CHAPTER 6

VARIANTS OF THE SIMPLEX METHOD

By a *variant* of the Simplex Method (in this chapter) we mean an algorithm consisting of a sequence of pivot steps in the primal system using alternative rules for the selection of the pivot. Historically these variants were developed to take advantage of a situation where an infeasible basic solution of the primal is available. In other applications there often occurs a set of problems differing from one another only in their constant terms and cost factors. In such cases, it is convenient to omit Phase I and to use the optimal basis of one problem as the initial basis for the next.

6.1 INTRODUCTION

Several methods have been proposed for varying the Simplex Algorithm to reduce the number of iterations. This is especially needed for problems involving many equations in order to reduce the computation time. It is also needed for problems involving a large number of variables n, for the number of iterations in practice appears to grow roughly proportional to n.

For example, instead of using the selection rule $\bar{c}_s = \min \bar{c}_j$, one could select $j = s$ such that introducing x_s into the basic set gives the largest decrease in the value of z in the next basic solution. This requires computing, for $\bar{c}_j < 0$, the largest in absolute value of $\bar{c}_j \theta_j$, where θ_j is determined so that if x_j replaced x_{j_r} then the solution will remain feasible. This rule is obviously not practical when using the revised Simplex Method with multipliers. Even using the standard canonical form, considerably more computations would be required per iteration. It is possible, however, in the nondegenerate case, to develop a modification of the canonical form in which the coefficient of the ith basic variable is allowed to be different from unity in the ith equation but $\bar{b}_i = 1$. In this form the selection of s by the steepest descent criterion would require a little more effort; moreover (by means of a special device), no more effort than that for the standard Simplex Algorithm would be required to maintain the tableau in proper modified form from iteration to iteration (see Section 6.2 for details).

The simplest variant occurs when the new problem differs from the original in the cost coefficients alone. In this case, the cost coefficients are replaced by the new ones, and Phase II of the Revised Simplex Method is applied. Another possibility is to use the parametric objective method to be described in Section 6.4.

An important variant occurs when the new problem differs from the original in the constant terms only. In this case the optimal basis of the first problem will still price out dual feasible, i.e., $\bar{c}_j \geq 0$, for the second, but the associated primal solution may not be feasible. For this situation, we could use the *Dual-Simplex* Algorithm, which is the variant of the standard Primal-Simplex Algorithm, to be discussed in Section 6.3, or the paramteric right-hand-side method to be described in Section 6.4, or the *Primal-Dual* method of Section 6.6.

However, when the problems differ by more than either the constant terms or the cost coefficient terms, the old basis may be neither primal feasible nor dual feasible. When neither the basic solution nor the dual solution generated by its simplex multipliers remains feasible, the corresponding algorithm is called *composite*. The *Self-Dual* parametric algorithm discussed in Section 6.5 is an example of such a composite algorithm.

Correspondence of Primal and Dual Bases.

In 1954, Lemke discovered a certain correspondence between the bases of the primal and dual systems that made it possible to interpret the Simplex Algorithm *as applied to the dual* as a sequence of basis changes in the primal; this interpretation is called the Dual-Simplex Algorithm. From a computational point of view, the Dual-Simplex Algorithm is advantageous because the size of the basis being manipulated in the computer is $m \times m$ instead of $n \times n$. In this case, however, the associated basic solutions of the primal are not feasible, but the simplex multipliers continue to price out optimal (hence, yield a basic feasible solution to the dual). It is good to understand the details of this correspondence, for it provides a means of easily dualizing a problem without transposing the constraint matrix.

Consider the standard form

$$\begin{array}{lrl} \text{Minimize} & c^T x &= z \\ \text{subject to} & Ax &= b, \qquad A:\ m \times n, \\ & x &\geq 0, \end{array} \tag{6.1}$$

and the dual of the standard form:

$$\begin{array}{lrl} \text{Maximize} & b^T \pi &= v \\ \text{subject to} & A^T \pi &\leq c, \qquad A:\ m \times n, \end{array} \tag{6.2}$$

PRIMAL-DUAL CORRESPONDENCES		
	Primal	Dual
Basis	B	$\bar{B} = \begin{pmatrix} B^T & 0 \\ N^T & I \end{pmatrix}$
Basic variables	x_B	$\pi,\ y_N = \bar{c}_N$
Nonbasic variables	x_N	$y_B = \bar{c}_B$
Feasibility condition	$Ax = b,\ x \geq 0$	$\bar{c} \geq 0$

Table 6-1: Primal-Dual Correspondences

where π_i is unrestricted in sign. We assume A is full rank. Adding slack variables $y \geq 0$ to the dual problem (6.2) we get:

$$\begin{array}{lrcl} \text{Maximize} & b^T\pi & = & v \\ \text{subject to} & A^T\pi + Iy & = & c, \qquad A:\ m \times n, \\ & y & \geq & 0. \end{array} \tag{6.3}$$

Clearly the dual has n basic variables and m nonbasic variables. The variables π are unrestricted in sign and, when A is full rank, always constitute m out of the n basic variables of the dual. The basis for the primal is denoted by B and the nonbasic columns are denoted by N. Note that $y = c - A^T\pi = \bar{c} \geq 0$ when the dual is feasible and that $\bar{c}_B = 0$ and $\bar{c}_N \geq 0$ when the primal is optimal. The basic and nonbasic columns for the dual are denoted by $\bar{B}$, an $n \times n$ matrix, and $\bar{N}$, an $n \times m$ matrix

$$\bar{B} = \begin{pmatrix} B^T & 0 \\ N^T & I_{n-m} \end{pmatrix}, \qquad \bar{N} = \begin{pmatrix} I_m \\ 0 \end{pmatrix}, \tag{6.4}$$

where I_{n-m} is an $(n-m)$-dimensional identity matrix and I_m is an m-dimensional identity matrix. Thus, (π, y_N) as basic variables and y_B as nonbasic variables constitute a basic solution to the dual if and only if $y_B = 0$.

▷ **Exercise 6.1** Show that the determinant of B has the same value as that of $\bar{B}$. Also show that if B^{-1} exists, then $\bar{B}^{-1}$ exists.

It is now clear that there is a correspondence between primal and dual bases. These correspondences are shown in Table 6-1. With these correspondences in mind, we shall discuss variations of the Simplex Method.

▷ **Exercise 6.2** How is the concept of complementary primal and dual variables related to the correspondence of primal and dual bases?

Definition (Dual Degeneracy): We have already defined degeneracy and nondegeneracy with respect to the primal. A basis is said to be *dual degenerate* if one or more of the $\bar{c}_j$ corresponding to nonbasic variables x_j are zero and to be *dual nondegenerate* otherwise.

6.2 MAX IMPROVEMENT PER ITERATION

In this section we present an alternative canonical form for efficiently determining the incoming column that yields the maximum improvement per iteration. It has been observed on many test problems that this often leads to fewer iterations. We will assume, to simplify the discussion, that all basic feasible solutions that are generated are nondegenerate.

Assume $x_j \geq 0$ for $j = 1, \ldots, n$ and x_j for $j = 1, \ldots, m$ are basic variables, in the *standard canonical form*:

$$\begin{array}{lllllll} -z & & + \ \bar{c}_{m+1}x_{m+1} + \cdots + & \bar{c}_j x_j + \cdots + & \bar{c}_n x_n & = & -\bar{z}_0 \\ x_1 & & + \ \bar{a}_{1,m+1}x_{m+1} + \cdots + & \bar{a}_{1j} x_j + \cdots + & \bar{a}_{1n} x_n & = & \bar{b}_1 \\ \quad \ddots & & \qquad \vdots & \vdots & \vdots & & \vdots \\ \qquad x_p & & + \ \bar{a}_{p,m+1}x_{m+1} + \cdots + & \bar{a}_{pj} x_j + \cdots + & \bar{a}_{pn} x_n & = & \bar{b}_p \\ \qquad \quad \ddots & & \qquad \vdots & \vdots & \vdots & & \vdots \\ & x_m & + \ \bar{a}_{m,m+1}x_{m+1} + \cdots + & \bar{a}_{mj} x_j + \cdots + & \bar{a}_{mn} x_n & = & \bar{b}_m \end{array} \tag{6.5}$$

where $\bar{a}_{ij}$, $\bar{c}_j$, $\bar{b}_i$, and $\bar{z}_0$ are constants and we assume that $\bar{b}_i > 0$ for $i = 1, \ldots, m$. If there are two or more $\bar{c}_j < 0$, our problem is to find which $j = s$ among them has the property that putting column s into the basis and driving some column $j = r$ out of the basis gives the greatest decrease in z while preserving feasibility. Using the standard canonical form (6.5), the *main work* is that of performing ratio tests for the several columns j such that $\bar{c}_j < 0$, plus the update work of pivoting, which takes mn operations where each operation consists of one addition (or subtraction) plus one multiplication. To do this more efficiently, consider the *alternative canonical form*:

$$\begin{array}{llllll} -z & & + \ \bar{c}_{m+1}x_{m+1} + \cdots + & \bar{c}_n x_n & = & -\bar{z}_0 \\ \alpha_{11}x_1 & & + \ \alpha_{1,m+1}x_{m+1} + \cdots + & \alpha_{1n} x_n & = & \mathbf{1} \\ \quad \ddots & & \qquad \vdots & \vdots & & \vdots \\ \qquad \alpha_{pp}x_p & & + \ \alpha_{p,m+1}x_{m+1} + \cdots + & \alpha_{pn} x_n & = & \mathbf{1} \\ \qquad \quad \ddots & & \qquad \vdots & \vdots & & \vdots \\ & \alpha_{mm}x_m & + \ \alpha_{m,m+1}x_{m+1} + \cdots + & \alpha_{mn} x_n & = & \mathbf{1} \end{array} \tag{6.6}$$

formed by rescaling the rows by dividing them by $\bar{b}_i > 0$ for all $i = 1, \ldots, m$. In this format we scan column j such that $\bar{c}_j < 0$ and find for each such j the row

$$r_j = \text{argmax } \alpha_{ij}.$$

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