Chapter 3

TWO PARADIGMS OF LOGICAL COMPUTATION IN AFFINE LOGIC?

Gianluigi Bellin*
Facoltà di Scienze
Università di Verona
bellin@sci.univr.it

Abstract We propose a notion of *symmetric reduction* for a system of proof-nets for *Multiplicative Affine Logic with Mix* (**MAL** + Mix) (namely, multiplicative linear logic with the mix-rule the unrestricted weakening-rule). We prove that such a reduction has the strong normalization and Church–Rosser properties. A notion of irrelevance in a proof-net is defined and the *possibility* of cancelling the irrelevant parts of a proof-net without erasing the entire net is taken as one of the *correctness conditions*; therefore purely *local* cut-reductions are given, minimizing cancellation and suggesting a paradigm of *"computation without garbage collection"*. Reconsidering Ketonen and Weyhrauch's decision procedure for affine logic [15, 4], the use of the mix-rule is related to the non-determinism of classical proof-theory. The question arises, whether these features of classical cut-elimination are really irreducible to the familiar paradigm of cut-elimination for intuitionistic and linear logic.

Keywords: affine logic, proof-nets

A Silvia Baraldini

*Paper submitted in 1999, revised in 2001. Research supported by EPSRC senior research fellowship on grant GL/L 33382. This research started during a visit to the University of Leeds in 1997: thanks to Stan Wainer, John Derrick and Michael Rathjen and Diane McMagnus for their hospitality. Thanks to Martin Hyland, Edmund Robinson and especially Arnaud Fleury for extremely useful discussions during the final revision.

R.J.G.B. de Queiroz (ed.), Logic for Concurrency and Synchronisation, 111–144.

1. Introduction

1. *Classical Multiplicative Affine Logic* is classical multiplicative linear logic with the unrestricted rule of *weakening*, but without the rule of *contraction*. Classical affine logic is a much simpler system than classical logic, but it provides similar challenges for *logical computation*, both in the sense of *proof-search* and of *proof normalization* (or *cut-elimination*). For instance, the problem of *confluence* of cut-elimination (the *Church–Rosser property*) is already present in affine logic, but here we do not have the problem of *non-termination*. Affine logic is also simpler than linear logic from the point of view of proof-search: e.g., propositional linear logic is undecidable, yet becomes decidable when the unrestricted rule of weakening is added. Provability in *constant-only multiplicative linear logic* is NP-complete, yet it is decidable in linear time for *constant-only multiplicative affine logic*, as it is shown below.

The tool we will use here, proof-nets for affine logic, is older than the notion of a proof-net for linear logic. In a 1984 paper [15], J. Ketonen and R. Weyhrauch presented a decision procedure for first-order affine logic (called then *direct logic*) which essentially consists in building cut-free proof-nets, using the unification algorithm to determine the axioms. The 1984 paper is sketchy and it has been corrected (see [3, 4], where the relation between the decision procedure and proof-nets for $\mathbf{MLL}^-$ are discussed), but it contains the main ideas exploited in the present paper, namely, the construction of proof-nets *free from irrelevance* through *basic chains*. Yet neither the 1984 paper nor its 1992 revisitation contained a treatment of cut-elimination.[1]

2. The problem of non-confluence for classical affine logic is non-trivial: the following well-known example (given in Lafont's Appendix to [14]) reminds us that the Church–Rosser property is non-deterministic under the familiar *asymmetric* cut-reductions.

Example 1

$$\dfrac{\dfrac{\begin{matrix} d_1 \\ \vdots \\ \vdash \Gamma \end{matrix}}{\vdash \Gamma, A}\,w_1 \qquad \dfrac{\begin{matrix} d_2 \\ \vdots \\ \vdash \Delta \end{matrix}}{\vdash \Delta, \neg A}\,w_2}{\vdash \Gamma, \Delta} \qquad \text{reduces to} \qquad \dfrac{\begin{matrix} d_1 \\ \vdots \\ \vdash \Gamma \end{matrix}}{\dfrac{\mathit{weakenings}}{\vdash \Gamma, \Delta}} \qquad \text{or to} \qquad \dfrac{\begin{matrix} d_2 \\ \vdots \\ \vdash \Delta \end{matrix}}{\dfrac{\mathit{weakenings}}{\vdash \Gamma, \Delta}}$$

Asymmetric reductions.

Indeed classical logic gives no justification for choosing between the two indicated reductions, the first commuting the cut-rule with the *left* application of the weakening-rule ("pushing d_2 up into d_1", thus erasing d_2), the second commuting the cut-rule with the *right* application of the weakening-rule ("pushing d_1 up into d_2", thus erasing d_1). Therefore the cut-elimination process in **MAL**, *a fortiori* in **LK**, is non-deterministic and non-confluent.

Compare this with normalization in intuitionistic logic. In the typed λ calculus a *cut / left weakening* pair corresponds to substitution of $t : A$ for a variable $x : A$ which does not occur in $u : B$; such a substitution is unambiguously defined as $u[t/x] = u$. Moreover in Prawitz's natural deduction **NJ** [19] the rule corresponding to *weakening-right* is the rule *"ex falso quodlibet"* and the normalization step for such a rule involves a form of η-expansion:

$$\dfrac{\begin{array}{c} d \\ \vdots \\ \bot \end{array}}{A \wedge B} \qquad \textit{reduces to} \qquad \dfrac{\dfrac{\begin{array}{c} d \\ \vdots \\ \bot \end{array}}{A} \quad \dfrac{\begin{array}{c} d \\ \vdots \\ \bot \end{array}}{B}}{A \wedge B}$$

Such a reduction does *not* yield cancellation. Thus the cut-elimination procedure for the intuitionistic sequent calculus **LJ** inherits one sensible reduction strategy from natural deduction: *"push the left derivation up into the right one"*. In the case of a *weakening / cut* pair it is always the *left* dedution to be erased.

3. Here we are interested in exploring an obvious remark: for classical logic in addition to the *asymmetric* reductions of Example 1, there is a *symmetric* possibility, the "*Mix*" of d_1 and d_2.

Example 1 cont.

$$\dfrac{\dfrac{\begin{array}{c} d_1 \\ \vdots \\ \vdash \Gamma \end{array}}{\vdash \Gamma, A}\, w_1 \qquad \dfrac{\begin{array}{c} d_2 \\ \vdots \\ \vdash \Delta \end{array}}{\vdash \Delta, \neg A}\, w_2}{\vdash \Gamma, \Delta} \qquad \text{reduces to} \qquad \dfrac{\begin{array}{c} d_1 \\ \vdots \\ \vdash \Gamma \end{array} \quad \begin{array}{c} d_2 \\ \vdots \\ \vdash \Delta \end{array}}{\vdash \Gamma, \Delta}\, Mix$$

Symmetric reduction.

Instead of choosing a direction where to "push up" the cut-rule, we do both asymmetric reductions, using the mix-rule.

The idea is loosely related to a procedure well-known in the literature for the case when both cut-formulas result from a contraction-rule, with the name *cross-cut reduction.* Let d_1 and d_2 be derivations of the left and right premises of the cut-rule:

Logic for Concurrency and Synchronisation
De Queiroz, R.J. (Ed.)
2003, XXI, 285 p., Hardcover
ISBN: 978-1-4020-1270-9