Chapter 2

CHU'S CONSTRUCTION: A PROOF-THEORETIC APPROACH

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Abstract The essential interaction between classical and intuitionistic features in the system of linear logic is best described in the language of category theory. Given a symmetric monoidal closed category $\mathcal{C}$ with products, the category $\mathcal{C} \times \mathcal{C}^{op}$ can be given the structure of a $*$-autonomous category by a special case of the Chu construction. The main result of the paper is to show that the intuitionistic translations induced by Girard's trips determine the functor from the free $*$-autonomous category $\mathcal{A}$ on a set of atoms $\{P, P', \ldots\}$ to $\mathcal{C} \times \mathcal{C}^{op}$, where $\mathcal{C}$ is the free monoidal closed category with products and coproducts on the set of atoms $\{P_O, P_I, P'_O, P'_I, \ldots\}$ (a pair P_O, P_I in $\mathcal{C}$ for each atom P of $\mathcal{A}$).

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1. Preface

An essential aim of linear logic [16] is the study of the dynamics of proofs, essentially normalization (cut elimination), in a system enjoying the good proof-theoretic properties of *intuitionistic* logic, but where the dualities of *classical* logic hold. Indeed *classical linear logic* **CLL** has a denotational semantics and a game-theoretic semantics; proofs are formalized in a sequent calculus, but also in a system of *proof-nets* and in the latter representation cut elimination not only has the strong normalizability property, but is also confluent. Although Girard's main system of linear logic is *classical*, considerable attention in the literature has also been given to the system of *intuitionistic linear logic*

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ILL, where proofs are also formalized in a sequent calculus and in a natural deduction system. A better understanding of the relations between **CLL** and **ILL** is one of the goals to which the present work is intended as a contribution.

The fact that intuitionistic logic plays an important role in the architecture of linear logic is not surprising: as indicated in the introductory section of Girard's fundamental paper [16], a main source of inspiration for the system was its denotational semantics of coherent spaces, a refinement of Scott's semantics for the λ-calculus. Fundamental decisions about the system **CLL** were made so that **CLL** has a semantics of proofs in coherent spaces in the same way as intuitionistic logic has a semantics of proofs in Scott's domains. But linear logic is not just a refinement of intuitionistic logic, such as **ILL**: there are expectations that **CLL** may tell us something fundamental about classical logic as well, indeed, that through linear logic a deep level of analysis may have been reached from which the "unity of logic" can be appreciated [17]. Therefore the relations between classical and intuitionistic components of linear logic deserve careful investigation.

A natural points of view to look at this issue is *categorical logic*. It has been known for years that monoidal closed categories provide a model for *intuitionistic linear logic*, though a fully adequate formulation of the syntax and of the categorical semantics of **ILL** especially with respect to the exponentials, has required considerable subtlety and effort [4, 5, 6]. It is also well known that $*$-autonomous categories give a model for *classical linear logic* [3]. The appendix to [2] provides a method, due to Barr's student Chu, to construct $*$-autonomous categories starting from monoidal closed ones.

In our proof-theoretic investigation we encounter a special case of Chu's construction, namely $\mathbf{Chu}(\mathcal{C}, \top)$ where $\mathcal{C}$ is a symmetric monoidal closed category with terminal object $\top$. More specifically, given the free $*$-autonomous category $\mathcal{A}$ on a set of objects (propositional variables) $\{P, P', \ldots\}$ and given the symmetric monoidal closed category $\mathcal{C}$ with products, free on the set $\{P_O, P_I, P'_O, P'_I, \ldots\}$ (a pair P_O, P_I in $\mathcal{C}$ for each atom P of $\mathcal{A}$), the category $\mathcal{C} \times \mathcal{C}^{op}$ can be given the structure of a $*$-autonomous category by Chu's construction. Indeed, since the dualizing object is the terminal object, $\mathbf{Chu}(\mathcal{C}, \top)$ is just $\mathcal{C} \times \mathcal{C}^{op}$ and the pullback needed to internalize the homsets is in fact a product. Here the tensor product $(X, X^{op}) \otimes (Y, Y^{op})$ must be an object of the form $(X \otimes Y, (X \multimap Y^{op}) \times (Y \multimap X^{op}))$ and the identity of the tensor must be $(\mathbf{1}, \top)$. Dually, the *par* $(X, X^{op}) \wp (Y, Y^{op})$ is defined as $((X^{op} \multimap Y) \times (Y^{op} \multimap X), X^{op} \otimes Y^{op})$ and the identity of the *par* must be $(\top, \mathbf{1})$. Now since $\mathcal{A}$ is free, there is a functor F of $*$-autonomous categories from $\mathcal{A}$ to $(\mathcal{C} \times \mathcal{C}^{op})$ taking P to (P_O, P_I). This is well-known, but so far no familiar construction had been shown to correspond to the functor F given by the abstract theory. The main contribution of this paper is to show that a familiar proof-theoretic construction, namely *Girard's trips* [16] on a proof-net, represent the action of such a functor on the morphisms of $\mathcal{A}$. Of course one could state the same result using Danos–Regnier graphs, as it was done in [8], but as we shall see a simpler definition of orientations is possible in terms of Girard's trips.

The key idea is simple enough and may be illustrated as a logical translation of formulas and proofs in **CMALL** into formulas and proofs in **IMALL**. In the translation a **CMALL** sequent S: $\vdash \Gamma, A$ becomes *polarized*: a selected formula-occurrence A is mapped to a *positive* formula-occurrence A_O in the succedent of an intuitionistic sequent S' (the *output* part of a logical computation); all other formula-occurrences C in Γ are mapped to *negative* C_I in the antecedent of S' (the *input* part). The polarized occurrences of an atom A become A_O, A_I, just two copies of A. Negation changes the polarity. For other complex polarized formulas, the polarization of the immediate subformulas is uniquely determined – for instance, $(A \wp B)_I$ becomes $A_I \otimes B_I$ – except in the cases of $(A \wp B)_O$ and $(A \otimes B)_I$. In these cases we take the product (logically, the *with*) of two possible choices (the "switches" in a proof-net): for instance, $(A \wp B)_O$ is encoded as $(A_I \multimap B_O) \& (B_I \multimap A_O)$. The intuitive motivation is clear: $A \wp B$ has a reading simultaneously as the internalization of the function space $\mathrm{Hom}_{\mathcal{A}}(A^{\perp}, B)$ and of the function space $\mathrm{Hom}_{\mathcal{A}}(B^{\perp}, A)$. The fact that the translation is functorial here means, roughly, that it is defined independently on the formulas (objects) and on the proofs (morphisms) and that it admits the rule of Cut (composition of morphisms); it is also compatible with cut-elimination. In this form the result can be easily proved within the formalisms of the sequent calculi for **CMALL** and **IMALL**. However, when we ask questions about the *faithfulness* and *fullness* of such a functor, thus also asking questions about the identity of proofs in linear logic, we find it convenient to consider the more refined syntax of proof-nets.

On the other hand, proof-nets are also useful to highlight the geometric aspect of certain logical properties; indeed ideas related to the present result have already proved quite useful in the study of what is sometimes called the *géometrie du calcul* (*geometry of computations*). Our own investigation has been motivated by the desire to understand and clarify the notion of a proof-net and the present result appears to reward many collective efforts in this direction. Given a proof-structure, i.e., a directed graph where edges are labeled with formulas, a *correctness criterion* characterizes those proof-structures which correspond to proofs in the sequent calculus. Girard's original condition (*"there are no short trips"*) [16] is *exponential* in time on the size of the proof-structure, but other *quadratic* criteria were found soon after (among others, one was given in [7]). Thus it is natural to ask *what additional information is contained in the construction of Girard's trips other than the correctness*

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