

DUALITY AND THEOREMS OF THE ALTERNATIVES

2.1 THE DUALITY THEOREM

The primal problem for a linear program stated in von Neumann “symmetric” form is:

$$\begin{array}{llll} & \text{Minimize} & c^T x = z & \\ \text{PRIMAL:} & \text{subject to} & Ax \geq b, & A : m \times n, \\ & & x \geq 0, & \end{array} \quad (2.1)$$

and the dual problem is

$$\begin{array}{llll} & \text{Maximize} & b^T y = v & \\ \text{DUAL:} & \text{subject to} & A^T y \leq c, & A : m \times n, \\ & & y \geq 0. & \end{array} \quad (2.2)$$

The von Neumann symmetric form is actually not symmetric but skew-symmetric because the full system of relations is:

$$\begin{pmatrix} 0 & A & -b \\ -A^T & 0 & c \\ b & -c & 0 \end{pmatrix} \begin{pmatrix} y \\ x \\ 1 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad x \geq 0, y \geq 0. \quad (2.3)$$

The *Duality Theorem* is a statement about the range of possible z values for the primal versus the range of possible v values for the dual. This is depicted graphically in Figure 2-1, for the case where the primal and dual are both feasible.

Von Neumann stated but did not prove the Duality Theorem: *If the primal (2.1) and dual (2.2) have feasible solutions, then there exist optimal feasible solutions to*

both the primal and the dual that are equal. We shall formally state and prove the Duality Theorem using the Infeasibility Theorem, which is proved using the Fourier-Motzkin Elimination Process; see *Linear Programming 1*. Here we state the Infeasibility Theorem without proof.

THEOREM 2.1 (Infeasibility Theorem) *The system of linear inequalities*

$$\sum_{j=1}^n a_{ij}x_j \geq b_j \quad \text{for } i = 1, \dots, m \quad (2.4)$$

is infeasible if and only if there exists a nonnegative linear combination of the inequalities that is an infeasible inequality. In matrix notation, the system $Ax \geq b$ is infeasible if and only if there exists a vector $y \geq 0$ such that $y^T Ax \geq y^T b$ is an infeasible inequality, namely one where $y^T A = 0$ and $y^T b > 0$.

▷ **Exercise 2.1** State the Infeasibility Theorem in terms of the system

$$\begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned} \quad (2.5)$$

and apply Phase I of the Simplex Algorithm to prove the Infeasibility Theorem.

COROLLARY 2.2 (Infeasible Equation) *If a system of linear equations in nonnegative variables is infeasible, there exists a linear combination of the equations that is an infeasible equation in nonnegative variables.*

Assuming that primal and dual solutions exist, the weaker form of the Duality Theorem, which follows, is obvious.

THEOREM 2.3 (Weak Duality Theorem) *If x^o is any feasible solution to the primal (2.1) and y^o is any feasible solution to the dual (2.2), then*

$$y^{oT}b = v^o \leq z^o = c^T x^o. \quad (2.6)$$

Proof. We have

$$\begin{aligned} Ax^o &\geq b & y^{oT}A &\leq c^T \\ c^T x^o &= z^o & y^{oT}b &= v^o \end{aligned}$$

Multiplying $Ax^o \geq b$ by y^{oT} on the left and multiplying $y^{oT}A \leq c^T$ by x^o on the right we obtain

$$\begin{aligned} y^{oT}Ax^o &\geq y^{oT}b = v^o \\ y^{oT}Ax^o &\leq c^T x^o = z^o \end{aligned}$$

Therefore,

$$v^o = y^{oT}b \leq y^{oT}Ax^o \leq c^T x^o = z^o.$$

This concludes our proof. ■

COROLLARY 2.4 (Bounds on the Objectives) *Every feasible solution y^o to the dual yields a lower bound $y^{oT}b$ to values of z^o for feasible solutions x^o to the primal. Conversely, every feasible solution x^o to the primal yields an upper bound c^Tx^o to values of v^o for feasible solutions y^o to the dual.*

▷ **Exercise 2.2** Prove Corollary 2.4.

COROLLARY 2.5 (Optimality) *If $v^o = z^o$ then $v^o = \max v$ and $z^o = \min z$.*

We can depict the relationship by plotting the points v^o and z^o on a line as shown in Figure 2-1.

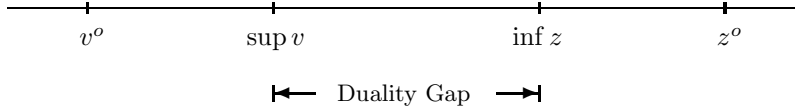


Figure 2-1: Illustration of the Duality Gap

We are now ready to formally state and prove Von Neumann's Duality Theorem which states that if feasible solutions to the primal and dual exist then the *duality gap* (depicted in Figure 2-1) is zero and $\sup v$ is actually attained for some choice of y , and $\inf z$ is attained for some choice of x .

THEOREM 2.6 (Strong Duality Theorem) *If the primal system $\min z = c^Tx$, $Ax \geq b$, $x \geq 0$ has a feasible solution and the dual system $\max v = b^Ty$, $A^Ty \leq c$, $y \geq 0$ has a feasible solution, then there exist optimal feasible solutions $x = x^*$ and $y = y^*$ to the primal and dual systems such that*

$$b^Ty^* = \max v = \min z = c^Tx^*. \quad (2.7)$$

Proof. Consider the system of inequalities and corresponding infeasibility multipliers:

$$Ax \geq b \quad : \quad \tilde{y} \quad (2.8)$$

$$Ix \geq 0 \quad : \quad \tilde{u} \quad (2.9)$$

$$-A^Ty \geq -c \quad : \quad \tilde{x} \quad (2.10)$$

$$Iy \geq 0 \quad : \quad \tilde{v} \quad (2.11)$$

$$b^Ty - c^Tx \geq 0 \quad : \quad \theta \quad (2.12)$$

We first show that (2.8) through (2.12) is a feasible system from which it follows by the Weak Duality Theorem 2.3 that strong duality holds. Assume, on the contrary, that (2.8) through (2.12) is an infeasible system. In general, if $Ms \geq d$



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