptions*. But there is a further epistemological gain which can be achieved by resorting to a system of natural deduction. In the words of Haskell B. Curry (1960, pp. 119-121):

In his doctoral thesis Gentzen presented a new approach to the logical calculus whose central characteristic was that it laid great emphasis on inferential rules which seemed to flow naturally from meanings as intuitively conceived. It is appropriate to call this mode of approach the *inferential approach* [...].

The essential content of the system is contained in the inferential (or deductive) rules. Except for a few rather trivial rules of special nature, these rules are associated with the separate operations; and those which are so associated with a particular operation express the meaning of that operation.

The outstanding novelty of Gentzen's standpoint, according to Curry, is thus a completely new approach to the issue of the meaning of logical constants. In axiomatic calculi, logical operations are implicitly defined by their mutual relationships as stated in the axioms of the system. No separate, operational meaning is ascribed to them. In the calculus of natural deduction, on the other hand, the emphasis is on laying down separate rules for each constant - rules which can be taken to express the operational content of logical symbols. In this way, any commitment to a holistic theory of the meaning of logical constants is avoided. It can be reasonably conjectured that this was the viewpoint of Gentzen himself, since he explicitly observed (p. 80):

The introductions represent, as it were, the "definitions" of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions.

We shall not dwell, for the time being, on this distinction between the respective roles of introduction and elimination inferences (but we shall return on this point). Suffice it to say that this fleeting remark by Gentzen was subsequently taken up and extensively developed by Dummett, Prawitz, Tennant, Schroeder-Heister and others, who started off a prolific trend of investigations into the relationships between natural deduction calculi and the

meaning of logical constants (see Sundholm 1986 for detailed references on this topic).

So much for the philosophical significance of natural deduction. What about sequent calculi? Gentzen seemed, *prima facie*, to award them a merely instrumental role, as these calculi appeared to him nothing more than an "especially suited" framework to the purpose of proving his *Hauptsatz*, a result whose importance we shall discuss at length¹. Looking in hindsight, however, we can legitimately say that Gentzen underestimated the philosophical status of his own creature, and that some issues concerning the meaning of logical operations can be framed and discussed in the context of sequent calculi just as well as (if not better than) in the context of natural deduction.

Well: we believe that by now the curiosity of the reader should have been sufficiently aroused and that a presentation of the calculus can no longer be deferred.

1.1. Structural rules, operational rules, and meaning

Gentzen's calculi **LK** (for classical logic) and **LJ** (for intuitionistic logic) are based on a first-order language; however, since the focus of this book is on propositional logic, we shall confine ourselves to their propositional fragments. Henceforth, then, by **LK** (**LJ**) we shall mean *propositional LK* (*LJ*). We shall now take on a slightly more formal tone for a short while, in order to state some definitions which will turn out useful throughout the rest of this volume.

Definition 1.1 (language of LK). Let \mathcal{L}_0 be a propositional language containing a denumerable stock of variables (p_1, p_2, \dots) and the connectives \neg, \wedge, \vee , and \rightarrow . We shall use p, q, \dots as metavariables for propositional variables. Formulae are constructed as usual; $A, B, C \dots$ will be used as metavariables for generic formulae.

Definition 1.2 (sequents in LK). The basic expressions of the calculus are inferences of the form $\Gamma \Rightarrow \Delta$ (read: Δ "follows", or "is derivable from" Γ), where Γ and Δ are finite, possibly empty, sequences of formulae of \mathcal{L}_0 , separated by commas. Such inferences are called *sequents*. Γ and Δ are called, respectively, the *antecedent* and the *succedent* of the sequent.

According to Gentzen, the sequent $A_1, \dots, A_n \Rightarrow B_1, \dots, B_m$ has the same informal meaning as the formula $A_1 \wedge \dots \wedge A_n \rightarrow B_1 \vee \dots \vee B_m$. This means that the comma must be read as a conjunction in the antecedent, and as a disjunction in the succedent, while the arrow corresponds to implication².

Definition 1.3 (postulates of **LK).** The *postulates* of the calculus are its *axioms* and *rules*. Intuitively speaking, the rules encode ways of transforming inferences in an acceptable way, i.e. without perturbing the derivability relation between the antecedent and the succedent. More precisely, they are ordered pairs or triples of sequents, arranged in either of these two forms:

$$\frac{S_1}{S_2} \qquad \frac{S_1 \quad S_2}{S_3}$$

The sequents above the horizontal line are called the *upper sequents*, or the *premisses*, of the rule; the sequent below the line is called the *lower sequent*, or the *conclusion*, of the rule. Rules, moreover, can be either *structural* or *operational*³. Here are the postulates of **LK**:

Axioms

$$A \Rightarrow A$$

Structural rules

Exchange

$$\frac{\Gamma, A, B, \Delta \Rightarrow \Pi}{\Gamma, B, A, \Delta \Rightarrow \Pi} (EL) \qquad \frac{\Gamma \Rightarrow \Delta, A, B, \Pi}{\Gamma \Rightarrow \Delta, B, A, \Pi} (ER)$$

Weakening

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} (WL) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} (WR)$$

Contraction

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} (CL) \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A} (CR)$$

Cut

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma} (Cut)$$

Operational rules

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta} (\neg L)$$

$$\frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A} (\neg R)$$

$$\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \quad \frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} (\wedge L) \quad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} (\wedge R)$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B} \quad \frac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \vee B} (\vee R) \quad \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} (\vee L)$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Pi \Rightarrow \Sigma}{A \rightarrow B, \Gamma, \Pi \Rightarrow \Delta, \Sigma} (\rightarrow L) \quad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B} (\rightarrow R)$$

Definition 1.4 (principal, side, and auxiliary formulae). In all these rules, the formula occurrences in $\Gamma, \Delta, \Pi, \Sigma$ are called *side formulae*; the formula occurrence in the conclusion which is not a side formula is called *principal*, and the formula occurrences in the premisses which are not side formulae are called *auxiliary*.

Definition 1.5 (proofs in **LK).** A *proof* in **LK** is a finite labelled tree whose nodes are labelled by sequents, in such a way that leaves are labelled by axioms and each sequent at a node is obtained from sequents at immediate predecessor(s) node(s) according to one of the rules of **LK**. We shall denote proofs by means of the metavariables $\mathcal{D}, \mathcal{D}', \dots$ If \mathcal{D} is a proof, a subtree \mathcal{D}' of \mathcal{D} which is itself a proof is called a *subproof* of \mathcal{D} .

A sequent S is *provable* in **LK** (or **LK**-*provable*, or a *theorem* of **LK**) iff it labels the root of some proof in **LK** (i.e., as we shall sometimes say, iff it is the *endsequent* of such a proof).

Definition 1.6 (sequents, postulates and proofs in **LJ).** The calculus **LJ** has the same language as **LK**, and all the concepts introduced in the Definitions 1.2-1.5 apply to it as well, with two sole exceptions. A *sequent* in **LJ** is an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite, possibly empty, sequences of formulae of \mathcal{L}_0 and Δ can contain *at most one formula*. The rules given for **LK**, therefore, must be adapted accordingly.

Definition 1.6 yields an immediate consequence as regards structural rules: the rules ER and CR have to be deleted from **LJ**, for they can only be applied to sequents with more than one formula in the succedent, while the rule WR



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