

DEGENERACY

We have seen that if degeneracy occurs, then it is possible to have a sequence of iterations with no decrease in the value of z . Under such circumstances, it may happen that a basic set will be repeated, thereby initiating an endless cycle of such repetitions. It turns out, for reasons not fully understood, that in practice almost all problems are degenerate and some are highly degenerate, but that in spite of degeneracy, cycling almost never happens. This is why early commercial software packages did not include any degeneracy resolving schemes. When there is degeneracy or “near” degeneracy it tends to slow the solution process, and this has given rise to a number of *anti-cycling* or *degeneracy resolving* schemes that have been very successfully used in commercial software packages to reduce the number of iterations.

5.1 EXAMPLES OF CYCLING

Example 5.1 (Hoffman) In 1951, A. J. Hoffman constructed an ingenious example to show that cycling can occur under degeneracy; it involves three equations and eleven variables; see Table 5-1 and Table 5-2. He showed, in the case of degeneracy, that if one *resolved* the ambiguity of choice regarding which variable to drop from the basic set by the rule of *selecting the first among them*, then the tableau at iteration 10 would turn out to be the same as that at iteration 0. Notice in Tables 5-1 and 5-2 that column 1, associated with the relation $x_1 = 1$, remains in the basis for all iterations. Next notice that the tableau for iteration 2 is exactly the same as that of iteration 0 if we relabel the indices $(2, 3, 4, \dots, 11)$ of iteration 0 as $(4, 5, 6, \dots, 11; 2, 3)$. Hence, eight more iterations will repeat iteration 0. It follows, in this case, using the first choice rule, that the same basic set would be repeated every ten iterations and the Simplex Algorithm would cycle forever without converging to an optimal solution.

x_1	x_2	x_3	x_4	x_5	x_6
Iteration 0					
			$-(1 - \cos \theta)/\cos \theta$	w	
1	1	1	$\boxed{\cos \theta}$ $\sin \theta \tan \theta/w$	$-w \cos \theta$ $\cos \theta$	$\cos 2\theta$ $\tan \theta \sin 2\theta/w$
Iteration 1					
	$(1 - \cos \theta)/\cos^2 \theta$			$w(2 \cos \theta - 1)/\cos \theta$	$-2 \sin \theta \tan \theta$
1	$\sec \theta$ $-\tan^2 \theta/w$	1	1	$-w$ $\boxed{\sec \theta}$	$4 \cos^2 \theta - 3$ $\tan^2 \theta/w$
Iteration 2					
	$4 \sin^2 \theta$	$w(1 - 2 \cos \theta)$			$-(1 - \cos \theta)/\cos \theta$
1	$\cos \theta$ $-\tan \theta \sin \theta/w$	$w \cos \theta$ $\cos \theta$	1	1	$\boxed{\cos \theta}$ $\sin \theta \tan \theta/w$

Where $\theta = 2\pi/5$, $w > (1 - \cos \theta)/(1 - 2 \cos \theta)$

Table 5-1: Hoffman's Example of Cycling (Continued on the Right)

▷ **Exercise 5.1** The purpose of this exercise is to demonstrate how many of the relations in Hoffman's examples are determined assuming that $\theta = 2\pi/5$.

1. Show that $\cos 2\theta = \cos 3\theta$.
2. Show that $\sin 2\theta = -\sin 3\theta$.
3. Use (1) to show that $\cos 2\theta + \cos \theta = \cos \theta \cos 2\theta$.
4. Use (1) and (3) to show that on iteration 1 the objective coefficient for x_6 is $-2 \sin \theta \tan \theta$.
5. Use (1) to show that the coefficient $\bar{a}_{26} = 4 \cos^2 \theta - 3$.

Example 5.2 (Beale's Three Equation, Seven Variable Example) In 1955, E. M. L. Beale constructed a second example, a version of which is shown in Table 5-3, which is remarkable for its simplicity. It also has three equations but only seven variables. Using the same rule for resolving a tie, the tableau at iteration 6 is the same as that at iteration 0; it has the same basic variables in the same order. It is conjectured that this is the simplest not totally degenerate example of cycling; i.e., none can be constructed with fewer variables regardless of the number of equations.

Example 5.3 (Tucker's Totally Degenerate Example) The following example, due to Tucker, is said to be the simplest of all examples constructed so far. However, this

x_7	x_8	x_9	x_{10}	x_{11}	RHS
Iteration 0					
$2w$	$4 \sin^2 \theta$	$-2w \cos 2\theta$	$4 \sin^2 \theta$	$w(1 - 2 \cos \theta)$	$z(\min)$
$-2w \cos^2 \theta$	$\cos 2\theta$	$2w \cos^2 \theta$	$\cos \theta$	$w \cos \theta$	1
$\cos 2\theta$	$-2 \sin^2 \theta/w$	$\cos 2\theta$	$-\tan \theta \sin \theta/w$	$\cos \theta$	0
Iteration 1					
$2w \cos \theta$	$(\cos \theta - 1)/\cos \theta$	$3w$	$2 \sin \theta \tan \theta$	$-w(4 \cos^2 \theta - 3)$	z
$-2w \cos \theta$	$4 \cos^2 \theta - 3$	$2w \cos \theta$	1	w	1
1	$2 \sin \theta \tan \theta/w$	$4 \cos^2 \theta - 3$	$-2 \sin \theta \tan \theta/w$	$4 \cos^2 \theta - 3$	0
Iteration 2					
$+w$		$2w$	$4 \sin^2 \theta$	$-2w \cos 2\theta$	z
$-w \cos \theta$	$\cos 2\theta$	$-2w \cos^2 \theta$	$\cos 2\theta$	$2w \cos^2 \theta$	1
$\cos \theta$	$\tan \theta \sin 2\theta/w$	$\cos 2\theta$	$-2 \sin^2 \theta/w$	$\cos 2\theta$	0

Where $\theta = 2\pi/5$, $w > (1 - \cos \theta)/(1 - 2 \cos \theta)$

Table 5-2: Hoffman's Example of Cycling (Continued from the Left)

example has a totally degenerate solution (see Exercise 5.4).

$$\begin{aligned}
 &\text{Minimize} && -2x_1 - 3x_2 + x_3 + 12x_4 = z \\
 &\text{subject to} && -2x_1 - 9x_2 + x_3 + 9x_4 \leq 0 \\
 &&& \frac{1}{3}x_1 + x_2 - \frac{1}{3}x_3 - 2x_4 \leq 0 \\
 &&& \text{and } x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0.
 \end{aligned} \tag{5.1}$$

On adding slack variables $x_5 \geq 0$ and $x_6 \geq 0$, a feasible solution is readily available as the slacks $(x_5, x_6) = (0, 0)$. Choose the initial basic variables as the slacks x_5, x_6 .

1. Assume the index s of the incoming variable is chosen as the nonbasic variable with the most negative reduced cost.
2. Assume the index j_r of the outgoing variable is determined by looking at all $\bar{a}_{is} > 0$ as specified in the Simplex Algorithm and choosing $r = i$ as the smallest index such that $\bar{a}_{is} > 0$.

Then the basic sets of indices generated at each iteration for the first six iterations are: $\{5, 2\}$, $\{1, 2\}$, $\{1, 4\}$, $\{3, 4\}$, $\{3, 6\}$, and $\{5, 6\}$, respectively. Observe that the basis repeats on the sixth iteration, leading to an endless cycle of iterations.

▷ **Exercise 5.2** Apply the tableau form of the Simplex Algorithm to Tucker's example, the linear program (5.1), to show that it cycles endlessly.



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