§4. Further Results and Applications

This paragraph is devoted primarily to applications of the main theorems in the text. We begin with two direct and simple applications of the Banach principle, first to establish some fixed point theorems for nonexpansive maps in Hilbert space and second to integral and differential equations. In Section 3, we give numerous applications of the elementary invariance of domain theorem. The last two sections are devoted to some further applications of the geometric KKM-principle.

1. Nonexpansive Maps in Hilbert Space

The Banach principle involves a contractive map in an arbitrary complete metric space. By giving the space a sufficiently rich structure, the contractiveness hypothesis on the map can be relaxed to nonexpansiveness; of course, uniqueness of the fixed point cannot be preserved, as the reflection of $\boldsymbol{R}^2$ in a line shows.

In this section we deal with (real) Hilbert space; the following proposition will play a basic role.

(1.1) PROPOSITION. *Let H be a Hilbert space, and let u, v be two elements of H. If there is an $x \in H$ such that $\|x-u\| \le R$, $\|x-v\| \le R$ and $\|x-(u+v)/2\| \ge r$, then $\|u-v\| \le 2\sqrt{R^2-r^2}$.*

PROOF. By the parallelogram law,

$$\begin{aligned}\|u-v\|^2 &= \|(x-v)-(x-u)\|^2 \\ &= 2\|x-v\|^2 + 2\|x-u\|^2 - \|x-v+x-u\|^2 \\ &= 2\|x-v\|^2 + 2\|x-u\|^2 - 4\left\|x - \frac{u+v}{2}\right\|^2,\end{aligned}$$

and the conclusion follows. □

We apply this proposition to study nonexpansive maps on bounded sets:

(1.2) LEMMA. *Let $C \subset H$ be a bounded set, and let $F : C \to C$ be nonexpansive. Assume that x, y and $a = (x+y)/2$ belong to C. If $\|x-F(x)\| \le \varepsilon$ and $\|y-F(y)\| \le \varepsilon$, then*

$$\|a-F(a)\| \le 2\sqrt{2\delta(C)}\sqrt{\varepsilon},$$

where $\delta(C) =$ diameter of C.

PROOF. Because

$$\|x-y\| \le \left\|x - \frac{a+F(a)}{2}\right\| + \left\|y - \frac{a+F(a)}{2}\right\|,$$

at least one of the terms on the right, say the first, must satisfy

$$\left\| x - \frac{a + F(a)}{2} \right\| \geq \frac{1}{2}\|x - y\|.$$

But also $\|x - a\| = \frac{1}{2}\|x - y\|$ and

$$\|x - F(a)\| \leq \|x - F(x)\| + \|F(x) - F(a)\| \leq \varepsilon + \|x - a\| = \varepsilon + \frac{1}{2}\|x - y\|.$$

By (1.1), we conclude that

$$\|a - F(a)\| \leq 2\sqrt{\left(\varepsilon + \tfrac{1}{2}\|x - y\|\right)^2 - \left(\tfrac{1}{2}\|x - y\|\right)^2} = 2\sqrt{\varepsilon}\sqrt{\varepsilon + \|x - y\|},$$

and since both ε and $\|x - y\|$ do not exceed $\delta(C)$, the proof is complete. □

This leads to the desired modification of the Banach theorem.

(1.3) THEOREM (Browder–Göhde–Kirk). *Let C be a nonempty closed bounded convex set in a Hilbert space. Then each nonexpansive map $F : C \to C$ has at least one fixed point.*

PROOF. There is no loss in generality to assume that $0 \in C$. For each integer $n = 2, 3, \ldots$ let $F_n = \left(1 - \frac{1}{n}\right)F$; because C is convex and contains the origin, each F_n maps C into itself. Moreover, each $F_n : C \to C$ is contractive, so by Banach's theorem, each F_n has a fixed point x_n, and

$$\|x_n - F(x_n)\| = \frac{1}{n}\|F(x_n)\| \leq \frac{1}{n}\delta(C).$$

For each $n \geq 2$, let $Q_n = \{x \in C \mid \|x - F(x)\| \leq \frac{1}{n}\delta(C)\}$; then $Q_2 \supset Q_3 \supset \cdots$ is a descending sequence of closed sets, and by what we have just shown, no Q_n is empty. We observe that if $x, y \in Q_{8n^2}$ and $a = (x + y)/2$, then according to the lemma

$$\|a - F(a)\| \leq 2\sqrt{2\delta(C)}\sqrt{\frac{\delta(C)}{8n^2}}, \quad \text{so that} \quad \frac{x + y}{2} \in Q_n.$$

Let $d_n = \inf\{\|x\| \mid x \in Q_n\}$; because the Q_n are descending, we see that $d_2 \leq d_3 \leq \cdots$ is a nondecreasing sequence of reals, which, being bounded by $\delta(C)$, converges to some d. Finally, let

$$A_n = Q_{8n^2} \cap \overline{B\left(0, d + \frac{1}{n}\right)}.$$

Then A_n is a descending sequence of nonempty closed sets. We calculate the diameter of A_n: if $x, y \in A_n$, then $\|0 - x\| \leq d + 1/n$, $\|0 - y\| \leq d + 1/n$, and by our observation above, $\|0 - (x + y)/2\| \geq d_n$; therefore, by (1.1) we find

$$\|x-y\| \le 2\sqrt{\left(d+\frac{1}{n}\right)^2 - d_n^2} = 2\sqrt{2dn^{-1}+n^{-2}+(d^2-d_n^2)}.$$

The term on the right is therefore an upper bound for $\delta(A_n)$, and shows that $\delta(A_n) \to 0$ as $n \to \infty$.

By Cantor's theorem, there is an $x_0 \in \bigcap_n A_n$; since $x_0 \in \bigcap_n Q_{8n^2}$, we find

$$\|x_0 - F(x_0)\| \le \delta(C)/(8n^2) \qquad \text{for all } n;$$

therefore, $\|x_0 - F(x_0)\| = 0$, and x_0 is a fixed point. □

Let C be a closed ball in a Hilbert space H; we will now consider the nonexpansive maps defined on C with values in H. For this purpose, we need the nonexpansiveness of the standard retraction of H on C:

(1.4) LEMMA. *Let H be a Hilbert space and C the closed ball $\{x \in H \mid \|x\| \le c\}$. Define a map $r : H \to C$ by*

$$r(x) = \begin{cases} x, & \|x\| \le c, \\ c\dfrac{x}{\|x\|}, & \|x\| \ge c. \end{cases}$$

Then $r : H \to C$ is nonexpansive.

PROOF. We first observe that if $u, v \ne 0$, then

$$(u - r(u), r(v) - r(u)) \le 0.$$

This is certainly true for $\|u\| \le c$ since $r(u) = u$; and if $\|u\| \ge c$, we have

$$(u - r(u), r(v) - r(u)) = \begin{cases} \left(1 - \dfrac{c}{\|u\|}\right)[(u,v) - c\|u\|], & \|v\| \le c, \\ \left(1 - \dfrac{c}{\|u\|}\right)\left[c\dfrac{(u,v)}{\|v\|} - c\|u\|\right], & \|v\| \ge c, \end{cases}$$

so that because $|(u,v)| \le \|u\|\|v\|$, our observation is established. To prove the lemma, write

$$x - y = r(x) - r(y) + x - r(x) + r(y) - y \equiv r(x) - r(y) + a;$$

then

$$\|x-y\|^2 = \|r(x) - r(y)\|^2 + \|a\|^2 + 2(a, r(x) - r(y));$$

because of our observation,

$$(a, r(x) - r(y)) = -(x - r(x), r(y) - r(x)) - (y - r(y), r(x) - r(y)) \ge 0,$$

so $\|x-y\|^2 \ge \|r(x) - r(y)\|^2$, and the proof is complete. □

This leads to the desired result:

(1.5) THEOREM (Nonlinear alternative for nonexpansive maps). *Let H be a Hilbert space and C the closed ball $\{x \in H \mid \|x\| \le c\}$. Then each nonexpansive $F : C \to H$ has at least one of the following two properties*:
(a) *F has a fixed point*,
(b) *there exist $x \in \partial C$ and $\lambda \in (0,1)$ such that $x = \lambda F(x)$.*

PROOF. By (1.4), the map $r : H \to C$ is nonexpansive, therefore so also is $r \circ F : C \to C$, and by (1.3), we have $rF(x) = x$ for some $x \in C$. Now we repeat the reasoning of (0.2.3): If $F(x) \in C$, then $x = rF(x) = F(x)$, so F has a fixed point; if $F(x)$ does not belong to C, then $x = rF(x) = cF(x)/\|F(x)\|$, so $x \in \partial C$, and taking $\lambda = c/\|F(x)\| < 1$ completes the proof. □

Several fixed point theorems are obtained from (1.5) by imposing conditions that prevent occurrence of the second possibility:

(1.6) COROLLARY. *Let $C = \{x \in H \mid \|x\| \le r\}$, and let $F : C \to H$ be nonexpansive. Assume that for all $x \in \partial C$, one of the following conditions holds*:
(a) $\|F(x)\| \le \|x\|$,
(b) $\|F(x)\| \le \|x - F(x)\|$,
(c) $\|F(x)\|^2 \le \|x\|^2 + \|x - F(x)\|^2$,
(d) $(x, F(x)) \le \|x\|^2$,
(e) $F(x) = -F(-x)$.
Then F has a fixed point.

The proof is strictly analogous to the proof of (1.4.2) and (1.4.3), and is left to the reader.

As a further application, we have

(1.7) COROLLARY. *Let H be a Hilbert space and $F : H \to H$ be nonexpansive. Assume $(x, x - F(x)) \ge \mu(\|x\|)\|x\|$, where $\mu(\|x\|) \to \infty$ as $\|x\| \to \infty$. Then the nonexpansive field $x \mapsto f(x) = x - F(x)$ is surjective.*

PROOF. Given a point $y_0 \in H$ let $g(x) = x - [F(x) + y_0]$ for $x \in H$. From

$$\frac{(x, g(x))}{\|x\|} = \frac{(x, f(x))}{\|x\|} - \frac{(x, y_0)}{\|x\|} \ge \mu(\|x\|) - \|y_0\|$$

it follows that for a sufficiently large $r > 0$,

$$(x, g(x)) \ge 0 \quad \text{for all } x \in H \text{ with } \|x\| = r.$$

By (1.6)(d), because $G(x) = F(x) + y_0$ is nonexpansive, we get $g(x_0) = 0$ for some x_0; hence $y_0 = x_0 - F(x_0) = f(x_0)$, and our assertion follows. □

2. Applications of the Banach Principle to Integral and Differential Equations

Use of the Banach principle requires that a given $F : Y \to Y$ be contractive relative to some complete metric d in Y. If the given F is not contractive with respect to one metric, it may be possible to find another complete metric with respect to which F is contractive. For example, the (linear) map $(x, y) \mapsto \frac{1}{10}(8x + 8y, x + y)$ of $\boldsymbol{R}^2$ into itself is not contractive with respect to the usual metric

$$d[(x, y), (z, w)] = \sqrt{(x - z)^2 + (y - w)^2};$$

but it is contractive, with contraction constant $\frac{9}{10}$, relative to the (equivalent, and complete) metric

$$\widehat{d}\,[(x, y), (z, w)] = |x - z| + |y - w|.$$

Thus, each complete metric d in Y determines a class $\mathscr{F}(d)$ of maps $F : Y \to Y$ that are contractive with respect to d, and in general, $\mathscr{F}(d) \neq \mathscr{F}(\widehat{d}\,)$ even for equivalent metrics d and $\widehat{d}$.

If E is a Banach space, recall that two norms $|x|$ and $\|x\|$ are *equivalent* if there are constants $m, M > 0$ with $m\|x\| \leq |x| \leq M\|x\|$, so that a map Lipschitzian in one norm is Lipschitzian in any equivalent norm. Thus, in Banach spaces, to study a Lipschitzian map $F : E \to E$, it is frequently very fruitful to seek a norm under which F is contractive.

These considerations are illustrated in the following proof of the existence of solutions for the Volterra integral equation of the second kind.

(2.1) THEOREM. *Let $K : [0, T] \times [0, T] \times \boldsymbol{R} \to \boldsymbol{R}$ be continuous and satisfy a Lipschitz condition*

$$|K(t, s, x) - K(t, s, y)| \leq L|x - y|$$

for all $(s, t) \in [0, T] \times [0, T]$, and $x, y \in \boldsymbol{R}$. Then for any $v \in C[0, T]$ the equation

$$u(t) = v(t) + \int_0^t K(t, s, u(s))\, ds \qquad (0 \leq t \leq T)$$

has a unique solution $u \in C[0, T]$. Moreover, if we define a sequence of functions $\{u_n\}$ inductively by choosing any $u_0 \in C[0, T]$ and setting

$$u_{n+1}(t) = v(t) + \int_0^t K(t, s, u_n(s))\, ds,$$

then the sequence $\{u_n\}$ converges uniformly on $[0, T]$ to the unique solution u.

PROOF. Let E be the Banach space of all continuous real-valued functions on $[0,T]$ equipped with the norm

$$\boldsymbol{|} g \boldsymbol{|} = \max_{0\leq t\leq T} e^{-Lt}|g(t)|.$$

This norm is in fact equivalent to the sup norm $\|x\|$, since

$$e^{-LT}\|x\| \leq \boldsymbol{|} x \boldsymbol{|} \leq \|x\|;$$

and moreover, it is also complete.

Define $F : E \to E$ by

$$F(g)(t) = v(t) + \int_0^t K(t,s,g(s))\,ds;$$

to prove that the integral equation has a solution, it is enough to show that $F : E \to E$ has a fixed point. We prove that, in fact, F is contractive: for

$$\begin{aligned}
\boldsymbol{|} F(g) - F(h) \boldsymbol{|} &\leq \max_{0\leq t\leq T} e^{-Lt}\int_0^t |K(t,s,g(s)) - K(t,s,h(s))|\,ds \\
&\leq L \max_{0\leq t\leq T} e^{-Lt}\int_0^t |g(s)-h(s)|\,ds \\
&= L \max_{0\leq t\leq T} e^{-Lt}\int_0^t e^{Ls}e^{-Ls}|g(s)-h(s)|\,ds \\
&\leq L\boldsymbol{|} g-h \boldsymbol{|} \max_{0\leq t\leq T} e^{-Lt}\int_0^t e^{Ls}\,ds \\
&= L\boldsymbol{|} g-h \boldsymbol{|} \max_{0\leq t\leq T} e^{-Lt}\frac{e^{Lt}-1}{L} \\
&\leq (1-e^{-LT})\boldsymbol{|} g-h \boldsymbol{|}.
\end{aligned}$$

Because $1-e^{-LT} < 1$, the map $F : E \to E$ is contractive; Banach's principle therefore guarantees first a unique fixed point $u \in E$, and then that the sequence $\{u_n\}$ determined by the iterations described in the statement of the theorem converges uniformly in the norm $\boldsymbol{|} x \boldsymbol{|}$, therefore also in the sup norm $\|x\|$, to that fixed point. □

Observe that if we had used the sup norm $\|x\|$ rather than $\boldsymbol{|} x \boldsymbol{|}$, then F would be contractive when regarded as a map $C[0,\lambda] \to C[0,\lambda]$, where $\lambda < \min\{T, 1/L\}$. Thus, if $T > 1/L$, then the Banach principle with the usual sup norm would have guaranteed a unique solution only on a subinterval of $[0,T]$, whereas by modifying the norm we have shown that in fact there is a unique solution on the entire interval $[0,T]$.

(2.2) Theorem. *Let $f : [0,T] \times \boldsymbol{R} \to \boldsymbol{R}$ satisfy the Lipschitz condition*

$$|f(s,x) - f(s,y)| \le L|x-y|$$

for $s \in [0,T]$, $x, y \in \boldsymbol{R}$. Then the initial value problem

$$\frac{du}{ds} = f(s,u), \qquad u(0) = 0,$$

has exactly one solution u defined on the entire interval $[0,T]$.

Proof. If $K(t,s,u) = f(s,u)$ and $v(t) = 0$ in (2.1), the Volterra equation becomes

$$u(t) = \int_0^t f(s,u(s))\,ds,$$

and the solution of this integral equation is precisely the solution of the present initial value problem. □

3. Applications of the Elementary Domain Invariance

We now give applications of the elementary domain invariance theorem in various fields such as linear functional analysis and the geometry of Banach spaces.

a. Domain invariance and invertibility of linear operators

We begin with two simple propositions:

(3.1) Proposition. *Let $T : E \to E$ be a linear operator in a Banach space. If $\|I - T\| < 1$, then T is invertible, and*

$$\|T^{-1}\| \le \frac{1}{1 - \|I - T\|}.$$

Proof. The map $I - T : E \to E$ is contractive, since

$$\|(I-T)(x-y)\| \le \|I-T\|\|x-y\|,$$

so by (1.2.2), the map $I - (I - T) = T$ is a homeomorphism, therefore invertible. The bound for the norm follows from

$$1 = \|TT^{-1}\| = \|T^{-1} - T^{-1}(I-T)\| \ge \|T^{-1}\| - \|T^{-1}\|\|I-T\|. \qquad \square$$

(3.2) Proposition. *Let $T : E \to E$ be an invertible linear operator in a Banach space. Then each linear operator S with $\|T - S\| < 1/\|T^{-1}\|$ is invertible, and*

$$\|S^{-1}\| \le \frac{\|T^{-1}\|}{1 - \|I - ST^{-1}\|}.$$

PROOF. Because T is invertible, it is enough to show that $J = S \circ T^{-1}$ is invertible, for then $S = J \circ T$ has $T^{-1}J^{-1}$ as inverse. From

$$\|I - J\| = \|I - ST^{-1}\| = \|(T - S) \circ T^{-1}\| \leq \|T - S\|\|T^{-1}\| < 1$$

and (3.1) we find that J is in fact invertible; and since $S^{-1} = T^{-1}J^{-1}$, the norm estimate follows from (3.1). □

These results lead to

(3.3) THEOREM. *Let E be a Banach space and $\mathscr{A} \subset \mathscr{L}(E, E)$ the set of all invertible linear operators. Let* $\operatorname{Inv} : \mathscr{A} \to \mathscr{A}$ *be the map $T \mapsto T^{-1}$. Then $\mathscr{A}$ is open in $\mathscr{L}(E, E)$ and* Inv *is a homeomorphism of $\mathscr{A}$ onto itself.*

PROOF. By (3.2), for each $T \in \mathscr{A}$, the ball $B(T, 1/(2\|T^{-1}\|))$ is also contained in $\mathscr{A}$; therefore $\mathscr{A}$ is open in $\mathscr{L}(E, E)$. To prove that Inv is continuous at any given $T \in \mathscr{A}$, it is enough to note that if $S \in B(T, 1/(2\|T^{-1}\|))$, so that $\|I - ST^{-1}\| < \frac{1}{2}$, then by (3.2) we have $S \in \mathscr{A}$ and

$$\begin{aligned}\|S^{-1} - T^{-1}\| &= \|T^{-1}(T - S)S^{-1}\| \\ &\leq \frac{\|T^{-1}\|^2}{1 - \|I - ST^{-1}\|}\|T - S\| \leq 2\|T^{-1}\|^2\|T - S\|;\end{aligned}$$

and since $\operatorname{Inv} \circ \operatorname{Inv} = \mathrm{id}$, it follows that Inv is a homeomorphism. □

Let E, F be Banach spaces and $S : E \to F$ a linear operator. If there is some $m > 0$ such that $\|Sx\|_F \geq m\|x\|_E$ for all $x \in E$, then it is immediate that S is injective; if such an S is also surjective, then it is invertible because $\|S^{-1}y\|_E \leq (1/m)\|y\|_F$ for all $y \in F$ shows that the inverse is continuous. The following result extends this observation to suitable perturbations of such operators, and is of importance in work with “a priori” estimates for linear differential operators.

(3.4) THEOREM (Schauder invertibility theorem). *Let E, F be Banach spaces and $S, T : E \to F$ two linear operators, with S invertible. Assume that there is an $m > 0$ such that for each $0 \leq t \leq 1$, the operator $L_t = (1-t)S + tT$ satisfies $\|L_t x\|_F \geq m\|x\|_E$ for all $x \in E$. Then L_t is invertible for all $0 \leq t \leq 1$, and in particular, T is invertible.*

PROOF. We begin by showing that if an operator L_s is invertible, then for each t in the open interval $J_s = \{t \mid |t - s| < m/\|T - S\|\}$, the operator L_t is invertible or, what is equivalent, that $L_s^{-1}L_t : E \to E$ is invertible for each $t \in J_s$. For this purpose, note that

$$L_t = S + s(T - S) + (t - s)(T - S) = L_s + (t - s)(T - S),$$

so that

$$L_s^{-1}L_t = I + (t-s)L_s^{-1}(T-S).$$

Because $\|L_s x\|_F \geq m\|x\|_E$, we have $\|L_s^{-1}\| \leq 1/m$, so for $t \in J_s$, we find

$$\|(t-s)L_s^{-1}(T-S)\| \leq |t-s|\frac{1}{m}\|T-S\|,$$

and by (3.1), the operator $L_s^{-1}L_t$ is therefore invertible.

To prove the theorem, let $\mathscr{J} = \{t \in [0,1] \mid L_t \text{ is invertible}\}$. By what we have just shown, $\mathscr{J}$ is an open set. If $t \notin \mathscr{J}$, then again by what we have just shown, no operator L_s with $s \in J_t$ can be invertible, so that $[0,1] - \mathscr{J}$ is also an open set. Because $[0,1]$ is connected and $\mathscr{J}$ is nonempty, we conclude that $\mathscr{J} = [0,1]$. □

b. The inverse function theorem

As another application we obtain a standard theorem in analysis:

(3.5) THEOREM (Inverse function theorem). *Let E be a Banach space, $U \subset E$ open, and $f : U \to E$ a C^1 map. Assume that at some $x_0 \in U$, the derivative $Df(x_0) : E \to E$ is an isomorphism. Then there exists a neighborhood V of x_0 and a neighborhood W of $f(x_0)$ such that:*

(1) *$Df(x) : E \to E$ is invertible for each $x \in V$,*

(2) *$f|V : V \to W$ is a homeomorphism of V onto W,*

(3) *the inverse $g : W \to V$ of $f|V$ is differentiable at each $w \in W$ and $Dg(w) = [Df(gw)]^{-1}$,*

(4) *the map $w \mapsto Dg(w)$ of W into $\mathscr{L}(E,E)$ is continuous.*

PROOF. We first consider the special case where $x_0 = 0$, $f(0) = 0$, and $Df(0) = I$. Because the set of invertible operators is open in $\mathscr{L}(E,E)$ and $x \mapsto Df(x)$ is continuous with $Df(0)$ invertible, we can find a ball B with $0 \in B \subset U$ on which $Df(x)$ is invertible.

Define $F : B \to E$ by $F(x) = x - f(x)$. Then F is a C^1 map, $DF(0) = I - Df(0) = 0$, and because F is continuously differentiable, there is a ball V with $0 \in V \subset B$ such that $M = \sup\{\|DF(x)\| \mid x \in V\} < \frac{1}{2}$. The map $F : V \to E$ is contractive: for, by the mean value theorem,

$$\|F(x_1) - F(x_2)\| \leq M\|x_1 - x_2\| \leq \tfrac{1}{2}\|x_1 - x_2\| \quad \text{for all } x_1, x_2 \in V.$$

Thus, by (1.2.1), the map $f : V \to E$ is a homeomorphism onto the open set $W = f(V)$ containing $f(0) = 0$, and the proof of both (1) and (2) is complete. We observe, for later reference, that if $x, a \in V$, then

$$\|x - a\| - \|f(x) - f(a)\| \leq \|F(x) - F(a)\| \leq \tfrac{1}{2}\|x - a\|,$$

so that

$$\|x-a\| \le 2\|f(x)-f(a)\|.$$

We now prove (3). Let $g : W \to V$ be the inverse of $f|V$. Given $y, b \in W$, let $g(b) = a$, $g(y) = x$, and write $Df(a) = T$. By the differentiability of f at a, we have

$$f(x) - f(a) = T(x-a) + \varphi(x,a),$$

where $\varphi(x,a)/\|x-a\| \to 0$ as $\|x-a\| \to 0$. Applying T^{-1} to this expression and noting that $f(x) = y$, $f(a) = b$, we find

$$T^{-1}(y-b) = g(y) - g(b) + T^{-1}\varphi[g(y), g(b)]$$

so it suffices to show that

$$R = \frac{\|T^{-1}\varphi[g(y), g(b)]\|}{\|y-b\|} \to 0 \quad \text{as } \|y-b\| \to 0.$$

Because $g|W : W \to V$ is bijective, we have

$$\begin{aligned} R &\le \frac{\|T^{-1}\|\|\varphi[g(y), g(b)]\|}{\|g(y)-g(b)\|} \frac{\|g(y)-g(b)\|}{\|y-b\|} \\ &\le 2\|T^{-1}\| \frac{\|\varphi[g(y), g(b)]\|}{\|g(y)-g(b)\|} = 2\|T^{-1}\| \frac{\|\varphi(x,a)\|}{\|x-a\|}, \end{aligned}$$

where we have used the observation above that $\|x-a\| \le 2\|f(x)-f(a)\|$ on V, i.e., $\|g(y)-g(b)\| \le 2\|y-b\|$ on W. Thus, as $\|y-b\| \to 0$, the continuity of g shows that $\|x-a\| \to 0$, and therefore $R \to 0$. This completes the proof of (3).

To prove (4), we note that $w \mapsto Dg(w)$ is the composition $\mathrm{Inv} \circ Df \circ g$ of three continuous maps, so it is continuous on W.

This completes the proof of the theorem in the special case where $f(0)=0$ and $Df(0) = I$. To prove the theorem as it is stated, apply this special case to the function

$$h(x) = [Df(x_0)]^{-1}(f(x+x_0) - f(x_0)). \qquad \square$$

c. Stability of open embeddings and monotone operators

We now apply elementary domain invariance to stability of open embeddings into Banach spaces.

(3.6) THEOREM. *Let X be any space, E a Banach space, and $F : X \to E$ an embedding of X onto an open $U \subset E$. Let $G : X \to E$ be a map such that $G \circ F^{-1} : U \to E$ is contractive. Then $x \mapsto Fx + Gx$ is also an open embedding of X into E.*

Proof. Consider $h = (F+G)\circ F^{-1} = I + GF^{-1} : U \to E$. By domain invariance this h maps U homeomorphically onto an open $h(U) \subset E$. Since $F + G = h \circ F$, the proof is complete. $\square$

As an evident consequence we obtain the stability property of Lipschitzian open embeddings into Banach spaces.

(3.7) Theorem. *Let X be a metric space and E a Banach space, and let $F : X \to E$ be an open embedding such that F^{-1} is Lipschitzian. Let $G : X \to E$ be a Lipschitzian map such that $L(G)L(F^{-1}) < 1$. Then $x \mapsto Fx + Gx$ is also an open embedding of X into E.*

Proof. It is enough to observe that $L(G \circ F^{-1}) \le L(G)L(F^{-1})$ and to apply (3.6). $\square$

We next derive some simple facts about monotone operators in Hilbert spaces.

Let H be a Hilbert space and $U \subset H$. A map $f : U \to H$ (not necessarily continuous) is said to be *monotone* if

$$\text{(i)} \qquad (fx - fy, x - y) \ge 0 \quad \text{for all } x, y \in U;$$

f is called *strongly monotone* if for some $C > 0$,

$$\text{(ii)} \qquad (fx - fy, x - y) \ge C\|x - y\|^2 \quad \text{for all } x, y \in U.$$

Clearly, every strongly monotone map is injective.

(3.8) Theorem. *Let $U \subset H$ be open, and let $f : U \to H$ be Lipschitzian and strongly monotone, i.e.,*

$$\|fx - fy\| \le M\|x - y\| \quad \textit{for all } x, y \in U \textit{ and some } M > 0$$

and

$$(fx - fy, x - y) \ge C\|x - y\|^2 \quad \textit{for all } x, y \in U.$$

Then f is an open map, in particular $f(U)$ is open in H, and f is a homeomorphism of U onto $f(U)$.

Proof. It is clearly enough to show that for a sufficiently small λ, the map λf is a contractive field. Given $\lambda > 0$, we have for $x, y \in U$,

$$\begin{aligned}
&\|(I - \lambda f)x - (I - \lambda f)y\|^2 \\
&\qquad = \|x - y\|^2 + \lambda^2\|fx - fy\|^2 - 2\lambda(fx - fy, x - y) \\
&\qquad \le \|x - y\|^2 + M^2\lambda^2\|x - y\|^2 - 2\lambda C\|x - y\|^2 \\
&\qquad \le (1 + M^2\lambda^2 - 2\lambda C)\|x - y\|^2.
\end{aligned}$$

Fix $\lambda < 2C/M^2$; then $1 + M^2\lambda^2 - 2\lambda C < 1$, and therefore $I - \lambda f$ is a contraction; our assertion now follows from (1.2.1). $\square$

As a corollary we have

(3.9) THEOREM. *Let $f : H \to H$ be a Lipschitzian and strongly monotone map. Then f is a homeomorphism of H onto itself.*

PROOF. By (3.8) it is enough to show that $f(H)$ is closed in H. Let $fx_n \to y$; from (ii) and the Cauchy inequality we get

$$\|fx_n - fx_m\| \geq C\|x_n - x_m\| \quad \text{for all } n, m \geq 1,$$

and hence $\{x_n\}$ is a Cauchy sequence. Let $x_n \to x$; then $fx_n \to fx$, and hence $y = fx$. □

d. Application to negligible sets

A subset B of a space Y is called *negligible* whenever $Y - B$ is homeomorphic to Y; a homeomorphism $h : Y - B \approx Y$ is called a *deleting homeomorphism*. We now show that any complete subset of a noncomplete normed linear space is negligible.

(3.10) THEOREM. *Let E be a noncomplete normed linear space and C a complete subset of E. Then there is a homeomorphism $h : E - C \approx E$ with $h(x) = x$ whenever $d(x, C) \geq 1$.*

PROOF. Let $\widehat{E}$ be the completion of E; taken with the natural extension of the given norm, $\widehat{E}$ is a Banach space. Let $\{x_n\}$ be a Cauchy sequence in E converging to some point in $\widehat{E} - E$, with $\|x_1\| + \sum_{n=1}^{\infty} \|x_n - x_{n+1}\| < \infty$; since no scalar multiple of a point in $\widehat{E} - E$ can belong to E, replacing all the x_n by a suitable scalar multiple, we can assume that $\{x_n\} \subset E$ converges to $y_0 \in \widehat{E} - E$ and $\|x_1\| + \sum_{n=1}^{\infty} \|x_n - x_{n+1}\| = \frac{1}{2}$.

Let $x_0 = 0$, and let $L \subset E$ be the jagged line in E consisting of the segments $[x_0, x_1] \cup [x_1, x_2] \cup \cdots$; we construct a piecewise linear map φ of the unit interval $[0, 1]$ onto $L \cup \{y_0\}$ as follows: let $s_0 = 0$ and for $n \geq 1$, let s_n be the nth partial sum of the series with sum $\frac{1}{2}$; for each $n \geq 0$, map the interval $[2s_n, 2s_{n+1}]$ linearly onto the segment $[x_n, x_{n+1}]$ and set $\varphi(1) = y_0$. It is clear that $|\varphi(t) - \varphi(t')| = \frac{1}{2}|t - t'|$ whenever t, t' belong to a common interval $[2s_n, 2s_{n+1}]$, so by the triangle inequality, we have $|\varphi(t) - \varphi(t')| \leq \frac{1}{2}|t - t'|$ for all $t, t' \in \boldsymbol{I}$. Extend φ to a map of $(-\infty, 1]$ into E by $\varphi(t) = 0$ for $t < 0$.

Now let $H : \widehat{E} \to \widehat{E}$ be the map $x \mapsto \varphi[1 - d(x, C)]$. This map is contractive, because

$$\|\varphi[1 - d(x, C)] - \varphi[1 - d(z, C)]\| \leq \tfrac{1}{2}\|d(z, C) - d(x, C)\| \leq \tfrac{1}{2}\|x - z\|;$$

therefore, by (1.2.1), the map $h(x) = x - H(x)$ is a homeomorphism of $\widehat{E}$ onto itself.

It is clear that $h(E - C) \subset E$; for if $x \in E - C$, then $d(x, C) > 0$, so $H(x) \in E$, and therefore $h(x) = x - H(x)$ also belongs to E. To establish the converse inclusion, assume $x \notin E - C$, so that $x \in (\widehat{E} - E) \cup C$; if $x \in \widehat{E} - E$, then $d(x, C) > 0$, so $H(x) \in E$, while if $x \in C$, then $x \in E$ and $H(x) = y_0 \notin E$; in both cases, exactly one of $x, H(x)$ belongs to E, so we conclude that $h(x) = x - H(x) \notin E$. Thus $h(E - C) \approx E$. □

The class of linear spaces for which such deleting homeomorphisms exist is very broad because of

(3.11) LEMMA. *Every infinite-dimensional Banach space* $(E, \|\cdot\|)$ *admits a noncomplete norm* $|\!|\!|\cdot|\!|\!|$ *with* $|\!|\!|x|\!|\!| \le \|x\|$ *for all* $x \in E$.

PROOF. Assume first that E is separable. Choose a countable separating family $\{f_n \mid n = 1, 2, \ldots\}$ of continuous linear functionals such that $|f_n(x)| \le (1/n)\|x\|$ for each n, and define $T : E \to l^2$ by $T(x) = \{f_n(x)\}$. This is a continuous linear operator, and $T : E \to T(E)$ is bijective because the family $\{f_n\}$ is separating. If the linear subspace $T(E) \subset l^2$ were complete, then because a bijective continuous linear map of Banach spaces is a homeomorphism, the inverse $T^{-1} : T(E) \to E$ would be continuous; but this is impossible because $\overline{T\{x \mid \|x\| \le 1\}}$ is easily seen to be compact, and E is infinite-dimensional. Thus, the l^2 norm on $T(E)$ is not complete, and defining $\|x\|_0 = \|T(x)\|$ gives an incomplete norm on E. We note that $\|x\|_0 \le \sqrt{\sum(1/n^2)}\|x\|$, so that $\|\cdot\|_0$ is continuous.

Now let E be arbitrary. Pick a separable infinite-dimensional closed linear subspace $L \subset E$ and an incomplete norm $\|\cdot\|_0$ on L. Let $A = \{x \in L \mid \|x\|_0 < 1\}$; because $\|\cdot\|_0$ is continuous, there is an $\varepsilon > 0$ such that $L \cap \{x \mid \|x\| < \varepsilon\} \subset A$. Let $C = \operatorname{conv}[A \cup \{x \mid \|x\| < \varepsilon\}]$; since C is a symmetric convex body with no rays, the Minkowski functional φ_C gives a norm, and because $\varphi_C(x) = \|x\|_0$ for $x \in L$, that norm is not complete. Finally, the continuity of φ_C implies that

$$b = \sup\{\varphi_C(x) \mid \|x\| \le 1\}$$

is finite, so $|\!|\!|x|\!|\!| = b^{-1}\varphi_C(x)$ is an incomplete norm with $|\!|\!|x|\!|\!| \le \|x\|$. □

Combining this with (3.10) gives

(3.12) THEOREM (Klee). *Let E be an arbitrary infinite-dimensional normed linear space, and $C \subset E$ compact. Then there is a homeomorphism $h : E - C \approx E$.*

PROOF. We can assume that E is a Banach space, else the result follows directly from (3.10). By (3.11), there is an incomplete norm $|\!|\!|x|\!|\!| \le \|x\|$; denote $(E, |\!|\!|\cdot|\!|\!|)$ by $\widehat{E}$; the identity map $j : E \to \widehat{E}$ is therefore continuous, so $\widehat{C} = j(C)$ is compact, therefore complete, in $\widehat{E}$. By (3.10), there is a deleting homeomorphism $\widehat{h} : \widehat{E} - \widehat{C} \approx \widehat{E}$ given by $\widehat{h}(\widehat{x}) = \widehat{x} - \widehat{\varphi}[1 - \widehat{d}(\widehat{x}, \widehat{C})]$, where $\widehat{d}$ is the norm-induced metric and $\widehat{\varphi}$ is a piecewise linear map $\widehat{\varphi} : (-\infty, 1) \to \widehat{E}$.

Because $\widehat{\varphi}$ is piecewise linear, it is continuous with any linear topology in the range space, so regarding it as a map $\varphi : (-\infty, 1) \to E$ we have φ continuous and $j \circ \varphi = \widehat{\varphi}$. Now define $h : E - C \to E$ by

$$h(x) = x - \varphi[1 - \widehat{d}(j(x), \widehat{C})];$$

this map is clearly continuous and makes the diagram

$$\begin{tikzcd} E - C \arrow[r, "h"] \arrow[d, "j"'] & E \arrow[d, "j"] \\ \widehat{E} - \widehat{C} \arrow[r, "\widehat{h}"'] & \widehat{E} \end{tikzcd}$$

commutative. Finally, define $g : E \to E - C$ by

$$g(x) = x + \varphi[1 - \widehat{d}(\widehat{h}^{-1} j(x), \widehat{C})];$$

this map is also continuous. We have

$$\begin{aligned} g[h(x)] &= h(x) + \varphi[1 - \widehat{d}(\widehat{h}^{-1} j h(x), \widehat{C})] \\ &= \{x - \varphi[1 - \widehat{d}(j(x), \widehat{C})]\} + \varphi[1 - \widehat{d}(\widehat{h}^{-1} \widehat{h} j(x), \widehat{C})] \\ &= x, \end{aligned}$$

and similarly $h \circ g = \mathrm{id}$. Thus, $h : E - C \to E$ is a homeomorphism having g as its inverse. $\square$

4. Elementary KKM-Principle and Its Applications

In this section we present a version of the geometric KKM-principle that permits us to establish in an elementary manner a large number of important results in Hilbert space theory. The approach, in which neither the weak topology nor compactness are used, is based on some simple intersection property of convex sets in Hilbert spaces.

a. Basic intersection property of convex sets in Hilbert spaces

Let $(H, \|\cdot\|)$ be a Hilbert space. From the parallelogram equality it follows immediately that the norm $\|\cdot\|$ in H is *uniformly convex*, i.e., if $\{x_n\}$ and $\{y_n\}$ are sequences in H such that the numerical sequences $\|x_n\|$, $\|y_n\|$, and $\frac{1}{2}\|x_n + y_n\|$ converge to 1, then the sequence $\{\|x_n - y_n\|\}$ tends to 0.

The following preliminary result will be of importance.

(4.1) LEMMA. *Let $(H, \|\cdot\|)$ be a Hilbert space and $\{C_n\}$ be a decreasing sequence of nonempty closed convex subsets of H. Suppose that $d = \sup_n d(0, C_n)$ is finite. Then there exists a unique point $x \in \bigcap C_n$ such that $\|x\| = d$.*

PROOF. Letting $P_n = C_n \cap K(0, d + 1/n)$ for each $n = 1, 2, \dots$, we obtain a decreasing sequence $\{P_n\}$ of nonempty closed and convex sets and we show that $\delta(P_n) \to 0$, where $\delta(P_n)$ is the diameter of P_n. Indeed, for any n, let $x_n, y_n \in P_n$ be such that $\delta(P_n) \le \|x_n - y_n\| + 1/n$; since the point $\frac{1}{2}(x_n + y_n)$ also belongs to P_n, the values $\|x_n\|$, $\|y_n\|$, and $\frac{1}{2}\|x_n + y_n\|$ lie between $d(0, C_n)$ and $d + 1/n$, and therefore the three sequences converge to d. By the uniform convexity of the norm, we infer that $\|x_n - y_n\| \to 0$, and consequently $\delta(P_n) \to 0$. Applying now the Cantor theorem, we get a unique point $x \in \bigcap P_n$; for this point we have $d(0, C_n) \le \|x\| \le d + 1/n$ for each n, implying that $\|x\| = d$. □

We are now in a position to prove the desired result:

(4.2) THEOREM (Intersection property). *Let $\{C_i \mid i \in I\}$ be a family of closed convex sets in a Hilbert space H with the finite intersection property. If C_{i_0} is bounded for some $i_0 \in I$, then the intersection $\bigcap\{C_i \mid i \in I\}$ is not empty.*

PROOF. Let $\langle I \rangle$ be the set of all finite subsets of I containing i_0. For any $J \in \langle I \rangle$, let $C_J = \bigcap\{C_j \mid j \in J\}$ and note that since each C_J is a nonempty closed convex subset of C_{i_0}, the supremum $d = \sup_{J \in \langle I \rangle} d(0, C_J)$ is finite.

Let $\{J_n\}$ be an increasing sequence of sets in $\langle I \rangle$ with $d(0, C_{J_n}) \ge d - 1/n$. Then $\{C_{J_n}\}$ is a decreasing sequence of nonempty closed convex sets in H such that $d = \sup_n d(0, C_{J_n})$. Applying Lemma (4.1), we get a unique point $x \in \bigcap_n C_{J_n}$ with $\|x\| = d$.

Next, let $J \in \langle I \rangle$ be arbitrary, and set $C_n = C_J \cap C_{J_n}$. Again, $\{C_n\}$ is a decreasing sequence of nonempty closed convex sets in H such that $d = \sup_n d(0, C_n)$, so by Lemma (4.1), there is a unique point $x' \in \bigcap_n C_n = C_J \cap \bigcap_n C_{J_n}$ with $\|x'\| = d$. By the uniqueness it is evident that $x = x'$ belongs to C_J.

Finally, we observe that the point x belongs to C_J for all $J \in \langle I \rangle$, which proves that the set $\bigcap\{C_i \mid i \in I\} \supset \bigcap\{C_J \mid J \in \langle I \rangle\}$ is not empty. □

Using (4.2) and the basic geometric property (3.1.4) of KKM-maps, we obtain the desired version of the geometric KKM-principle:

(4.3) THEOREM (Elementary KKM-principle). *Let H be a Hilbert space, X a nonempty subset of H, and $G : X \to 2^H$ a KKM-map with closed convex values such that Gx_0 is bounded for some $x_0 \in X$. Then the intersection $\bigcap\{Gx \mid x \in X\}$ is not empty.* □

In the remaining part of this section we give a number of applications of Theorem (4.3).

b. Theorem of Stampacchia

A function $\varphi : X \to \boldsymbol{R}$ on a subset of a normed linear space is said to be *coercive* if $\{x \in X \mid \varphi(x) \leq r\}$ is bounded for each $r \in \boldsymbol{R}$. If H is a Hilbert space, a bilinear form $a : H \times H \to \boldsymbol{R}$ is *coercive* if the function $x \mapsto a(x,x)/\|x\|$ is coercive on $H - \{0\}$.

(4.4) THEOREM (Stampacchia). *Let C be a nonempty closed convex subset of a Hilbert space H, $a : H \times H \to \boldsymbol{R}$ a continuous coercive bilinear form, and $l : H \to \boldsymbol{R}$ a continuous linear form. Then there exists a unique point $y_0 \in C$ such that $a(y_0, y_0 - x) \leq l(y_0 - x)$ for all $x \in C$.*

PROOF. It follows from the coercivity of a that $a(x,x) > 0$ for all $x \neq 0$, so there can be at most one solution. Consider the map $G : C \to 2^C$ given by

$$Gx = \{y \in C \mid a(y, y - x) \leq l(y - x)\}.$$

It is easy to check that the values of G are closed, convex (since $x \mapsto a(x,x)$ is continuous and convex) and bounded (because a is coercive); furthermore, since $x \in Gx$, and the cofibers of G are convex, it follows from (3.1.2) that G is a KKM-map. Therefore, by (4.3), there is a $y_0 \in \bigcap_{x \in C} Gx$, which was to be proved. □

We note that the special case $C = H$ of (4.4) yields

(4.5) THEOREM (Lax–Milgram–Vishik). *Let $a : H \times H \to \boldsymbol{R}$ be a continuous coercive bilinear form. Then for any continuous linear form $l : H \to \boldsymbol{R}$ there exists a unique point $y_0 \in H$ such that $a(y_0, x) = l(x)$ for all $x \in H$.*

PROOF. By the Stampacchia theorem, because $C = H$, there exists a unique $y_0 \in H$ such that $a(y_0, z) \leq l(z)$ for all $z \in H$; replacing in this inequality z by $-z$ we obtain $a(y_0, z) \geq l(z)$ for all $z \in H$, and the conclusion follows. □

c. Variational inequalities. Theorem of Hartman–Stampacchia

We now extend the above results to a certain class of nonlinear operators that we describe below. Let H be a Hilbert space and C be any subset of H. We recall that an operator $f : C \to H$ is said to be *monotone* on C

if $(f(y) - f(x), y - x) \geq 0$ for all $x, y \in C$. We say that $f : C \to H$ is *hemicontinuous* if the function $[0,1] \ni t \mapsto (f(y + t(x-y)), x - y)$ is continuous at 0 for all $x, y \in C$, and *coercive* if for some $x_0 \in C$ the function $x \mapsto (f(x), x - x_0)/\|x - x_0\|$ is coercive on $C - \{x_0\}$.

(4.6) THEOREM (Hartman–Stampacchia). *Let C be a nonempty closed convex subset of H, $f : C \to H$ monotone coercive hemicontinuous, and $l : H \to \boldsymbol{R}$ a continuous linear form. Then there exists a point $y_0 \in C$ such that $(f(y_0), y_0 - x) \leq l(y_0 - x)$ for all $x \in C$.*

PROOF. We consider only the case of a bounded C; an easy proof of the general case is left to the reader. Define $G, F : C \to 2^C$ by

$$Gx = \{y \in C \mid (f(y), y - x) \leq l(y - x)\},$$
$$Fx = \{y \in C \mid (f(x), y - x) \leq l(y - x)\}.$$

Because f is monotone, we have

$$(f(y), y - x) \geq (f(x), y - x) \quad \text{for all } x, y \in C,$$

and therefore $Gx \subset Fx$ for each $x \in C$. Observe that $x \in Gx$ and that the cofibers of G are convex; thus, by (3.1.2), G is a KKM-map; consequently, so is the map F. Since by definition the values of F are convex and closed, we infer by (4.3) that for some $y_0 \in C$ we have $y_0 \in \bigcap_{x \in C} Fx$, and thus

$$(f(z), y_0 - z) \leq l(y_0 - z) \quad \text{for all } z \in C.$$

Choose any $x \in C$ and let $z_t = y_0 + t(x - y_0)$ for $t \in [0,1]$. We have

$$(f(y_0 + t(x - y_0)), y_0 - x) \leq l(y_0 - x) \quad \text{for } t > 0.$$

Now let $t \to 0$; the hemicontinuity of f gives $(f(y_0), y_0 - x) \leq l(y_0 - x)$. Since x was arbitrary, the conclusion follows. □

As an immediate consequence we obtain

(4.7) THEOREM (Minty–Browder). *Let $f : H \to H$ be a monotone coercive hemicontinuous operator. Then for any continuous linear form $l : H \to \boldsymbol{R}$ there exists a point $y_0 \in H$ such that $(f(y_0), x) = l(x)$ for all $x \in H$.* □

As another consequence of Theorem (4.6), we get a version of the fixed point theorem of Browder–Göhde–Kirk:

(4.8) THEOREM. *Let C be a nonempty closed convex bounded subset of H, and let $F : C \to H$ be nonexpansive (i.e., $\|F(x) - F(y)\| \leq \|x - y\|$ for all $x, y \in C$). Suppose that for each $x \in C$ with $x \neq F(x)$ the line segment $[x, F(x)]$ contains at least two points of C. Then F has a fixed point.*

PROOF. Since F is nonexpansive, the operator $f(x) = x - F(x)$ from C to H is monotone continuous. Applying Theorem (4.6) we get a point $y_0 \in C$ such that $(y_0 - F(y_0), y_0 - x) \leq 0$ for all $x \in C$. Since for some $t > 0$ the point $y_0 + t(F(y_0) - y_0)$ lies in C, we can insert that value into the above inequality to get $(y_0 - F(y_0), F(y_0) - y_0) \geq 0$, showing that y_0 is a fixed point for F. □

d. Maximal monotone operators

We conclude this section by deriving some basic facts in the theory of maximal monotone operators in a Hilbert space H. A set-valued operator $T : H \to 2^H$ is said to be *monotone* if $(y^* - x^*, y - x) \geq 0$ whenever $x^* \in Tx$ and $y^* \in Ty$, and *maximal monotone* if it is monotone and maximal in the set of all monotone operators from H into 2^H ordered by $S \leq T$ if $Sx \subset Tx$ for all $x \in H$. In what follows, we denote by $D(T)$ the domain of T, i.e., $D(T) = \{y \in H \mid Ty \neq \emptyset\}$.

It is clear from the definitions that:

(1) if T is monotone, then

$$\sup_{x^* \in Tx} (x^*, y - x) < \infty \qquad \text{for all } x \in D(T) \text{ and } y \in \operatorname{Conv} D(T),$$

(2) if T is maximal monotone, then $y^* \in Ty$ whenever

$$(x^* - y^*, x - y) \geq 0 \qquad \text{for all } x \in D(T) \text{ and } x^* \in Tx.$$

For the proof of our main result, we need

(4.9) LEMMA. *Let E be a vector space, $C \subset E$ convex, and D an arbitrary subset of C. Let $g : D \times C \to \boldsymbol{R}$ be a function such that:*
(a) *$g(x, y) + g(y, x) \leq 0$ for all $(x, y) \in D \times D$,*
(b) *$y \mapsto g(x, y)$ is convex on C for each $x \in D$.*
Then the map $\mathcal{G} : D \to 2^C$ given by $\mathcal{G}x = \{y \in C \mid g(x, y) \leq 0\}$ is a KKM-map.

PROOF. Let $A = \{x_1, \ldots, x_n\} \subset D$, and let $y_0 = \sum_{i=1}^n \lambda_i x_i$ be a convex combination of the x_i's; we are going to show that $y_0 \in \mathcal{G}(A)$.

In view of (a), we have

$$g(x_i, x_j) + g(x_j, x_i) \leq 0 \qquad \text{for all } i, j \in [n].$$

So, multiplying by λ_i and summing over i, we find

$$\sum_{i=1}^n \lambda_i g(x_i, x_j) + \sum_{i=1}^n \lambda_i g(x_j, x_i) \leq 0 \qquad \text{for every } j \in [n],$$

and therefore, because $y \mapsto g(x, y)$ is convex, we get

$$\sum_{i=1}^n \lambda_i g(x_i, x_j) + g(x_j, y_0) \leq 0 \qquad \text{for every } j \in [n].$$

By multiplying each of the above inequalities by λ_j, then summing over all j, and using the convexity of $y \mapsto g(x,y)$, we finally get

$$\sum_{i=1}^{n} \lambda_i g(x_i, y_0) + \sum_{j=1}^{n} \lambda_j g(x_j, y_0) \leq 0.$$

This implies that $g(x_i, y_0) \leq 0$ for at least one point x_i, i.e., $y_0 \in \bigcup_{i=1}^{n} \mathcal{G}x_i$, as asserted. $\square$

We are now able to prove the desired result:

(4.10) THEOREM. *Let $T : H \to 2^H$ be a monotone set-valued operator and $u : H \to H$ be a single-valued, linear, monotone, and bounded operator. Set $D = D(T)$ and $C = \operatorname{Conv} D(T)$. Assume that for some $x_0 \in D$ the set*

$$\{y \in C \mid \sup_{x^* \in Tx_0} (u(y) + x^*, y - x_0) \leq 0\}$$

is bounded. Then there is a point $y_0 \in C$ such that

$$\sup_{x^* \in Tx} (u(y_0) + x^*, y_0 - x) \leq 0 \quad \textit{for all } x \in D.$$

PROOF. We show that the map $G : D \to 2^C$ defined by

$$Gx = \{y \in C \mid \sup_{x^* \in Tx} (u(y) + x^*, y - x) \leq 0\} \quad \text{for } x \in D$$

satisfies all the conditions of the elementary KKM-principle (4.3).

First, we show that G is a KKM-map. To this end consider the function $f : C \times D \times C \to \boldsymbol{R}$ given by

$$f(\xi, x, y) = \sup_{x^* \in Tx} (u(\xi) + x^*, y - x).$$

Because T is monotone, f is well defined and satisfies the following conditions:

(a) $f(\xi, x, y) + f(\xi, y, x) \leq 0$ for all $(x, y) \in D \times D$ and all $\xi \in C$,

(b) $y \mapsto f(\xi, x, y)$ is convex on C for each $x \in D$ and each $\xi \in C$.

Now, observe that using f, we can equivalently describe $G : D \to 2^C$ as

$$Gx = \{y \in C \mid f(y, x, y) \leq 0\}.$$

To show that G is KKM, let $A = \{x_1, \ldots, x_n\} \subset D$ and let $y_0 \in [A]$. Define $g : A \times [A] \to \boldsymbol{R}$ by $g(x, y) = f(y_0, x, y)$. It follows from (a) and (b) that g satisfies the conditions of (4.9), so the map $\mathcal{G} : A \to 2^{[A]}$ given by $\mathcal{G}x = \{y \in [A] \mid g(x, y) \leq 0\}$ is KKM. This implies in particular that $y_0 \in \mathcal{G}x_i$ for some $x_i \in A$, which means that $f(y_0, x_i, y_0) = g(x_i, y_0) \leq 0$, that is, $y_0 \in Gx_i$. The proof that G is KKM is complete.

On the other hand, the values of G are closed and convex (because the function $y \mapsto (u(y), y)$ is continuous and convex on C), and the value Gx_0 is bounded by assumption. By the elementary KKM-principle (4.3), we get $\bigcap\{Gx \mid x \in D\} \neq \emptyset$. □

(4.11) COROLLARY (Minty). *Let $T : H \to 2^H$ be a maximal monotone operator. Then*:

(a) *$I + T$ is surjective (I denotes the identity operator on H)*,

(b) *if $D(T)$ is bounded, then T is surjective.*

PROOF. (a) It is clearly enough to show that $0 \in (I+T)(H)$. By (4.10) with $u = I$, there is $y_0 \in C = \operatorname{Conv} D(T)$ such that

$$(x^* - (-y_0), x - y_0) \geq 0 \quad \text{for all } x \in D(T) \text{ and } x^* \in Tx.$$

Because T is maximal monotone, we derive that $-y_0 \in Ty_0$, or equivalently, that $0 \in y_0 + Ty_0$.

(b) As in (a), it is sufficient to show that $0 \in T(H)$. Since T is maximal, $D(T)$ is not empty, and therefore C is closed, convex, bounded, and nonempty. By (4.10) with $u = 0$, we find $y_0 \in C$ such that

$$(x^*, x - y_0) \geq 0 \quad \text{for all } x \in D(T) \text{ and } x^* \in Tx.$$

Since T is maximal monotone, this implies that $0 \in Ty_0$. □

5. Theorems of Mazur–Orlicz and Hahn–Banach

In this section, using the Markoff–Kakutani theorem, Theorem (3.2.2) and the fact that a Tychonoff cube is compact, we derive some basic facts of linear functional analysis.

Let E be a vector space and E' the algebraic dual of E. We recall that a functional $p : E \to \boldsymbol{R}$ is said to be *sublinear* if

(i) $$p(x + y) \leq p(x) + p(y) \quad \text{for all } x, y \in E,$$

(ii) $$p(\alpha x) = \alpha p(x) \quad \text{for all } \alpha \geq 0 \text{ and } x \in E.$$

Note that if p is sublinear, then $0 = p(0) = p(x + (-x)) \leq p(x) + p(-x)$, and therefore

(iii) $$-p(-x) \leq p(x) \quad \text{for each } x \in E.$$

(5.1) LEMMA (Banach). *Let $p : E \to \boldsymbol{R}$ be a sublinear functional. Then there exists an $f \in E'$ such that $f(x) \leq p(x)$ for all $x \in E$.*

PROOF. Let $X = \boldsymbol{R}^E$ be the linear topological space of maps $E \to \boldsymbol{R}$ equipped with the product topology; clearly, X has sufficiently many linear functionals (the evaluation maps $f \mapsto f(x)$ are in fact linear and continuous from $\boldsymbol{R}^E$ to $\boldsymbol{R}$). Consider now the sets

$$\begin{aligned} X_0 &= \prod_{x \in E} [-p(-x), p(x)], \\ X_1 &= \{g \in X_0 \mid -p(-x) \leq g(x+y) - g(y) \leq p(x) \text{ for all } x, y \in E\}; \end{aligned}$$

clearly, both X_0 and X_1 are nonempty (because from $-p(y-x) \leq p(x)-p(y) \leq p(x-y)$ it follows that $p \in X_1$). They are both convex and compact (by the Tychonoff theorem).

We define a family $\{T_y \mid y \in E\}$ of maps $T_y : X_1 \to X_1$ by

$$(T_y g)(x) = g(x+y) - g(y) \quad \text{ for } x \in E.$$

Clearly, the family $\{T_y\}$ consists of continuous affine maps and is commuting. By the Markoff–Kakutani fixed point theorem, there exists an $f \in X_1$ such that

$$T_y f = f \quad \text{ for all } y \in E,$$

i.e., $f(x+y) = f(x) + f(y)$ for all $x, y \in E$.

Note that the additivity of f gives $f(rx) = rf(x)$ for each $r \in \boldsymbol{Q}$. Let λ be any real number, and $\{r_n\}$ be a sequence of rational numbers such that $r_n \to \lambda$ and $r_n < \lambda$. Because f is in X_1, we have

$$-(\lambda - r_n)p(-x) \leq f(\lambda x) - f(r_n x) \leq (\lambda - r_n)p(x),$$

and therefore $f(\lambda x) = \lim f(r_n x) = \lim r_n f(x) = \lambda f(x)$. Thus, $f \in E'$ and $f(x) \leq p(x)$ for all $x \in E$. ☐

(5.2) LEMMA. *Let $p : E \to \boldsymbol{R}$ be a sublinear functional and $x_0 \in E$. Then there exists an $f \in E'$ such that $f(x_0) = p(x_0)$ and $f(x) \leq p(x)$ for all $x \in E$.*

PROOF. Define $p^* : E \to \boldsymbol{R}$ by

$$p^*(x) = \inf\{p(x + \lambda x_0) - \lambda p(x_0) \mid \lambda \geq 0\}, \quad x \in E.$$

Clearly, because $-p(-x) \leq p^*(x) \leq p(x)$ for all $x \in E$, p^* is well defined and $p^* \leq p$. Since for each $\alpha > 0$,

$$\begin{aligned} p^*(\alpha x) &= \inf\{p(\alpha x + \lambda x_0) - \lambda p(x_0) \mid \lambda \geq 0\} \\ &= \inf\{\alpha[p(x + (\lambda/\alpha)x_0) - (\lambda/\alpha)p(x_0)] \mid \lambda \geq 0\} \\ &= \alpha \inf\{p(x + \lambda' x_0) - \lambda' p(x_0) \mid \lambda' = \lambda\alpha^{-1} \geq 0\} = \alpha p^*(x), \end{aligned}$$

p^* is positively homogeneous. Now, for $x_1, x_2 \in E$ and fixed $\varepsilon > 0$ take $\lambda_1, \lambda_2 \geq 0$ so that $p^*(x_i) \geq p(x_i + \lambda_i x_0) - \lambda_i p(x_0) - \varepsilon$ for $i = 1, 2$. Letting

$\mu = \lambda_1 + \lambda_2$, and adding the above inequalities, we obtain

$$\begin{aligned} p^*(x_1) + p^*(x_2) &\geq p(x_1 + \lambda_1 x_0) + p(x_2 + \lambda_2 x_0) - \mu p(x_0) - 2\varepsilon \\ &\geq p(x_1 + x_2 + \mu x_0) - \mu p(x_0) - 2\varepsilon \\ &\geq p^*(x_1 + x_2) - 2\varepsilon, \end{aligned}$$

and this (because ε was arbitrary) implies that p^* is sublinear. By Lemma (5.1), there is a linear functional $f \in E'$ such that

$$f(x) \leq p^*(x) \leq p(x) \quad \text{for all } x \in E.$$

Then

$$-f(x_0) = f(-x_0) \leq p^*(-x_0) \leq p(-x_0 + \lambda x_0) - \lambda p(x_0)$$

for all $\lambda \geq 0$, which implies by putting $\lambda = 1$ that $-f(x_0) \leq -p(x_0)$, and hence $f(x_0) = p(x_0)$. □

We are now in a position to prove the following fundamental result:

(5.3) THEOREM (Mazur–Orlicz). *Let $p : E \to \boldsymbol{R}$ be a sublinear functional. Assume that we are given a family $\{x_t \mid t \in T\}$ of points in E and a family $\{\beta_t \mid t \in T\}$ of real numbers, both indexed by the same abstract set T. Then the following two conditions are equivalent*:

(A) *there exists a linear functional $f \in E'$ such that $f(x) \leq p(x)$ for all $x \in E$ and $\beta_t \leq f(x_t)$ for all $t \in T$,*

(B) *for every convex combination $\sum_{i=1}^n \lambda_i x_{t_i}$ of points $x_{t_1}, \dots, x_{t_n}$ in E,*

$$\sum_{i=1}^n \lambda_i \beta_{t_i} \leq p\Big(\sum_{i=1}^n \lambda_i x_{t_i}\Big).$$

PROOF. Clearly, (A)⇒(B); we are going to show that (B) implies (A). Consider the convex sets

$$\begin{aligned} X_0 &= \prod_{x \in E} [-p(-x), p(x)], \\ Y &= \{f \in E' \mid -p(-x) \leq f(x) \leq p(x) \text{ for all } x \in E\}. \end{aligned}$$

By Lemma (5.1), Y is nonempty, and because Y is closed in X_0, it is also compact.

Consider now the family $\varPhi = \{\varphi_t \mid t \in T\}$ of continuous affine functions $\varphi_t : Y \to \boldsymbol{R}$ defined by $\varphi_t(f) = \beta_t - f(x_t)$ for $f \in Y$, and examine the following two conditions:

(C) there exists an $f \in Y$ such that

$$\varphi_t(f) = \beta_t - f(x_t) \leq 0 \quad \text{for all } \varphi_t \in \varPhi,$$

(D) for every convex combination $\psi = \sum \lambda_i \varphi_{t_i} \in [\Phi] = \operatorname{conv} \Phi$ of elements of Φ, there is an $f \in Y$ such that

$$\psi(f) = \sum \lambda_i [\beta_{t_i} - f(x_{t_i})] \leq 0.$$

We observe that (C) is clearly equivalent to (A), and by (3.2.2), also (C) and (D) are equivalent.

It follows that it is enough to show that (B) implies (D). So assume that (B) is true, and let $\psi = \sum \lambda_i \varphi_{t_i} \in \operatorname{conv} \Phi$. By assumption we have $\sum \lambda_i \beta_{t_i} \leq p(\sum \lambda_i x_{t_i})$. Let $x_0 = \sum \lambda_i x_{t_i}$; by Lemma (5.2), there is an $f \in Y$ such that $f(x_0) = p(x_0)$, so we have

$$\sum \lambda_i \beta_{t_i} \leq f\Big(\sum \lambda_i x_{t_i}\Big),$$

and thus $\psi(f) = \sum \lambda_i [\beta_{t_i} - f(x_{t_i})] \leq 0$; hence (D) is true, and the proof is complete. □

As an immediate consequence we obtain a refined version of the Hahn–Banach theorem:

(5.4) THEOREM. *Let $p : E \to \boldsymbol{R}$ be a sublinear functional, C a convex subset of E and $g : C \to \boldsymbol{R}$ a concave function such that $g(y) \leq p(y)$ for all $y \in C$. Then there is a linear functional $f \in E'$ such that $g(y) \leq f(y)$ for all $y \in C$ and $f(x) \leq p(x)$ for all $x \in E$.*

PROOF. Take $T = C$, $\beta_t = g(t)$, and $x_t = t$ for $t \in C$; then condition (B) of (5.3) is satisfied. Consequently, by the Mazur–Orlicz theorem, there exists a linear functional $f \in E'$ such that $f(x) \leq p(x)$ for all $x \in E$ and $g(t) \leq f(t)$ for all $t \in C$. □

Another consequence is concerned with an extended version of the classical moments problem:

(5.5) THEOREM. *Let E be a normed linear space, $\{x_m\}_{m=1}^{\infty}$ a given sequence in E, and $\{c_m\}_{m=1}^{\infty}$ a sequence of real numbers. Then the following two conditions are equivalent:*

(A) *there exists a linear functional $f \in E^*$ such that $f(x_m) \geq c_m$ for all $m = 1, 2, \ldots$ and $\|f\| \leq M$, where $M > 0$,*

(B) *for every convex combination $\sum_{i=1}^{n} \lambda_i x_i$ of the points $x_1, \ldots, x_n$ we have*

$$\sum_{i=1}^{n} \lambda_i c_i \leq M \Big\| \sum_{i=1}^{n} \lambda_i x_i \Big\|.$$

PROOF. Clearly, it is enough to show that (B)⇒(A). To this end, letting $p(x) = M\|x\|$ for $x \in E$, we apply (5.3) to the set $T = \boldsymbol{N}$ and the sequences $\{x_m\}$ and $\beta_m = c_m$. □

6. Miscellaneous Results and Examples

A. Applications of the Banach theorem and of related results to analysis

(A.1) (*Systems of linear equations*) Consider the infinite system of linear equations

$$(*)\qquad x_i = \sum_{j=1}^{\infty} a_{ij} x_j + b_i, \qquad i = 1, 2, \ldots, \qquad b_i, a_{ij} \in \boldsymbol{R},$$

and assume that one of the following conditions is satisfied:

(a) for some constants $0 \leq \alpha < 1$ and $\beta > 0$ we have

$$\sum_{j=1}^{\infty} |a_{ij}| \leq \alpha, \qquad |b_i| \leq \beta \qquad \text{for each } i = 1, 2, \ldots,$$

(b) for some $p > 1$ we have

$$\sum_{i=1}^{\infty} \Big(\sum_{j=1}^{\infty} |a_{ij}|^{p/(p-1)} \Big)^{p-1} < 1, \qquad \sum_{i=1}^{\infty} |b_i|^p < \infty,$$

(c) letting $c_i = \sup\{|a_{ij}| \mid j = 1, 2, \ldots\}$, we have

$$\sum_{i=1}^{\infty} c_i < 1, \qquad \sum_{i=1}^{\infty} |b_i| < \infty.$$

Prove: The system $(*)$ has a unique solution in the space, respectively: m (the space of bounded sequences with the sup norm), l^p and l^1.

(A.2) (*Integral equations*) Let $K : [a, b] \times [a, b] \to \boldsymbol{R}$ be a measurable and square integrable function. Assume that for a real parameter λ,

$$|\lambda| \Big(\int K(s, t)^2 \, ds \, dt \Big)^{1/2} < 1.$$

Show: The integral equation $u(s) = f(s) + \lambda \int K(s, t) u(t) \, dt$ $(a \leq s \leq b)$, where $f \in L^2[a, b]$, has a unique solution $u \in L^2[a, b]$.

(A.3) (*Application of the nonlinear alternative for contractive maps*) We seek the solutions to the initial value problem

$$(\mathrm{P})\qquad \begin{cases} x'(t) = f(t, x(t)), & t \in [0, T], \\ x(0) = 0, \end{cases}$$

where $f : [0, T] \times \boldsymbol{R} \to \boldsymbol{R}$ is continuous. Suppose

(a) for each $r > 0$ there is an $l_r \in \boldsymbol{R}$ such that

$$|f(t, x) - f(t, y)| \leq l_r |x - y| \qquad \text{for all } t \in [0, T] \text{ and } x, y \in [-r, r],$$

(b) there is a continuous function $\varphi : [0, \infty) \to (0, \infty)$ such that

$$|f(t, x)| \leq \varphi(|x|) \qquad \text{for all } t \in [0, T] \text{ and } x \in \boldsymbol{R}.$$

Prove: If $T < \int_0^\infty ds/\varphi(s)$, then the initial value problem (P) has a unique solution $x \in C^1([0, T])$.

[Consider the family of problems

$$(\mathrm{P}_\lambda) \qquad \begin{cases} x'(t) = \lambda f(t, x(t)), & t \in [0, T], \\ x(0) = 0, \end{cases}$$

depending on a parameter $\lambda \in [0, 1]$. Fix an M so that $T < \int_0^M ds/\varphi(s)$ and show that if x is a solution of the problem (P_λ) for some λ, then $|x(t)| < M$ for all $t \in [0, T]$. Let $L = l_M$ be the constant given in (a), and define the norm $\| \; \|_L$ in $C([0, T])$ by

$$\|x\|_L = \sup\{e^{-tL}|x(t)| \mid t \in [0, T]\}.$$

Set $U = \{x \in C([0, T]) \mid |x(t)| < M \text{ for all } t \in [0, T]\}$ and prove that $G : \overline{U} \to C([0, T])$ given by $G(x)(t) = \int_0^t f(s, x(s))\, ds$ is contractive in the norm $\| \; \|_L$; conclude the argument by showing that λG has no fixed points on the boundary of U.]

B. Nonexpansive maps and monotone operators in Hilbert space

(B.1) Let $f : A \to H$ be a map (not necessarily continuous) on a subset $A \subset H$.
(a) f is *monotone* $\Leftrightarrow$ $(fx - fy, x - y) \geq 0$ for all $x, y \in A$.
(b) f is *strictly monotone* $\Leftrightarrow$ $(fx - fy, x - y) > 0$ for all $x, y \in A$, $x \neq y$.
(c) f is *strongly monotone* $\Leftrightarrow$ $(fx - fy, x - y) \geq C\|x - y\|^2$ for all $x, y \in A$ and some $C > 0$.

Show:
(i) Every contractive field is strictly monotone.
(ii) Every nonexpansive field is monotone.
(iii) Every strongly monotone map is injective.
(iv) f is strongly monotone with constant C if and only if $\frac{1}{C} f - I$ is monotone.
(v) If f is strongly monotone, then $\|fx - fy\| \geq C\|x - y\|$.
(vi) If $f : H \to H$ is differentiable, then f is strongly monotone if and only if $(Df(x)h, h) \geq C\|h\|^2$ for some $C > 0$ and all $h \in H$.

(B.2) (*Minimizing convex functionals*) Let $C \subset H$ be a closed convex set.
(a) Let $\varphi : C \to \boldsymbol{R}$ be a quasi-convex l.s.c. coercive function. Prove: φ attains its minimum at some $y_0 \in C$.
[For each $x \in C$, let $\Gamma x = \{y \in C \mid \varphi(y) \leq \varphi(x)\}$ and apply (4.2).]
(b) Let $a : H \times H \to \boldsymbol{R}$ be a coercive continuous bilinear form, and let $l : H \to \boldsymbol{R}$ be a continuous linear form. Show: There is a unique $y_0 \in C$ such that the following equivalent properties hold:
(i) $\frac{1}{2}[a(y_0, y_0 - x) + a(y_0 - x, y_0)] \leq l(y_0 - x)$ for all $x \in C$,
(ii) the quadratic form $x \mapsto \varphi(x) = \frac{1}{2}a(x, x) - l(x)$ on C attains its minimum at y_0.
[Verify first that (i)$\Leftrightarrow$(ii); to prove (ii), apply (a) to the coercive convex function φ.]

(B.3) (*Nikodym theorem*) Let $C \subset H$ be closed convex. Show: There exists a retraction $r : H \to C$ with the following properties:
(i) If $x_0 \in H$, then $r(x_0)$ is the unique point in C with $\|x_0 - r(x_0)\| = \inf\{\|x_0 - x\| \mid x \in C\} = d(x_0, C)$.
(ii) For each $x_0 \in H$, the point $r(x_0)$ is a solution of the variational inequality $(r(x_0) - x_0, r(x_0) - x) \leq 0$ for all $x \in C$.
(iii) The retraction r is nonexpansive.
(iv) If $C = H_0$ is a linear subspace of H, then $r : H \to H_0$ is the orthogonal projection (i.e., for each $x \in H$, $(x - r(x), y) = 0$ for all $y \in H_0$).

(Parts (i) and (iv) are due to Nikodym [1931].)

(B.4) Let C be a closed convex set in $\boldsymbol{R}^n$ and $f : C \to \boldsymbol{R}$ be a C^1 function that attains its minimum at $y_0 \in C$. Let $F(x) = \operatorname{grad} f(x)$. Show: $(F(y_0), y_0 - x) \leq 0$ for all $x \in C$.

(B.5) (*Complementarity problem*) Let $\boldsymbol{R}^n_+ = \{x = (x_1, \dots, x_n) \in \boldsymbol{R}^n \mid x_i \geq 0 \text{ for all } i\}$, and let $F : \boldsymbol{R}^n \to \boldsymbol{R}^n$. The *complementarity problem* is to find $y_0 \in \boldsymbol{R}^n_+$ such that $F(y_0) \in \boldsymbol{R}^n_+$ and $(F(y_0), y_0) = 0$.

(a) Show: The following statements are equivalent: (i) $y_0 \in \boldsymbol{R}^n_+$ is a solution of the complementarity problem, (ii) $(F(y_0), y_0 - x) \leq 0$ for all $x \in \boldsymbol{R}^n_+$.

[For (ii)⇒(i), note that if $y_0 \in \boldsymbol{R}^n_+$ solves (ii), then letting $e_i \in \boldsymbol{R}^n$ be the standard unit basis, we have $x = y_0 + e_i \in \boldsymbol{R}^n_+$ for each $i \in [n]$, and therefore $F(y_0) \in \boldsymbol{R}^n_+$ by (ii); deduce that $(F(y_0), y_0) \leq 0$ and finally that $(F(y_0), y_0) = 0$.]

(b) Let $F : \boldsymbol{R}^n \to \boldsymbol{R}^n$ be strongly monotone and continuous on the cone $\boldsymbol{R}^n_+$. Show: The complementarity problem for F has a unique solution.

(The above results are due to Karamardian [1972].)

(B.6) Let $C = \{x \in H \mid \|x\| \leq r\}$ and $f : C \to H$ be monotone and hemicontinuous. Prove: if $f(y) \neq \lambda y$ for all $\lambda < 0$ and $\|y\| = r$, then there is $y_0 \in C$ such that $f(y_0) = 0$.

[Apply (4.6) to get $y_0 \in C$ with $(fy_0, y_0 - x) \leq 0$ for all $x \in X$, consider two cases: $\|y_0\| = r$ and $\|y_0\| < r$.]

(B.7) Let $f : H \to H$ be monotone and hemicontinuous. Prove: If $(fx, x)/\|x\| \to \infty$ uniformly as $\|x\| \to \infty$, then f is surjective (G. Minty).

[Given $y_0 \in H$ consider $x \mapsto g(x) = f(x) - y_0$ and apply (B.6) to the map $g : H \to H$ on a sufficiently large ball.]

(B.8) Let $C \subset H$ be a closed convex set, and let $r : H \to C$ be the map sending each $x \in H$ to its nearest point in C. Show: $r : H \to C$ is nonexpansive.

[Use the reasoning in the latter half of (1.4).]

(B.9) Show: The nonlinear alternative (1.5) for nonexpansive maps and its corollaries (1.6)(a)–(d) remain valid if in (1.5) and (1.6) the closed ball in H is replaced by any closed convex bounded subset $C \subset H$ with $0 \in \operatorname{Int}(C)$.

(B.10) Let C be a bounded closed convex subset of a Hilbert space and $\mathscr{F}$ be a family of commuting nonexpansive maps of C into C. Show: The maps in $\mathscr{F}$ have a common fixed point (F. Browder).

C. Nonexpansive maps in Banach spaces

(C.1) A Banach space is *uniformly convex* if there is a monotone increasing surjection $\varphi : [0, 2] \to [0, 1]$ continuous at 0, with $\varphi(0) = 0, \varphi(2) = 1$, such that $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x - y\| \geq \varepsilon$ implies $\|(x + y)/2\| \leq 1 - \varphi(\varepsilon)$. Let $\eta : [0, 1] \to [0, 2]$ be the inverse of φ.

(a) Let E be uniformly convex, and u, v two elements of E. Assume that there is an $x \in E$ with $\|x - u\| \leq R$, $\|x - v\| \leq R$, $\|x - (u + v)/2\| \geq r > 0$. Prove:

$$\|u - v\| \leq R\eta\left[\frac{R - r}{R}\right].$$

[Write $r = (1 - (R - r)/R)R$.]

(b) Let E be a uniformly convex Banach space and $C \subset E$ a closed bounded convex set. Prove: Every nonexpansive $F : C \to C$ has a fixed point (Browder [1965], Göhde [1965], Kirk [1965]).

(c) Let C be a closed convex set in a uniformly convex Banach space E. Show: For each $x_0 \in E$, there is a unique $u \in C$ with $\|x_0 - u\| = \inf_{c \in C} \|x_0 - c\|$.

(C.2) Let $K^\infty = \{x = \{x_i\} \in c_0 \mid \|x\| = \sup_{1 \le i < \infty} |x_i| \le 1\}$ be the unit ball in c_0. Show that $\varphi : K^\infty \to K^\infty$ given by $(x_1, x_2, \ldots) \mapsto (1, x_1, x_2, \ldots)$ is a nonexpansive map without fixed points (Beals).

(C.3) Let E be a Banach space. A set $S \subset E$ is *star-shaped* if there is some $p \in S$ such that $tx + (1-t)p \in S$ for all $x \in S$ and $0 \le t \le 1$. A set $A \subset S$ is called an *attractor* for a map $F : S \to S$ provided

$$\overline{\bigcup_{n \ge 1} F^n(x)} \cap A \ne \emptyset \qquad \text{for each } x \in S.$$

(a) Let S be a compact star-shaped subset of a Banach space. Prove: Every nonexpansive $F : S \to S$ has a fixed point.

(b) Let S be a star-shaped subset of a Banach space and $F : S \to S$ a nonexpansive map with a compact attractor. Show: F has a fixed point (Göhde [1965]).

[Assuming, without loss of generality, that $0 \in S$, establish that given $\lambda \in (0,1)$ there is a point $x_\lambda \in S$ satisfying $\|x_\lambda - Fx_\lambda\| \le d(1-\lambda)$, where $d = \delta(S)$; then find a point y_λ in a compact attractor A of F such that $\|y_\lambda - F^{n(\lambda)}(x_\lambda)\| \le (1-\lambda)$ for a sufficiently large integer $n(\lambda)$. Establish the inequality $\|y_\lambda - Fy_\lambda\| \le (1-\lambda)(d+2)$ and use compactness of A to conclude the proof.]

(C.4) Let E be a Banach space, and $A \subset E$ any nonempty subset. Let

$$\begin{aligned} r_a(A) &= \inf\{r \mid A \subset B(a,r)\} \qquad (a \in A), \\ r(A) &= \inf\{r_a(A) \mid a \in A\}, \\ \check{C}(A) &= \{a \in A \mid r_a(A) = r(A)\}. \end{aligned}$$

(a) Let A be a bounded closed set. Show: $\delta[\check{C}(A)] \le r(A)$.

(b) Let K be a bounded closed convex set and $T : K \to K$ nonexpansive. Prove: If $\operatorname{Conv} T(K) = T(K)$, then $T[\check{C}(K)] \subset \check{C}(K)$.

(c) A convex set K in a Banach space is said to have *normal structure* if $r(D) < \delta(D)$ for each bounded closed convex $D \subset K$ with $\delta(D) > 0$.

Let E be a reflexive Banach space, and K a nonempty bounded closed convex set with normal structure. Prove: If $\delta(K) > 0$, then $\check{C}(K)$ is a nonempty proper closed convex subset of K (Brodskiĭ–Milman [1948]).

[Observe $K \subset B(u,r) \Leftrightarrow u \in \bigcap\{\overline{B}(x,r) \mid x \in K\}$. Next note that for each $\varepsilon > 0$, the set $C_\varepsilon(K) = \bigcap\{\overline{B}(x, r(K)+\varepsilon) | x \in K\} \ne \emptyset$ and that $\check{C}(K) = \bigcap\{C_\varepsilon(K) \mid \varepsilon > 0\}$. Now use the Mazur–Šmulian theorem.]

(d) The Hilbert space l^2 renormed by $\|x\| = \sup_n\{\frac{1}{2}\sqrt{\sum_i x_i^2}, |x_n|\}$ is reflexive. Let $K = \{x \mid \|x\| \le 1 \text{ and } x_i \ge 0 \text{ for all } i\}$. Show: K is a closed bounded convex set that does not have normal structure.

(e) Prove: If E is a uniformly convex Banach space, then every bounded closed convex set has normal structure (Brodskiĭ–Milman [1948]).

(C.5) Let E be a reflexive Banach space, and K a nonempty bounded closed convex set with normal structure. Prove: Every nonexpansive $T : K \to K$ has a fixed point (Kirk [1965]).

[Use the Kuratowski–Zorn lemma to find a minimal nonempty closed convex $K_0 \subset K$ with $T(K_0) \subset K_0$; show that $\operatorname{Conv} T(K_0) = K_0$; then apply (C.4)(b) and (c).]

(C.6) The following example of Alspach [1981] shows that if C is a weakly compact convex set in a Banach space, then a nonexpansive $T : C \to C$ need not have a fixed point.

Let $T : \boldsymbol{I}^2 \to \boldsymbol{I}^2$ be the "baker's transformation"

$$T(x,y) = \begin{cases} (x/2, 2y), & 0 \le y \le \frac{1}{2}, \\ (x/2 + 1/2, 2y - 1), & \frac{1}{2} < y \le 1, \end{cases}$$

which can be visualized as first squeezing $\boldsymbol{I}^2$ into the rectangle $\{(x,y) \mid 0 \le x \le \frac{1}{2},\ 0 \le y \le 2\}$, then cutting off the top half and placing it next to the lower half. It is known that T is measure-preserving, i.e., $\mu(TA) = \mu(A)$ for every measurable $A \subset \boldsymbol{I}^2$.

In $L^1(\boldsymbol{I})$ with the usual norm $\|f\| = \int_{\boldsymbol{I}} |f|$, consider the weakly compact subset

$$C = \{f \in L^1(\boldsymbol{I}) \mid 0 \le f \le 1,\ \textstyle\int_{\boldsymbol{I}} f = \frac{1}{2}\}.$$

For each $f \in C$, let

$$\widehat{T} f(x) = \begin{cases} \min[2f(2x), 1], & 0 \le x \le \frac{1}{2}, \\ \max[2f(2x-1) - 1, 0], & \frac{1}{2} < x \le 1 \end{cases}$$

(the graph of $\widehat{T} f$ is that obtained from the graph of f after the top half of the squeezed rectangle is placed next to the lower half). Prove: (a) $\widehat{T}$ is an isometry $C \to C$, (b) $\widehat{T}$ has no fixed point.

[(a) Observe that if A_f is the ordinal set $\{(x,y) \in \boldsymbol{I}^2 \mid y \le f(x)\}$, then $\|f - g\| =$ the measure of the symmetric difference $A_f \triangle A_g$ of the ordinal sets, and recall that T is measure-preserving. (b) If $\widehat{T} f = f$, then either $f = 0$ or $f = 1$ a.e.; but then $\int_{\boldsymbol{I}} f \ne \frac{1}{2}$.]

(C.7) Let C be a compact convex set in a normed linear space and $\mathscr{F}$ be a family of commuting nonexpansive maps of C into itself. Prove: There is a common fixed point for the family $\mathscr{F}$ (DeMarr [1964]).

D. Geometric and elementary KKM-theory

(D.1) (*Intersection property in superreflexive spaces*) Let E be a Banach space. We call E *superreflexive* if it admits an equivalent uniformly convex norm. Let $\{C_i \mid i \in I\}$ be a family of closed convex sets in a superreflexive Banach space with the finite intersection property. Show: If C_0 is bounded for some $i_0 \in I$, then $\bigcap\{C_i \mid i \in I\} \ne \emptyset$.

[First assume E to be uniformly convex; follow the proof of (4.1) and (4.2).]

(D.2) (*Mazur–Schauder theorem*) Let E be a reflexive Banach space and C a closed convex subset of E. Let $\varphi : C \to \boldsymbol{R}$ be a lower semicontinuous, quasi-convex, and coercive (i.e., $\varphi(x) \to \infty$ as $\|x\| \to \infty$) functional on C. Show: The functional φ attains its minimum at some $x_0 \in C$ (Mazur–Schauder [1936]).

[For E superreflexive, use (D.1); in the general case equip E with the weak topology.]

(D.3) (*KKM-maps in superreflexive spaces*) Let E be a superreflexive Banach space and $X \subset E$. Let $G : X \to 2^E$ be a KKM-map with closed convex values such that one the sets Gx_0 is bounded. Show: $\bigcap\{Gx \mid x \in X\} \ne \emptyset$.

[Use (3.1.4) and (D.1).]

(D.4) (*Hartman–Stampacchia theorem in reflexive Banach spaces*) Let E be a Banach space, E^* its dual space and for $(\xi, \nu) \in E^* \times E$ denote $\xi(\nu)$ by $\langle \xi, \nu \rangle$. A map $f : C \to E^*$ defined on a subset $C \subset E$ is called *monotone* if $\langle f(x) - f(y), x - y \rangle \ge 0$ for all $x, y \in C$; f is *hemicontinuous* if for all $x, y \in C$ the mapping $[0,1] \ni t \mapsto \langle f(y + t(x-y)), x - y \rangle$ is continuous at 0.

Let E be a reflexive Banach space, C a nonempty closed bounded convex subset of E, and let $f : C \to E^*$ be monotone and hemicontinuous. Prove: There exists a $y_0 \in C$ such that $\langle f(y_0), y_0 - x\rangle \le 0$ for all $x \in C$ (Hartman–Stampacchia [1965]).

[Equip E with the weak topology and using the geometric KKM-principle follow the proof of (4.6).]

(D.5) Let C be a nonempty bounded closed convex subset of a superreflexive Banach space E, and let $S, T : C \to 2^C$ be such that:

(i) $Sx \subset Tx$ for all $x \in C$,
(ii) S has convex cofibers,
(iii) T has closed and convex values.

Show: If $x \in Sx$ for each $x \in C$, then $\bigcap\{Tx \mid x \in C\} \neq \emptyset$.

(D.6) Let C be a nonempty bounded closed convex subset of a superreflexive Banach space E, and let $f, g : C \times C \to \boldsymbol{R}$ satisfy:

(i) $g(x, y) \le f(x, y)$ for all $x, y \in C$,
(ii) $x \mapsto f(x, y)$ is quasi-concave on C for each $y \in C$,
(iii) $y \mapsto g(x, y)$ is l.s.c. and quasi-convex on C for each $x \in C$.

Prove:

(a) For any $\lambda \in \boldsymbol{R}$, either (i) there exists a $y_0 \in C$ such that $g(x, y_0) \le \lambda$ for all $x \in C$, or (ii) there exists a $w \in C$ such that $f(w, w) > \lambda$.
(b) The following minimax inequality holds:

$$\inf_{y\in C} \sup_{x\in C} f(x, y) \le \sup_{x\in C} g(x, x).$$

[For (a), define $S, T : C \to 2^C$ by $Sx = \{y \in C \mid f(x, y) \le 0\}$ and $Tx = \{y \in C \mid g(x, y) \le 0\}$ and apply (D.5).]

(D.7) (*Maximal monotone operators in reflexive spaces*) Let E be a reflexive Banach space. A set-valued operator $T : E \to 2^{E^*}$ is *monotone* if $\langle y^* - x^*, y - x\rangle \ge 0$ whenever $x^* \in Tx$ and $y^* \in Ty$; T is called *maximal monotone* if it is monotone and maximal in the set of all monotone operators from E into 2^{E^*}. Show: If $T : E \to 2^{E^*}$ is maximal and $D(T) = \{x \in E \mid Tx \neq \emptyset\}$ is bounded, then T is surjective (F. Browder).

[Follow the proof of (4.10), (4.11). If E is superreflexive, use (4.3); if not, equip E with the weak topology and apply the geometric KKM-principle.]

E. Selected results

(E.1) (*Elementary implicit function theorem*) Let $(E_1, \|\ \|)$ and $(E_2, |\ |)$ be Banach spaces, and let $U \subset E_1$ be open and $V \subset E_2$ be open connected. Assume $H : \overline{U} \times V \to E_1$ is a continuous map with the following properties:

(i) $\|H(u, v) - H(u', v)\| \le \alpha\|u - u'\|$, where $0 \le \alpha < 1$ and α is independent of v,
(ii) $H(u, v) \neq u$ for any $(u, v) \in \partial U \times V$,
(iii) for some $v_0 \in V$ the equation $H(u, v_0) = u$ has a unique solution $u \in U$.

Show:

(a) For each $v \in V$ the equation $H(u, v) = u$ has a unique solution u_v.
(b) The assignment $v \mapsto u_v$ is a continuous map from V to U.
(c) If the restriction $H|(U \times V) : U \times V \to E_1$ is C^1, then the assignment $v \mapsto u_v$ is a C^1 map from V to U.

(E.2) (*Miranda theorem*) Let E_1, E_2 be Banach spaces and $H : E_1 \times [0, 1] \to E_2$ be a continuous map with the following properties:

(i) $(x,t) \mapsto H(x,t)$ admits a continuous partial derivative $H_x : E_1 \times [0,1] \to \mathscr{L}(E_1, E_2)$ with respect to $x \in E_1$,
(ii) the set $\{x \in E_1 \mid H(x,t) = 0 \text{ for some } t \in [0,1]\}$ is compact,
(iii) if $H(x,t) = 0$ for some x and t, then the linear operator $H_x(x,t)$ is invertible,
(iv) for some $t_0 \in [0,1]$ the equation $H(x,t_0) = 0$ has a unique solution.

Show: The equation $H(x,t) = 0$ has a unique solution for each $t \in [0,1]$ (Miranda [1971]).

(E.3) (*Hartman theorem*) Let $(E, |\ |)$ be a Banach space and $(\mathscr{C}(E,E), \|\ \|)$ the Banach space of bounded uniformly continuous functions $p : E \to E$ with the sup norm $\|\ \|$. Let $L \in \mathrm{GL}(E)$ be a given hyperbolic isomorphism: $E = E_1 \oplus E_2$, $L|E_i = L_i \in \mathrm{GL}(E_i)$, $i = 1,2$, with $\|L_1\| < 1$ and $\|L_2^{-1}\| < 1$. We assume that $\alpha = \max(\|L_1\|, \|L_2^{-1}\|) < 1$ and that E is given the norm $|x_1 + x_2| = \max(|x_1|, |x_2|)$ for $x_i \in E_i$, $i = 1,2$.

(a) For $\nu > 0$ let

$$\mathscr{L}_\nu(L) = \{\Lambda = L + \lambda \mid \lambda \in \mathscr{C}(E,E) \text{ is bounded by } \nu \text{ and Lipschitz with constant } \leq \nu\}.$$

Call $\mathscr{L}_\nu(L)$ *admissible* if its elements are Lipschitz isomorphisms of E onto itself. Show: 1° $\mathscr{L}_\nu(L) \subset \mathscr{C}(E,E)$ is a complete metric space. 2° $\mathscr{L}_\nu(L)$ is admissible for small ν.

[For 2°, use elementary domain invariance.]

(b) Let $\mathscr{L}_\nu(L)$ be admissible and assume that $\Lambda = L + \lambda$ and $\Lambda' = L + \lambda'$ are two elements of $\mathscr{L}_\nu(L)$. Consider the equation

$$\text{(i)} \qquad h\Lambda = \Lambda' h \quad \text{for } h \in \mathscr{H},$$

where $\mathscr{H} = \{h = 1 + p \mid p \in \mathscr{C}(E,E)\}$ with $1 = \mathrm{id}_E$. Show: The equation (i) is equivalent to the equation

$$\text{(ii)} \qquad p = L^{-1}[p\Lambda + \lambda - \lambda'(1+p)] \quad \text{for } p \in \mathscr{C}(E,E).$$

(c) Show: The equation (ii) is equivalent to the system

$$\text{(iii)} \qquad \begin{cases} p_1 = [L_1 p_1 + \lambda_1'(1+p) - \lambda_1]\Lambda^{-1}, \\ p_2 = L_2^{-1}[p_2\Lambda + \lambda_2 - \lambda_2'(1+p)], \end{cases}$$

where p_i, L_i, λ_i, λ_i' $(i = 1,2)$ are the components of p, L, λ, λ', respectively, with respect to the splitting $E = E_1 \oplus E_2$.

(d) Define a map $H : \mathscr{C}(E,E) \times \mathscr{L}_\nu(L) \times \mathscr{L}_\nu(L) \to \mathscr{C}(E,E)$ by

$$H((p_1,p_2), \Lambda, \Lambda') = ([L_1 p_1 + \lambda_1'(1+p) - \lambda_1]\Lambda^{-1}, L_2^{-1}[p_2\Lambda + \lambda_2 - \lambda_2'(1+p)]).$$

Show: If $\alpha + \nu < 1$, then H is $(\alpha+\nu)$-contractive with respect to (p_1,p_2) and continuous with respect to (Λ, Λ').

(e) Assume $\mathscr{L}_\nu(L)$ is admissible with $\alpha + \nu < 1$. Show: To each pair $\Lambda, \Lambda' \in \mathscr{L}_\nu(L)$ there corresponds a unique homeomorphism $h_{\Lambda\Lambda'} = h \in \mathscr{H}$ such that $h\Lambda = \Lambda' h$; this homeomorphism depends continuously on Λ, Λ'.

[Use (b), (c), (d) and the parametrized version of the Banach theorem (1.6.A.2).]

(The above proof of a theorem of Hartman [1964] is due to Pugh [1969].)

(E.4) (*Bruhat–Tits theorem*) Let (X,d) be a complete metric space that satisfies the following *semiparallelogram law*: for any $a, b \in X$ there is a point $z \in X$ such that for all $x \in X$,

$$(*) \qquad d(a,b)^2 + 4d(x,z)^2 \leq 2d(x,a)^2 + 2d(x,b)^2.$$

(a) Prove: The point z in $(*)$ is the midpoint between a and b, i.e., $d(a,z)+d(b,z)=\frac{1}{2}d(a,b)$.

(b) Let A be a bounded subset of X. Show: There exists a unique closed ball $K(a,r)$ in X of minimal radius containing A (J.-P. Serre).

[For uniqueness, use (a); for existence, consider a sequence of closed balls $K(a_n, r_n) \supset A$ with $r_n \to r$. Prove that $\{a_n\}$ is a Cauchy sequence and that $K(a,r)$, where $a = \lim a_n$, is the desired ball.]

(c) Let $\mathscr{G}$ be a group of isometries of X. Show: If $\mathscr{G}$ has a bounded orbit $\mathscr{G}(x)$, then $\mathscr{G}$ has a common fixed point (Bruhat–Tits [1972]).

[Letting $K(x,r)$ be the unique closed ball of minimal radius containing $\mathscr{G}(x)$, prove that x is a common fixed point of $\mathscr{G}$.]

7. Notes and Comments

Fixed points for nonexpansive maps

Nonexpansive maps appear for the first time in Kolmogoroff [1933], where they were used in the axiomatic treatment of measure theory. Pontrjagin and Schnirelmann [1932] used the notion in dimension theory and established the following result: *If X is a compact metric space with $\dim X \geq r$, then there exists a nonexpansive map $\varphi : X \to \boldsymbol{R}^r$ such that $\dim \varphi(X) = r$.*

In Section 1 we give only a few theorems that are related to the contraction principle. Theorem (1.3) is a special case of more general results (see (C.1(b))) obtained independently by Browder [1965], Göhde [1965], and Kirk [1965]. Earlier, a general fixed point result for isometries was obtained by Brodskiĭ–Milman [1948]. All the above authors used weak-topology arguments in the proofs of their results; the elementary proof of (1.3) given in the text is due to Goebel [1969]. We remark that in (1.3), the sequence of iterates $\{F^n(x)\}$ does not necessarily converge to a fixed point of F; it can be proved, however, that for each $x \in C$ the sequence $\frac{1}{n}(x + Fx + \cdots + F^n x)$ converges weakly to a fixed point of F (Baillon [1975]).

We also remark that the nonlinear alternative for nonexpansive maps (1.5) remains valid for nonexpansive set-valued maps in uniformly convex spaces (Frigon [1995]). For single-valued maps, (1.5) and its corollaries (1.6) can be easily deduced (as observed by Z. Guennoun) from the following result given in Browder's survey [1976]: *Let E be a uniformly convex Banach space, $C \subset E$ be closed, convex, and bounded, and $F : C \to E$ be nonexpansive. Then the nonexpansive field $f(x) = x - Fx$ is demiclosed on C, i.e., if $\{x_n\}$ in C converges weakly to x and $\{f(x_n)\}$ converges strongly to y, then $x \in C$ and $f(x) = y$.*

For further results on nonexpansive maps (including some applications as well as some iterative techniques for approximating fixed points) the reader is referred to "Miscellaneous Results and Examples", the surveys of Opial [1967], Petryshyn [1975], Browder [1976], and to the books by Goebel–Reich [1984] and Goebel–Kirk [1991].

Applications of the Banach theorem

The fundamental idea of applying fixed point results to produce theorems in analysis is due to Poincaré [1884], [1912] and was developed further in the works of Birkhoff [1913], Birkhoff–Kellogg [1922] and then Schauder [1927a], [1927b], [1930]. Systematic applications of the Banach principle to various existence theorems in analysis were initiated by Caccioppoli [1930]. An expository account of many such applications may be found in the surveys by Niemytzki [1936] and Miranda [1949]. For applications to differential and integral equations the reader is referred to Pogorzelski's book [1966] and to Griffel [1985]. The renorming technique used in Section 2 was introduced by Bielecki [1956]. Numerous (and diverse) applications of the Banach theorem are given in "Miscellaneous Results and Examples".

Applications of the elementary domain invariance

Elementary domain invariance permits a simple and unified treatment of a number of familiar results in various fields. Theorem (3.4), established by Schauder [1934], is an abstraction (in the linear case) of Poincaré's method of continuation of solutions along a parameter and underlies the general idea due to Bernstein that obtaining suitable a priori bounds for solutions of a class of problems is frequently sufficient to establish their existence. For many uses of Theorem (3.4) in partial differential equations the reader is referred to the book of Gilbarg–Trudinger [1977].

The proof of the inverse function theorem presented in the text is an adaptation of that given in H. Cartan's book [1967].

Monotone operators were introduced independently by Kachurovskiĭ [1960], Zarantonello [1960], and Minty [1962]. Kachurovskiĭ observed that the gradient maps of convex functions are monotone and introduced the term "monotonicity". Theorem (3.9), due to Zarantonello [1960], is one of the simplest results of the theory that is related to the contraction principle. More information on monotone operators and their applications to integral and differential equations can be found in the surveys by Kachurovskiĭ [1968] and Browder [1976] and also in Brézis's book [1973].

Theorem (3.12) is due to Klee [1956], who was the first to study the negligibility of sets in Banach spaces. The method of proof presented in the text is based on the noncomplete norm technique due to Bessaga. By refining this technique, Bessaga [1966] proved that *every infinite-dimensional Hilbert space is diffeomorphic to its unit sphere,* and as a consequence established the following theorem: *There exists a C^∞ retraction of the closed unit ball in an infinite-dimensional Hilbert space onto its boundary.* The last result implies the existence of a fixed point free C^∞ self-map of the closed unit ball in an infinite-dimensional Hilbert space. For more details on negligibility of sets the reader is referred to the book of Bessaga–Pełczyński [1975].

Other invertibility results

We mention some global invertibility theorems that are not proved in the text. In the differentiable case the following result is due to Hadamard [1906]: *If $f : E \to F$ is a C^1 map between finite-dimensional Banach spaces and if f is a local homeomorphism such that $\|[f'(x)]^{-1}\| \leq M$ for some $M > 0$ and all $x \in E$, then f is a diffeomorphism.* For a proof of the Hadamard theorem for arbitrary Banach spaces, the reader is referred to the lecture notes by J.T. Schwartz [1969].

S. Mazur and S. Ulam, Lwów, 1935

A general invertibility theorem is due to Banach–Mazur [1934]: *Let X and Y be metric spaces, where X is connected and Y is locally arcwise connected and simply connected. Let $f : X \to Y$ be a proper map. Then f is invertible if and only if it is a local homeomorphism.*

A special case of the Banach–Mazur theorem was established earlier by Caccioppoli [1932]: *Let $f : E \to F$ be a C^1 proper map between Banach spaces. Then f is a diffeomorphism if and only if it is a local diffeomorphism.*

The proofs of the above two results can be found in Berger's book [1977].

For some other related results the reader is referred to Carathéodory–Rademacher [1917] and Ambrosetti–Prodi [1972].

Applications of the elementary KKM-principle

The presentation in Section 4 follows Granas–Lassonde [1995]. Variational inequalities (the systematic study of which began around 1965) are of importance in many applied problems (see Kinderlehrer–Stampacchia [1980], where an introductory account of the theory and further references can be found). Theorem (4.6) is due to Hartman–Stampacchia [1966]; its proof is a simplification of the one in Dugundji–Granas [1978]. Theorem (4.7) is due to Minty [1962]. The significance of maximality of set-valued monotone operators was brought to light by Minty [1965], to whom the theory of maximal monotone operators in a Hilbert space is due.

For more general or related results the reader is referred to Brézis's book [1973], Browder's survey [1976] and also to "Miscellaneous Results and Examples".

Mazur–Orlicz theorem

Kakutani [1938] proved the Hahn–Banach theorem using the Markoff–Kakutani theorem and the compactness of the Tychonoff cube. The same idea is used in §4 for the proof of Banach's lemma (5.1) on which the proof of the Mazur–Orlicz theorem (5.3) is based. This proof follows Granas–Lassonde [1991] and is due to F. C. Liu (unpublished). We remark that the formulation of (5.3) (obtained from the original one by replacing "linear combinations" with "convex combinations") permits getting at once a refined version of the Hahn–Banach theorem (5.4). Theorem (5.5), due to Mazur–Orlicz [1953], is a generalization of the classical "moments problem" theorem. The classical separation theorems of Mazur and Eidelheit (called by Bourbaki the "geometric forms of the Hahn–Banach theorem") also follow at once from Theorem (5.3).

S. Mazur observed that the Mazur–Orlicz and Hahn–Banach theorems remain valid if in their formulation a sublinear functional p is replaced by a convex functional; the corresponding proofs can be found in Alexiewicz [1969]. For more recent applications of the Mazur–Orlicz theorem the reader is referred to Liu [1993].

http://www.springer.com/978-0-387-00173-9

Fixed Point Theory
Granas, A.; Dugundji, J.
2003, XVI, 690 p., Hardcover
ISBN: 978-0-387-00173-9