Chapter 1

GEOMETRY OF DEDUCTION VIA GRAPHS OF PROOFS

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Abstract We are here concerned with the study of proofs from a geometric perspective. By first recalling the pioneering work of Statman in his doctoral thesis *Structural Complexity of Proofs* (1974), we review two recent research programmes which approach the study of structural properties of formal proofs from a geometric perspective: (i) the notion of *proof-net*, given by Girard in 1987 in the context of linear logic; and (ii) the notion of *logical flow graph* given by Buss in 1991 and used as a tool for studying the exponential blow up of proof sizes caused by the cut-elimination process, a recent programme (1996–2000) proposed by Carbone in collaboration with Semmes.

Statman's geometric perspective does not seem to have developed much further than his doctoral thesis, but the fact is that it looks as if the main idea, *i.e.* extracting structural properties of proofs in natural deduction (ND) using appropriate geometric intuitions, offers itself as a very promising one. With this in mind, and having at our disposal some interesting and rather novel techniques developed for *proof-nets* and *logical flow graphs*, we have tried to focus our investigation on a research for an alternative proposal for looking at the geometry of ND systems. The lack of symmetry in ND presents a challenge for such a kind of study. Of course, the obvious alternative is to look at multiple-conclusion

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calculi. We already have in the literature different approaches involving such calculi. For example, Kneale's (1958) *tables of development* (studied in depth by Shoesmith & Smiley (1978)) and Ungar's (1992) multiple-conclusion ND.

After surveying the main research programmes, we sketch a proposal which is similar to both Kneale's and Ungar's in various aspects, mainly in the presentation of a multiple conclusion calculus in ND style. Rather than just presenting yet another ND proof system, we emphasise the use of 'modern' graph-theoretic techniques in tackling the 'old' problem of adequacy of multiple-conclusion ND. Some of the techniques have been developed for *proof-nets* (e.g. splitting theorem, soundness criteria, sequentialisation), and have proved themselves rather elegant and useful indeed.

Keywords: proofs as graphs, natural deduction, multiple-conclusion, geometry of deduction

1. Motivation

In 1980's various studies in "Logic and Computation" were pursued with the intention of giving a logical treatment of computer programming issues. Some of these studies have brought in a number of interesting proof-theoretic developments, such as for example:

- the functional interpretation of logical connectives[1] via deductive systems which use some sort of labelling mechanism:

  (i) Martin-Löf's *Intuitionistic Type theory* [53], which contributed to a better understanding of the foundations of computer science from a type-theoretic perspective, drawing on the connections between constructive mathematics and computer programming;

  and

  (ii) the *Labelled Deductive Systems*, introduced by Gabbay [34], which, arising from the need of computer science applications to handle "meta-level" aspects of logical system in harmony with object-level, helped providing a more general alternative to the "formulae-as-types" paradigm;

- *Linear Logic*, introduced by Girard in [38]. Since then it has become very popular in the theoretical computer science research community. The novelty here is that the logic comes with new connectives forming a new logical system with various interesting features for computer science, such as the possibility of interpreting a sequent as the state of a system and the treatment of a formula as a resource.

In recent years, linear logic has been established as one of the most widely used formalisms for the study of the interface between logic and computation. One of its key aspects represents a rather interesting novelty for studying the geometry of deductions: the concept of *proof-nets*. The theory of proof-nets developed out of a comparison between the sequent calculus and natural deduction (ND) Gentzen systems [36] as well as from an analysis of the importance of studying the structural properties of proofs through a *geometrical* perspective.

Another recent work which also presents a *geometrical* analysis in the study of structural properties of proofs has been developed by Carbone in collaboration with Semmes [14, 15, 16, 17, 18, 22]. Again in the context of "Logic and Computation", the analysis of Carbone and Semmes is motivated by questions which involve the middle ground between mathematical logic and computational complexity. In the beginning of the 1970's, Cook used the notion of satisfiability (a concept from logic) to study one of the most fundamental dichotomies in theoretical computer science: **P** versus **NP**. By the end of the decade Cook and Reckhow had established an important observation which puts emphasis on a relevant direction in complexity theory: **NP** is closed under complementation iff there is a propositional proof system in which all tautologies have a polynomial size proof [27]. This represents an important result linking mathematical logic and computational complexity since it relates classes of computational problems with proof systems. Motivated by questions such as the length of proofs in certain classical proof systems (in the style of Gentzen sequent calculus), Carbone set out to study the phenomenon of expansion of proofs, and for this purpose she found in concept of *logical flow graphs*, introduced by S. Buss [13], a rather convenient mathematical tool. Using the notion of *logical flow graph*, Carbone was able to obtain results such as, for example, providing an explanation for the exponential blow up of proof sizes caused by the cut-elimination process. With appropriate geometrical intuitions associated with the concept of *logical flow graph*, Carbone and Semmes developed a combinatorial model to study the evolution of graphs underlying proofs during the process of cut-elimination.

Now, if on the one hand we have

- Girard's proposal of studying the geometry of deductions through the concept of *proof-nets*, (in [40] he presents various arguments in defense of his programme, emphasizing the importance of "finding out the geometrical meaning of the *Hauptsatz*, *i.e.* what is hidden behind the somewhat boring syntactical manipulations it involves"),

on the other hand, there is

- Carbone's systematic use of *logical flow graph* in a geometrical study of the cut-elimination process, yielding a combinatorial model which uncovers the exponential expansion of proofs after cut-elimination.[2]

Although with different ends and means these two works concern the study of structural features of proofs by a geometric perspective. Back in the 1970's

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