2
Mathematical Preliminaries

This chapter covers a fairly wide array of topics from mathematics that we shall need in later chapters. We do not pretend to give all the needed background for the reader to learn these things in a comprehensive way from scratch. However, we hope that this summary will be helpful to set the notation, fill in some gaps the reader may have, and to provide a guide to the literature for needed background and proofs.

2.1 Vector Fields, Flows, and Differential Equations

This section introduces vector fields on Euclidean space and the flows they determine. This topic puts together and globalizes two basic ideas learned in undergraduate mathematics: vector fields and differential equations.

2.1.1 Example (A Basic Example)**.** An example that illustrates many of the concepts of dynamical systems is the ball in a rotating hoop. Refer to Figure 2.1.1.

This system consists of a rigid hoop that hangs from the ceiling with a small ball resting in the bottom of the hoop. The hoop rotates with frequency ω about a vertical axis through its center.

Consider varying ω, keeping the other parameters (radius of the hoop, mass of the ball, ect.) fixed. For small values of ω, the ball stays at the bottom of the hoop, and correspondingly, that position is stable (Figure 2.1.1 (left)). Accept this in an intuitive sense for the moment; eventually,

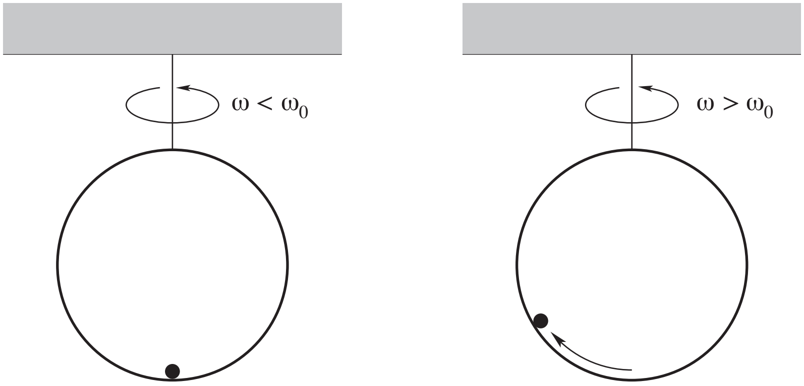

FIGURE 2.1.1. The ball in the hoop system; the equilibrium is stable for $\omega < \omega_c$ and unstable for $\omega > \omega_c$.

one has to define this concept carefully. However, when ω reaches a certain critical value ω_0, this point becomes unstable and the ball rolls up the side of the hoop to a new position $x(\omega)$, which is stable. The ball may roll to the left or to the right, depending perhaps upon the side of the vertical axis to which it was initially leaning. (See Figure 2.1.1 (right).) The position at the bottom of the hoop is still a fixed point, but it has become *unstable*. The solutions to the initial value problem governing the ball's motion are unique for all values of ω. Despite this uniqueness, because of uncertainties in the initial condition, for $\omega > \omega_0$ we cannot predict which way the ball will roll.

Using the basic principles of mechanics given in Chapter 1, we start with the Lagrangian function for this problem (the kinetic energy in an inertial frame minus the gravitational potential energy). Then the associated Euler–Lagrange equations *with forces* are given by

$$mR^2\ddot{\theta} = mR^2\omega^2 \sin\theta\cos\theta - mgR\sin\theta - \nu R\dot{\theta}, \tag{2.1.1}$$

where R is the radius of the hoop, θ is the angle from the bottom vertical, m is the mass of the ball, g is the acceleration due to gravity, and ν is a coefficient of friction.[1]

[1]This does not represent a realistic friction law, but is an ad hoc one for illustration only; even for this simple problem friction laws are controversial, depending on the exact nature of the mechanical system. If one were to suppose Coulomb friction, one would make the tangential force proportional to the normal force.

To analyze the system (2.1.1), we use a ***phase plane analysis***; that is, we write the equation as a system:

$$\begin{aligned}\dot{x} &= y\,,\\ \dot{y} &= \frac{g}{R}(\alpha\cos x - 1)\sin x - \beta y\,,\end{aligned}$$

where $\alpha = R\omega^2/g$ and $\beta = \nu/mR$. This system of equations produces for each initial point in the xy-plane a unique trajectory. That is, given a point (x_0, y_0) there is a unique solution $(x(t), y(t))$ of the equation that equals (x_0, y_0) at $t = 0$. This statement is proved by using general existence and uniqueness theory, which we give in Theorem 2.1.4. When we draw these curves in the plane, we get figures like those shown in Figure 2.1.2. ♦

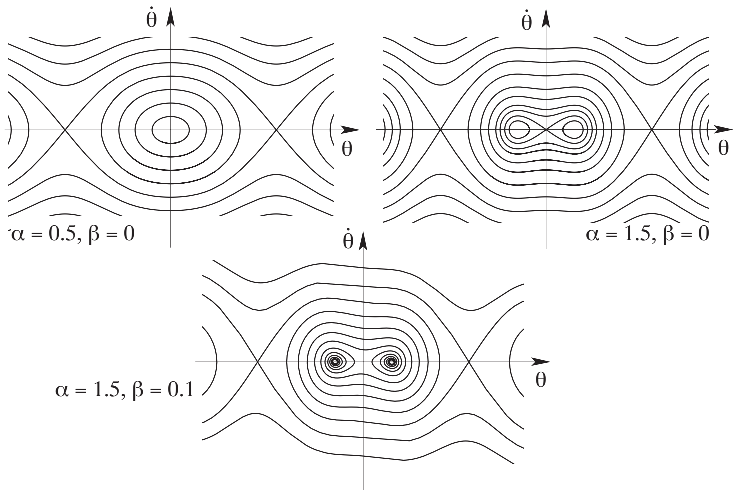

FIGURE 2.1.2. The phase portrait for the ball in the hoop before and after the onset of instability for the case $g/R = 1$.

The above example has ***system parameters*** such as g, R, and ω. In many problems one takes the point of view, as we have done in the preceding discussion, of looking at how the phase portrait of the system changes as the parameters change. Sometimes these parameters are called ***control parameters***, since one can readily imagine changing them. However, this is still a *passive point of view*, since we imagine sitting back and watching the dynamics unfold for each value of the parameters.

In control theory, on the other hand, we take a more *active point of view*, and try to *intervene directly* with the dynamics to achieve a desired end. For example, we might imagine manipulating ω as a *function of time* to make the ball move in a desired way.

Dynamical Systems. More generally than in the above example, in Euclidean space $\mathbb{R}^n$, whose points are denoted by $x = (x^1, \ldots, x^n)$, we are concerned with a system of the form

$$\dot{x} = F(x)\,, \qquad (2.1.2)$$

which, in components, reads

$$\dot{x}^i = F^i(x^1, \ldots, x^n), \qquad i = 1, \ldots, n,$$

where F is an n-component function of the n variables $x^1, \ldots, x^n$. Sometimes the function, or *vector field*, F depends on time or on other parameters (such as the mass or angular velocity in the example), and keeping track of this dependence is important. For general dynamical systems one needs some theory to develop properties of solutions; roughly, we draw curves in $\mathbb{R}^n$ emanating from initial conditions, just as we did in the preceding example.

Equilibrium Points, Stability, and Bifurcation. Equilibrium points are points where the right-hand side of the system (2.1.2) vanishes. In the ball in the hoop example, as ω increases, we see that the original stable fixed point becomes *unstable* and two *stable* fixed points split off at a critical value that we denoted above by ω_0, as indicated in Figure 2.1.2. One can use some basic stability theory that we shall develop to show that $\omega_0 = \sqrt{g/R}$. This is one of the simplest situations in which *symmetric problems can have asymmetric solutions* and in which *there can be multiple stable equilibria*, so there is nonuniqueness of equilibria (even though the solution of the initial value problem is unique).

This example shows that in some systems the phase portrait can change as certain parameters are changed. Changes in the qualitative nature of phase portraits as parameters are varied are called ***bifurcations***. Consequently, the corresponding parameters are often called ***bifurcation parameters***. These changes can be simple, such as the formation of new fixed points, called ***static bifurcations***, or more complex ***dynamic bifurcations*** such as the formation of ***periodic orbits***, that is, an orbit $x(t)$ with the property that $x(t+T) = x(t)$ for some T and all t, or even more complex dynamical structures. Thus, the ball in the hoop example exhibits a static bifurcation called a ***pitchfork bifurcation*** as the parameter ω crosses the critical value $\omega_0 = \sqrt{g/R}$.

Another important bifurcation, called the ***Hopf bifurcation***, or more properly, the ***Poincaré–Andronov–Hopf*** bifurcation, occurs in a number of examples. This is a dynamic bifurcation in which, roughly speaking, a periodic orbit rather than another fixed point is formed when an equilibrium loses stability. In this case, too, there will be a bifurcation parameter, say μ, that crosses a critical value μ_0, as indicated in Figure 2.1.3, while the original critical point loses stability.

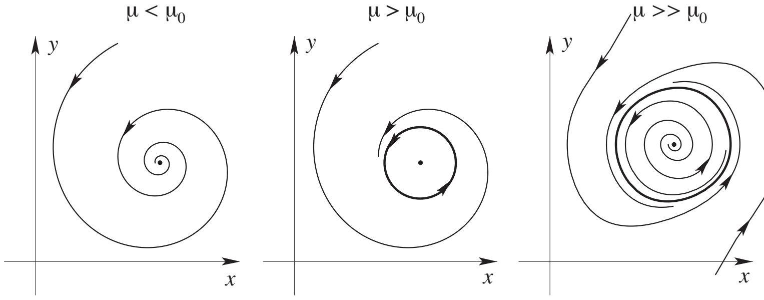

FIGURE 2.1.3. A periodic orbit appears for μ close to μ_0.

Depending on the nonlinear terms, in this bifurcation the periodic orbits can appear above (supercritical) or below (subcritical) the critical value. Unless a special degeneracy occurs, the subcritical case gives rise to unstable periodic orbits, and the supercritical case gives rise to stable orbits. See Figure 2.1.4.

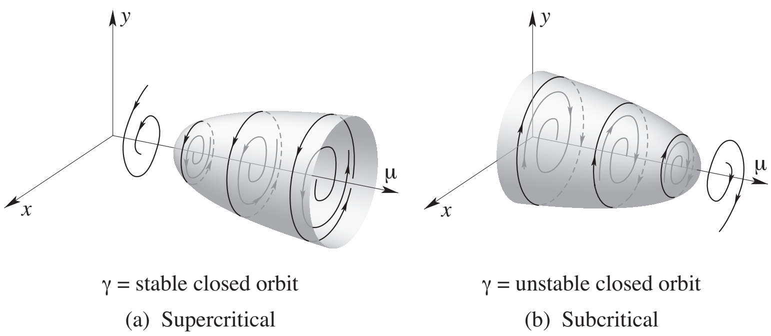

FIGURE 2.1.4. The periodic orbit appears for μ close to μ_0 can be super- or subcritical.

An everyday example of a Hopf bifurcation is ***flutter***. For example, when venetian blinds flutter in the wind or a television antenna "sings" as the wind velocity increases, there is probably a Hopf bifurcation occurring. A related example that is physically easy to understand is flow through a hose: Consider a straight vertical rubber tube conveying fluid. The lower end is a nozzle from which the fluid escapes. This is called a ***follower-load*** problem, since the water exerts a force on the free end of the tube that follows the movement of the tube. Those with any experience in a garden will not be surprised by the fact that the hose will begin to oscillate if the water velocity is high enough.

Vector Fields. With the above example as motivation, we can begin the more formal treatment of vector fields and their associated differential equations. Of course, we will eventually add the concept of controls to these vector fields, but we need to understand the notion of vector field itself first.

2.1.2 Definition. *Let $r \geq 0$ be an integer. A C^r* ***vector field*** *on an open set $U \subset \mathbb{R}^n$ is a mapping $X : U \to \mathbb{R}^n$ of class C^r from $U \subset \mathbb{R}^n$ to $\mathbb{R}^n$. The set of all C^r vector fields on U is denoted by $\mathfrak{X}^r(U)$, and the C^∞ vector fields by $\mathfrak{X}^\infty(U)$ or $\mathfrak{X}(U)$.*

We think of a vector field as assigning to each point $x \in U$ a vector $X(x)$ based (i.e., bound) at that same point.

Newton's Law of Gravitation. Here the set U is $\mathbb{R}^3$ minus the origin, and the vector field is defined by

$$\mathbf{F}(x, y, z) = -\frac{mMG}{r^3}\mathbf{r},$$

where m is the mass of a test body, M is the mass of the central body, G is the constant of gravitation, $\mathbf{r}$ is the vector from the origin to (x, y, z), and $r = (x^2 + y^2 + z^2)^{1/2}$; see Figure 2.1.5.

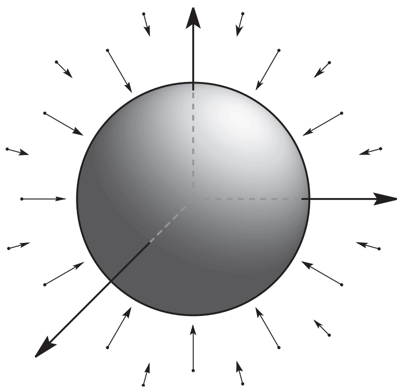

FIGURE 2.1.5. The gravitational force field.

Evolution Operators. Consider a general physical system that is capable of assuming various "states" described by points in a set Z. For example, Z might be $\mathbb{R}^3 \times \mathbb{R}^3$, and a state might be the position and velocity $(q, \dot{q})$ of a particle. As time passes, the state evolves. If the state is $z_0 \in Z$ at time t_0 and this changes to z at a later time t, we set

$$F_{t,t_0}(z_0) = z$$

and call F_{t,t_0} the ***evolution operator***; it maps a state at time t_0 to what the state would be at time t. "Determinism" is expressed by the law

$$F_{t_2,t_1} \circ F_{t_1,t_0} = F_{t_2,t_0}\,, \quad F_{t,t} = \text{ identity},$$

sometimes called the ***Chapman–Kolmogorov law***.

The evolution laws are called ***time-independent*** when F_{t,t_0} depends only on the elapsed time interval $t - t_0$; i.e.,

$$F_{t,t_0} = F_{s,s_0} \quad \text{if} \quad t - t_0 = s - s_0.$$

Setting $F_t = F_{t,0}$, the preceding law becomes the ***group property***:

$$F_\tau \circ F_t = F_{\tau+t}, \quad F_0 = \text{ identity}.$$

We call such an F_t a ***flow*** and F_{t,t_0} a ***time-dependent flow***, or an evolution operator. If the system is defined only for $t \geq 0$, we speak of a ***semiflow***.

It is usually not F_{t,t_0} that is given, but rather the ***laws of motion***. In other words, differential equations are given that we must solve to find the flow. In general, Z is a manifold (a generalization of a smooth surface), but we confine ourselves for this section to the case that $Z = U$ is an open set in some Euclidean space $\mathbb{R}^n$. These equations of motion have the form

$$\frac{dx}{dt} = X(x), \quad x(0) = x_0\,,$$

where X is a (possibly time-dependent) vector field on U.

Newton's Second Law. The motion of a particle of mass m moving in $\mathbb{R}^3$ under the influence of the gravitational force field is determined by Newton's second law:

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F}\,,$$

i.e., by the ordinary differential equations

$$\begin{aligned} m\frac{d^2x}{dt^2} &= -\frac{mMGx}{r^3}, \\ m\frac{d^2y}{dt^2} &= -\frac{mMGy}{r^3}, \\ m\frac{d^2z}{dt^2} &= -\frac{mMGz}{r^3}. \end{aligned}$$

Letting $\mathbf{q} = (x, y, z)$ denote the position and $\mathbf{p} = m(d\mathbf{r}/dt)$ denote the linear momentum, these equations become

$$\frac{d\mathbf{q}}{dt} = \frac{\mathbf{p}}{m}, \quad \frac{d\mathbf{p}}{dt} = \mathbf{F}(\mathbf{q})\,.$$

The phase space here is the open set $U = (\mathbb{R}^3 \backslash \{\mathbf{0}\}) \times \mathbb{R}^3$. The right-hand side of the preceding equations define a vector field by

$$X(\mathbf{q}, \mathbf{p}) = ((\mathbf{q}, \mathbf{p}), (\mathbf{p}/m, \mathbf{F}(\mathbf{q}))).$$

In many courses on mechanics or differential equations, it is shown how to integrate these equations explicitly, producing trajectories, which are planar conic sections. These trajectories comprise the flow of the vector field.

Of course, these equations are special cases of the Euler–Lagrange equations, and so we see how dynamical systems are relevant to the study of mechanics, and this relevance is both for holonomic and nonholonomic systems of the sorts we saw in Chapter 1.

Integral Curves of Vector Fields. Relative to a chosen set of Euclidean coordinates, we can identify a vector field X defined on an open set in $\mathbb{R}^n$ with an n-component vector function $(X^1(x), \ldots, X^n(x))$, the components of X.

2.1.3 Definition. *Let $U \subset \mathbb{R}^n$ be an open set and $X \in \mathfrak{X}^r(U)$ a vector field on U. An **integral curve** of X with initial condition x_0 is a differentiable curve c defined on some open interval $I \subset \mathbb{R}$ containing 0 such that $c(0) = x_0$ and $c'(t) = X(c(t))$ for each $t \in I$.*

Clearly, c is an integral curve of X when the following system of ordinary differential equations is satisfied:

$$\begin{aligned}
\frac{dc^1}{dt}(t) &= X^1(c^1(t), \ldots, c^n(t)), \\
&\;\;\vdots \\
\frac{dc^n}{dt}(t) &= X^n(c^1(t), \ldots, c^n(t)).
\end{aligned}$$

We shall often write $x(t) = c(t)$, an admitted abuse of notation. The preceding system of equations is called ***autonomous*** when X is time-independent. If X were time-dependent, time t would appear explicitly on the right-hand side. As we have already seen, the preceding system of equations includes equations of higher order (such as second-order Euler–Lagrange equations) by the usual reduction to first-order systems.

Existence and Uniqueness Theorems. One of the basic theorems concerning the existence and uniqueness of solutions of ordinary differential equations of the above sort is the following.

2.1.4 Theorem (Local Existence, Uniqueness, and Smoothness)**.** *Let $U \subset \mathbb{R}^n$ be open and X be a C^r vector field on U for some $r \geq 1$. For each $x_0 \in U$, there is a curve $c : I \to U$ with $c(0) = x_0$ such that $c'(t) = X(c(t))$ for all $t \in I$. Any two such curves are equal on the intersection of their domains. Furthermore, there are a neighborhood U_0 of the point $x_0 \in U$, a real number $a > 0$, and a C^r mapping $F : U_0 \times I \to U$, where I is the open interval $]-a, a[$, such that the curve $c_u : I \to U$ defined by $c_u(t) = F(u, t)$ is a curve satisfying $c_u(0) = u$ and the differential equations $c_u'(t) = X(c_u(t))$ for all $t \in I$.*

This theorem has many variants. We refer to Coddington and Levinson [1955], Hartman [1982], and Abraham, Marsden, and Ratiu [1988] for these variants and for proofs.[2]

Here is an example of a variant: with just continuity of X one can get existence (the ***Peano existence theorem***) but *without uniqueness*. The equation in one dimension given by $\dot{x} = \sqrt{x}, x(0) = 0$ has the two C^1 solutions $x_1(t) = 0$ and $x_2(t)$, which is defined to be 0 for $t \leq 0$ and $x_2(t) = t^2/4$ for $t > 0$. This example shows that one can indeed have existence without uniqueness for continuous vector fields.

The standard proof of Theorem 2.1.4 starts with a Lipschitz assumption on the vector field and proceeds to show existence and uniqueness by showing that there is a unique solution to the integral equation

$$x(t) = x_0 + \int_0^t X(x(s))\, ds.$$

One way to do this is by using the contraction mapping principle on a suitable space of curves[3] or by showing that the sequence of curves given by ***Picard iteration*** converges: Let $x_0(t) = x_0$ and define inductively

$$x_{n+1}(t) = x_0 + \int_{t_0}^t X(x_n(s))\, ds.$$

The existence and uniqueness theory also holds if X depends explicitly on t or on a parameter ρ, is jointly continuous in (t, ρ, x), and is Lipschitz or class C^r in x uniformly in t and ρ.

Dependence on Initial Conditions and Parameters. The following inequality is of basic importance not only in existence and uniqueness theorems, but also in making estimates on solutions.

2.1.5 Theorem (Gronwall's Inequality). *Let $f, g : [a, b[\to \mathbb{R}$ be continuous and nonnegative.*[4] *Suppose there is a constant $A \geq 0$ such that for all t satisfying $a \leq t \leq b$,*

$$f(t) \leq A + \int_a^t f(s)\, g(s)\, ds.$$

Then

$$f(t) \leq A \exp\left(\int_a^t g(s)\, ds \right) \quad \text{for all} \quad t \in [a, b[\,.$$

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[2] This last reference also has a proof based directly on the implicit function theorem applied in suitable function spaces. This proof has a technical advantage: It works easily for other types of differentiability assumptions on X or on F_t, such as Hölder or Sobolev differentiability; this result is due to Ebin and Marsden [1970].

[3] The contraction mapping principle is a standard result in basic real analysis, with which we assume the reader is familiar; see, for example, Marsden and Hoffman [1993].

[4] We denote an interval that is open on the right and closed on the left by either $[a, b[$ or by $[a, b)$.

We refer to the preceding references for the proof. This result is one of the key ingredients in showing that the solutions depend in a Lipschitz or smooth way on initial conditions. Specifically, let $F_t(x_0)$ denote the solution (= integral curve) of $x'(t) = X(x(t)), x(0) = x_0$. Then for Lipschitz vector fields, $F_t(x)$ depends in a continuous, and indeed Lipschitz, manner on the initial condition x and is jointly continuous in (t, x). Again, the same result holds if X depends explicitly on t and on a parameter ρ, is jointly continuous in (t, ρ, x), and is Lipschitz in x uniformly in t and ρ. We let $F^{\rho}_{t,\lambda}(x)$ be the unique integral curve $x(t)$ satisfying $x'(t) = X(x(t), t, \rho)$ and $x(\lambda) = x$. Then $F^{\rho}_{t,t_0}(x)$ is jointly continuous in the variables (t_0, t, ρ, x), and is Lipschitz in x, uniformly in (t_0, t, ρ).

Additional work along these same lines shows that F_t is C^r if X is. Again, there is an analogous result for the evolution operator $F^{\rho}_{t,t_0}(x)$ for a time-dependent vector field $X(x, t, \rho)$, which depends on extra parameters ρ in some other Euclidean space, say $\mathbb{R}^m$. If X is C^r, then $F^{\rho}_{t,t_0}(x)$ is C^r in all variables and is C^{r+1} in t and t_0.

Suspension Trick. The parameter ρ can be dealt with by suspending X to a new vector field obtained by appending the trivial differential equation $\rho' = 0$; this defines a vector field on $\mathbb{R}^n \times \mathbb{R}^m$, and the basic existence and uniqueness theorem may be applied to it. The flow on $\mathbb{R}^n \times \mathbb{R}^m$ is just $F_t(x, \rho) = (F^{\rho}_t(x), \rho)$.

Rectification. An interesting result, called the ***rectification theorem***, shows that near a point x_0 satisfying $X(x_0) \neq 0$, the flow can be transformed by a change of variables so that the integral curves become straight lines moving with unit speed.[5] This shows that in effect, nothing interesting happens with flows away from equilibrium points *as long as one looks at the flow only locally and for short time.*

The mapping F gives a locally unique integral curve c_u for each $u \in U_0$, and for each $t \in I, F_t = F|(U_0 \times \{t\})$ maps U_0 to some other set. It is convenient to think of each point u being allowed to "flow for time t" along the integral curve c_u (see Figure 2.1.6). This is a picture of a U_0 "flowing," and the system (U_0, a, F) is a local flow of X, or ***flow box***.

Global Uniqueness. The first global issue concerns uniqueness. Recall that *local* uniqueness was already addressed in Theorem 2.1.4; now we are concerned with *global* uniqueness. The following is readily proved by combining local uniqueness with a connectedness argument.

2.1.6 Proposition (Global Uniqueness)**.** *Suppose c_1 and c_2 are two integral curves of X in U and that for some time t_0, $c_1(t_0) = c_2(t_0)$. Then $c_1 = c_2$ on the intersection of their domains.*

[5] The proof can be found in Abraham and Marsden [1978], Arnold [1983], and Abraham, Marsden, and Ratiu [1988], but of course the result goes back to the classical literature.

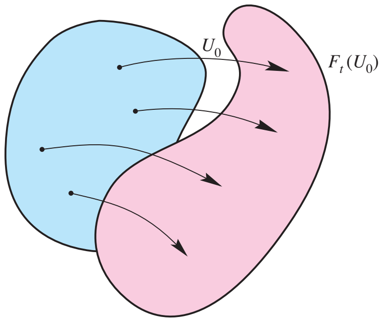

FIGURE 2.1.6. The flow of a vector field.

Completeness. Other global issues center on considering the flow of a vector field as a whole, extended as far as possible in the t-variable.

2.1.7 Definition. *Given an open set U and a vector field X on U, let $\mathcal{D}_X \subset U \times \mathbb{R}$ be the set of $(x,t) \in U \times \mathbb{R}$ such that there is an integral curve $c : I \to U$ of X with $c(0) = x$ with $t \in I$. The vector field X is* ***complete*** *if $\mathcal{D}_X = U \times \mathbb{R}$. A point $x \in U$ is called σ-**complete**, where $\sigma = +, -,$ or $\pm$, if $\mathcal{D}_X \cap (\{x\} \times \mathbb{R})$ contains all (x,t) for $t > 0$, $t < 0$, or $t \in \mathbb{R}$, respectively. Let $T^+(x)$ (respectively $T^-(x)$) denote the sup (respectively inf) of the times of existence of the integral curves through x; $T^+(x)$ respectively $T^-(x)$ is called the* ***positive (negative) lifetime of*** *x.*

Thus, X is complete iff each integral curve can be extended so that its domain becomes $]-\infty, \infty[$; i.e., $T^+(x) = \infty$ and $T^-(x) = -\infty$ for all $x \in U$.

2.1.8 Examples.

A. For $U = \mathbb{R}^2$, let X be the constant vector field $X(x,y) = (0,1)$. Then X is complete, since the integral curve of X through (x,y) is $t \mapsto (x, y+t)$.

B. On $U = \mathbb{R}^2 \backslash \{0\}$, the same vector field is not complete, since the integral curve of X through $(0,-1)$ cannot be extended beyond $t = 1$; in fact, as $t \to 1$ this integral curve tends to the point $(0,0)$. Thus $T^+(0,-1) = 1$, while $T^-(0,-1) = -\infty$.

C. On $\mathbb{R}$ consider the vector field $X(x) = 1 + x^2$. This is not complete, since the integral curve c with $c(0) = 0$ is $c(\theta) = \tan\theta$, and thus it cannot be continuously extended beyond $-\pi/2$ and $\pi/2$; i.e., $T^{\pm}(0) = \pm\pi/2$. ♦

2.1.9 Proposition. *Let $U \subset \mathbb{R}^n$ be open and $X \in \mathfrak{X}^r(U)$, $r \geq 1$. Then:*

(i) *$\mathcal{D}_X \supset U \times \{0\}$.*

(ii) *$\mathcal{D}_X$ is open in $U \times \mathbb{R}$.*

(iii) *There is a unique C^r mapping $F_X : \mathcal{D}_X \to U$ such that the mapping $t \mapsto F_X(x,t)$ is an integral curve at x for all $x \in U$.*

(iv) *For* $(x, t) \in \mathcal{D}_X, (F_X(x,t), s) \in \mathcal{D}_X$ *iff* $(m, t+s) \in \mathcal{D}_X$*; in this case*

$$F_X(x, t+s) = F_X(F_X(x,t), s).$$

2.1.10 Definition. *Let $U \subset \mathbb{R}^n$ be open and $X \in \mathfrak{X}^r(U)$, $r \geq 1$. Then the mapping F_X is called the* ***integral*** *of X, and the curve $t \mapsto F_X(x,t)$ is called the* ***maximal integral curve*** *of X at x. If X is complete, F_X is called the* ***flow*** *of X.*

Thus, if X is complete with flow F, then the set $\{F_t \mid t \in \mathbb{R}\}$ is a group of diffeomorphisms on U, sometimes called a ***one-parameter group of diffeomorphisms***. Since $F_n = (F_1)^n$ (the nth power), the notation F^t is sometimes convenient and is used where we use F_t. For incomplete flows, (iv) says that $F_t \circ F_s = F_{t+s}$ wherever it is defined. Note that $F_t(x)$ is defined for $t \in \,]T^-(x), T^+(x)[$. The reader should write out similar definitions for the time-dependent case and note that the lifetimes depend on the starting time t_0.

Criteria for Completeness. A useful criterion for global existence or completeness is the following:

2.1.11 Proposition. *Let X be a C^r vector field on an open subset U of $\mathbb{R}^n$, where $r \geq 1$. Let $c(t)$ be a maximal integral curve of X such that for every finite open interval $]a,b[$ in the domain $]T^-(c(0)), T^+(c(0))[$ of c, $c(]a,b[)$ lies in a compact subset of U. Then c is defined for all $t \in \mathbb{R}$. If $U = \mathbb{R}^n$, this holds, provided that $c(t)$ lies in a bounded set.*

For example, this is used to prove the following:

2.1.12 Corollary. *A C^r vector field on an open set U with compact support contained in U is complete.*

Completeness corresponds to well-defined dynamics persisting eternally. In some circumstances (shock waves in fluids and solids, singularities in general relativity, etc.) one has to live with incompleteness, realize that one may be dealing with an overly idealized model, or overcome it in some other way.

2.1.13 Examples.

A. Let X be a C^r vector field, $r \geq 1$, on the open set $U \subset \mathbb{R}^n$ admitting a ***first integral***, i.e., a C^r function $f : U \to \mathbb{R}$ such that

$$\sum_{i=1}^n X^i(x^1, \ldots, x^n) \frac{\partial f}{\partial x^i}(x^1, \ldots, x^n) = 0.$$

If all level sets $f^{-1}(r)$, $r \in \mathbb{R}$, are compact, then X is complete. Indeed, by the chain rule, it follows that f is constant along integral curves of X, and so each integral curve lies on a level set of f. Thus, the result follows by the preceding proposition. Of course, in mechanics we often turn to quantities like energy and linear and angular momentum to find first integrals.

B. Suppose $X(x) = A \cdot x + B(x)$ where A is a linear operator of $\mathbb{R}^n$ to itself and B is ***sublinear***; i.e., $B : \mathbb{R}^n \to \mathbb{R}^n$ is C^r with $r \geq 1$ and satisfies $\|B(x)\| \leq K\|x\| + L$ for constants K and L. We shall show that X is complete. Let $x(t)$ be an integral curve of X on the bounded interval $[0, T]$. Then

$$x(t) = x(0) + \int_0^t (A \cdot x(s) + B(x(s)))\, ds\,.$$

Hence

$$\|x(t)\| \leq \|x(0)\| + \int_0^t (\|A\| + K)\|x(s)\|\, ds + Lt.$$

By Gronwall's inequality,

$$\|x(t)\| \leq (Lt + \|x(0)\|)e^{(\|A\|+K)t}$$

for $0 \leq t \leq T$. Hence, $x(t)$ remains bounded on bounded t-intervals, so the result follows by Proposition 2.1.11. ♦

A further example on the global existence of solutions for a particle in a potential field is given in the web supplement.

The following is proved by a study of the local existence theory; we state it for completeness only.

2.1.14 Proposition. *Let X be a C^r vector field on U, $r \geq 1$, $x_0 \in U$, and $T^+(x_0)(T^-(x_0))$ the positive (negative) lifetime of x_0. Then for each $\varepsilon > 0$, there exists a neighborhood V of x_0 such that for all $x \in V$, $T^+(x) > T^+(x_0) - \varepsilon$ (respectively, $T^-(x) < T^-(x_0) + \varepsilon$). (One says that $T^+(x_0)$ is a lower semicontinuous function of x.)*

2.1.15 Corollary. *Let X_t be a C^r time-dependent vector field on U, $r \geq 1$, and let x_0 be an* ***equilibrium*** *of X_t; i.e., $X_t(x_0) = 0$, for all t. Then for any T there exists a neighborhood V of x_0 such that any $x \in V$ has integral curve existing for time $t \in [-T, T]$.*

Linear Equations. Flows of linear equations $\dot{x} = Ax$, where A is an $n \times n$ matrix, are given by $F_t(x) = e^{tA}$, where the exponential is defined, for example, by a power series

$$e^{tA} = I + tA + \frac{1}{2}t^2A^2 + \frac{1}{3!}t^3A^3 + \cdots\,.$$

Of course, one has to show that this series converges and is differentiable in t and that the derivative is given by Ae^{tA}, but this is learned in courses on real analysis. One also learns how to carry out exponentiation in courses in linear algebra by bringing A into a canonical form.

Exercises

◇ **2.1-1.** Derive equation (2.1.1) using Lagrangian mechanics.

◇ **2.1-2.** Is the flow of the vector field

$$X(x, y) = \left(x + y, \frac{1}{1 + x^2 + y^2}\right)$$

on $\mathbb{R}^2$ complete?

◇ **2.1-3.** Consider the matrix

$$A = \begin{bmatrix} -2 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Solve the system $\dot{\mathbf{x}} = A\mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^4$ with initial condition $\mathbf{x} = (1, -1, 2, 3)$.

2.2 Differentiable Manifolds

Modern analytical mechanics and nonlinear control theory are most naturally discussed in the mathematical language of differential geometry. The present chapter is meant to serve as an introduction to the elements of geometry that we shall use in the remainder of the book. Since this is not primarily a text on geometry, there is a great deal that must be left out.[6]

Studying the motion of physical systems leads immediately to the study of rates of change of position and velocity, i.e., to calculus. Differentiable manifolds provide the most natural setting in which to study calculus. Roughly speaking, differentiable manifolds are topological spaces that locally look like Euclidean space, but that may be globally quite different from Euclidean space. Since taking a derivative involves only a local computation—carried out in a neighborhood of the point of interest—it would appear that derivatives should be computable on any topological space that is infinitesimally indistinguishable from Euclidean space. This is indeed the case. What makes differentiable manifolds most important in the study of analytical mechanics, however, is the global features and their implications for the large-scale behavior of trajectories of the corresponding equations of motion.

With these remarks in mind, we begin with a definition of manifold that relates these objects to Euclidean space in small neighborhoods of each point. Questions about important global features of differentiable manifolds will be discussed in subsequent sections.

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[6]Some references are Abraham, Marsden, and Ratiu [1988], Auslander and MacKenzie [1977], Boothby [1986], Dubrovin, Fomenko and Novikov [1984], and Warner [1983].

2.2.1 Definition. *An n-dimensional **differentiable manifold** M is a set of points together with a finite or countably infinite set of subsets $U_\alpha \subset M$ and 1-to-1 mappings $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ such that:*

1. $\bigcup_\alpha U_\alpha = M$.
2. *For each nonempty intersection $U_\alpha \cap U_\beta$, the set $\varphi_\alpha(U_\alpha \cap U_\beta)$ is an open subset of $\mathbb{R}^n$, and the 1-to-1 and onto mapping $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is a smooth function.*
3. *The family $\{U_\alpha, \varphi_\alpha\}$ is maximal with respect to conditions 1 and 2.*

The situation is illustrated in Figure 2.2.1.

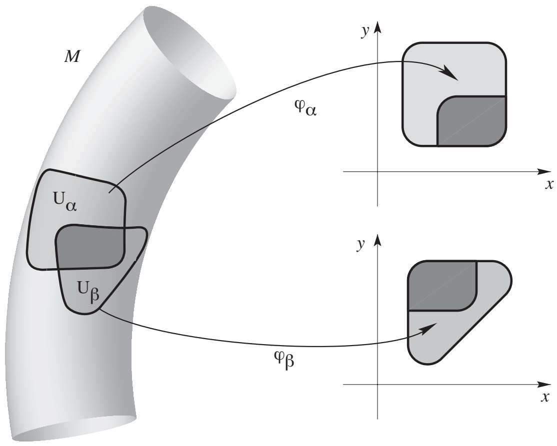


FIGURE 2.2.1. Coordinate charts on a manifold.

The sets U_α in the definition are called ***coordinate charts***. The mappings φ_α are called ***coordinate functions*** or ***local coordinates***. A collection of charts satisfying 1 and 2 is called an ***atlas***. The notion of a C^k-differentiable (respectively analytic) manifold is defined similarly, wherein the ***coordinate transformations*** $\varphi_\alpha \circ \varphi_\beta^{-1}$ are required only to have continuous partial derivatives of all orders up to k (respectively be analytic). We remark that condition 3 is included merely to make the definition of manifold independent of a choice of atlas. A set of charts satisfying 1 and 2 can always be extended to a maximal set, and in practice, 1 and 2 define the manifold.

A ***coordinate neighborhood*** V of a point x in a manifold M is a subset of the domain U of a coordinate chart $\varphi : U \subset M \to \mathbb{R}^n$ such that $\varphi(V)$ is open in $\mathbb{R}^n$. Unions of coordinate neighborhoods define the open sets in M, and one checks that these open sets in M define a topology. *Usually we assume without explicit mention that the topology is Hausdorff*: Two different points x, x' in M have nonintersecting neighborhoods.

A useful viewpoint is to think of M as a set covered by a collection of coordinate charts with local coordinates $(x^1, \ldots, x^n)$ with the property that all mutual changes of coordinates are smooth maps.

We can also extend the definition of manifold to ***manifold with boundary***, in which case we take the maps φ to be either into $\mathbb{R}^n$ or $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \mid x_n \geq 0\}$ (see, e.g., Spivak [1979] or Abraham, Marsden, and Ratiu [1988] for details). In doing this, one must define the notion of a smooth map from a half-open set in $\mathbb{R}^n$ to another, and this is done by requiring the map to be the restriction of a smooth map on a containing open set. See Figure 2.2.2.

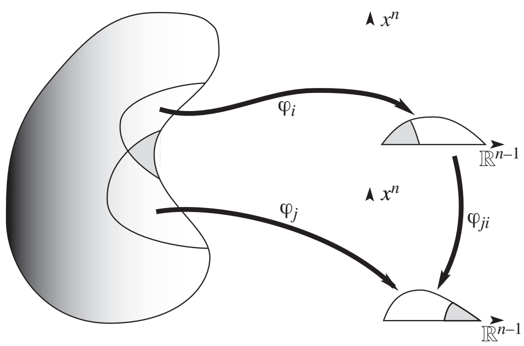


FIGURE 2.2.2. For a manifold with boundary, charts map into either open sets or half-open sets, and the overlap maps are still required to be smooth.

Level Sets as Differentiable Manifolds in $\mathbb{R}^n$. A typical and important way that manifolds arise is as follows. Let $p_1, p_2, \ldots, p_m : \mathbb{R}^n \to \mathbb{R}$. The zero (or level) set

$$M = \{x \mid p_i(x) = 0,\, i = 1, \ldots, m\}$$

is called a ***differentiable variety*** in $\mathbb{R}^n$. If the $n \times m$ matrix

$$\left(\frac{\partial p_1}{\partial x^i} \vdots \cdots \vdots \frac{\partial p_m}{\partial x^i} \right)$$

has a (constant) rank ρ at each point $x \in M$, then M admits the structure of a differentiable manifold of dimension $n - \rho$. We call this the ***rank condition***. In particular, if the rank is m (so that the matrix is onto $\mathbb{R}^m$) then we say that the map $p = (p_1, \ldots, p_m)$ is a ***submersion***. This is a common case that is often encountered.

The idea of the proof of the rank criterion (or its special case, the submersion criterion) is that under our rank assumption an argument using the implicit function theorem shows that an $(n-\rho)$-dimensional coordinate chart may be defined in a neighborhood of each point on M. In this situation, we say that the level set is a ***submanifold*** of $\mathbb{R}^n$.

Matrix Groups. We briefly discuss matrix groups as examples of differentiable manifolds. More details are given in Section 2.8.

Let $\mathbb{R}^{n\times n}$ be the set of $n\times n$ matrices with entries in $\mathbb{R}$, and let $\mathrm{GL}(n,\mathbb{R})$ denote the set of all $n\times n$ invertible matrices with entries in $\mathbb{R}$. Clearly, $\mathrm{GL}(n,\mathbb{R})$ is a group, called the ***general linear group***. Let $A\in\mathrm{GL}(n,\mathbb{R})$ be symmetric (and invertible). Consider the subset

$$\mathcal{S}=\{X\in\mathrm{GL}(n,\mathbb{R})\mid XAX^T=A\}.$$

It is easy to see that if $X\in\mathcal{S}$, then $X^{-1}\in\mathcal{S}$, and if $X,Y\in\mathcal{S}$, then the product XY is also in $\mathcal{S}$. Hence, $\mathcal{S}$ *is a subgroup of* $\mathrm{GL}(n,\mathbb{R})$.

We can also show that $\mathcal{S}$ *is a submanifold of* $\mathbb{R}^{n\times n}$. Indeed, $\mathcal{S}$ is the zero locus of the mapping $X\mapsto XAX^T-A$. Let $X\in\mathcal{S}$, and let δX be an arbitrary element of $\mathbb{R}^{n\times n}$. Then

$$\begin{aligned}(X+\delta X)A(X+\delta X)^T-A=&\\ &XAX^T-A+\delta XAX^T+XA\delta X^T+O(\delta X)^2.\end{aligned}$$

We can conclude that $\mathcal{S}$ is a submanifold of $\mathbb{R}^{n\times n}$ if we can show that the linearization of the locus map, namely the linear mapping L defined by $\delta X\mapsto\delta XAX^T+XA\delta X^T$ of $\mathbb{R}^{n\times n}$ to itself, has constant rank for all $X\in\mathcal{S}$. We see that both the original map and the image of L lie in the subspace of $n\times n$ symmetric matrices. We claim that the map L is onto this space and hence the original map is a submersion. Indeed, given X and any symmetric matrix S we can find δX such that $(\delta X)AX^T+XA(\delta X)^T=S$, namely $\delta X=SA^{-1}X/2$. Thus, the original map to the space of symmetric matrices is a submersion. For a submersion, the dimension of the level set is the dimension of the domain minus the dimension of the range space. In this case, this dimension is $n^2-n(n+1)/2=n(n-1)/2$. In summary, we have established the following fact.

2.2.2 Proposition. *Let $A\in\mathrm{GL}(n,\mathbb{R})$ be symmetric. Then the subgroup $\mathcal{S}$ of $\mathrm{GL}(n,\mathbb{R})$ defined by*

$$\mathcal{S}=\{X\in\mathrm{GL}(n,\mathbb{R})\mid XAX^T=A\}$$

is a submanifold of $\mathbb{R}^{n\times n}$ of dimension $n(n-1)/2$.

The Orthogonal Group. Of special interest in mechanics is the case $A=I$. Here $\mathcal{S}$ specializes to $\mathrm{O}(n)$, the group of $n\times n$ orthogonal matrices. It is both a subgroup of $\mathrm{GL}(n,\mathbb{R})$ and a submanifold of the vector space $\mathbb{R}^{n\times n}$. $\mathrm{GL}(n,\mathbb{R})$ is an open, dense subset of $\mathbb{R}^{n\times n}$ that inherits the topology and manifold structure from $\mathbb{R}^{n\times n}$. Thus, $\mathrm{O}(n)$ (or any $\mathcal{S}$ defined as above) is both a subgroup and a submanifold of $\mathrm{GL}(n,\mathbb{R})$. Subgroups of $\mathrm{GL}(n,\mathbb{R})$ that are also submanifolds are called ***matrix Lie groups***. We shall discuss Lie groups more abstractly later on.

Tangent Vectors to Manifolds. Two curves $t \mapsto c_1(t)$ and $t \mapsto c_2(t)$ in an n-manifold M are called ***equivalent at*** $x \in M$ if

$$c_1(0) = c_2(0) = x \quad \text{and} \quad (\varphi \circ c_1)'(0) = (\varphi \circ c_2)'(0)$$

in some chart φ, where the prime denotes the derivative with respect to the curve parameter. It is easy to check that this definition is chart independent. A ***tangent vector*** v to a manifold M at a point $x \in M$ is an equivalence class of curves at x. One proves that the set of tangent vectors to M at x forms a vector space. It is denoted by T_xM and is called the ***tangent space*** to M at $x \in M$. Given a curve $c(t)$, we denote by $c'(s)$ the tangent vector at $c(s)$ defined by the equivalence class of $t \mapsto c(s+t)$ at $t = 0$.

Let U be a chart of an atlas for the manifold M with coordinates $(x^1, \ldots, x^n)$. The ***components*** of the tangent vector v to the curve $t \mapsto (\varphi \circ c)(t)$ are the numbers $v^1, \ldots, v^n$ defined by

$$v^i = \left. \frac{d}{dt} (\varphi \circ c)^i \right|_{t=0},$$

$i = 1, \ldots, n$. The ***tangent bundle*** of M, denoted by TM, is the differentiable manifold whose underlying set is the disjoint union of the tangent spaces to M at the points $x \in M$; that is,

$$TM = \bigcup_{x \in M} T_xM.$$

Thus, a point of TM is a vector v that is tangent to M at some point $x \in M$. To define the differentiable structure on TM, we need to specify how to construct local coordinates on TM. To do this, let $x^1, \ldots, x^n$ be local coordinates on M and let $v^1, \ldots, v^n$ be components of a tangent vector in this coordinate system. Then the $2n$ numbers $x^1, \ldots, x^n, v^1, \ldots, v^n$ give a local coordinate system on TM. Notice that $\dim TM = 2 \dim M$.

The Tangent Bundle Projection. The ***tangent bundle***, or ***natural projection***, is the map $\tau_M : TM \to M$ that takes a tangent vector v to the point $x \in M$ at which the vector v is attached (that is, $v \in T_xM$). The inverse image $\tau_M^{-1}(x)$ of a point $x \in M$ under the natural projection τ_M is the tangent space T_xM. This space is called the ***fiber*** of the tangent bundle over the point $x \in M$.

Manifolds with Boundary. A manifold M with boundary is the union of two other manifolds, the interior and the boundary, denoted by ∂M. The boundary has its own tangent space, which is a subspace of the tangent space to the entire manifold at that point. See Figure 2.2.3.

Tangent Spaces to Level Sets. Let $M = \{x \mid p_i(x) = 0, i = 1, \ldots, m\}$ be a differentiable variety in $\mathbb{R}^n$. For each $x \in M$,

$$T_xM = \left\{ v \in \mathbb{R}^n \;\middle|\; \frac{\partial p_i}{\partial x}(x) \cdot v = 0 \right\}$$

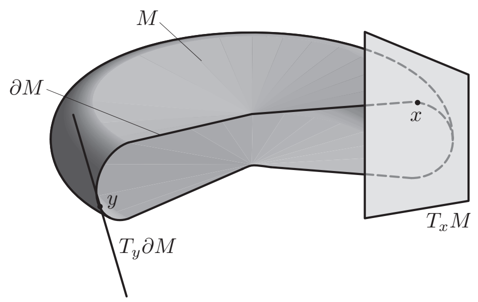


FIGURE 2.2.3. An example of a manifold M showing its boundary ∂M. In this example, M is two-dimensional, while the bounding curve is one-dimensional.

is called the ***tangent space to*** M ***at*** x. Clearly, T_xM is a vector space. If the rank condition holds, so that M is a differentiable manifold, then this definition may be shown to be equivalent to the one given earlier. For example, the reader may show that the tangent spaces to spheres are what they should intuitively be, as in Figure 2.2.4.

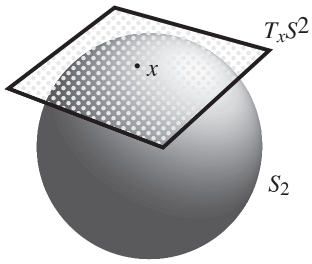


FIGURE 2.2.4. The tangent space to a sphere.

Tangent Spaces to Matrix Groups. Let $A \in \mathrm{GL}(n, \mathbb{R})$ be a symmetric matrix. We wish to explicitly describe the tangent space at a typical point of the group $\mathcal{S} = \{X \in \mathrm{GL}(n, \mathbb{R}) \mid X^T AX = A\}$. Given our definition, it is clear that the tangent space $T_X\mathcal{S}$ is a subspace of the linear space of all $n \times n$ matrices, $\mathbb{R}^{n\times n}$. Let $V \in \mathbb{R}^{n\times n}$. Then V is in $T_X\mathcal{S}$ precisely when it is tangent to a curve in the group:

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \left[(X + \epsilon V)^T A(X + \epsilon V) - A\right] = 0.$$

This condition is equivalent to $V^T AX + X^T AV = 0$.

This shows that if $X \in \mathcal{S}$, then

$$T_X\mathcal{S} = \{V \in \mathbb{R}^{n\times n} \mid V^T AX + X^T AV = 0\}.$$

Differentiable Maps. Let E and F be vector spaces (for example, $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively), and let $f : U \subset E \to V \subset F$, where U and V are open sets, be of class C^{r+1}. We define the ***tangent map*** [7] of f to be the map $Tf : TU = U \times E \to TV = V \times F$ defined by

$$Tf(u, e) = (f(u), Df(u) \cdot e), \tag{2.2.1}$$

where $u \in U$ and $e \in E$. This notion from calculus may be generalized to the context of manifolds as follows. Let $f : M \to N$ be a map of a manifold M to a manifold N. We call f ***differentiable*** (or C^k) if in local coordinates on M and N it is expressed, or represented, by differentiable (or C^k) functions. The ***derivative*** of a differentiable map $f : M \to N$ at a point $x \in M$ is defined to be the linear map

$$T_x f : T_x M \to T_{f(x)} N$$

constructed in the following way. For $v \in T_x M$, choose a curve $c : \,]-\epsilon, \epsilon[\, \to M$ with $c(0) = x$, and velocity vector $dc/dt\,|_{t=0} = v$. Then $T_x f \cdot v$ is the velocity vector at $t = 0$ of the curve $f \circ c : \mathbb{R} \to N$; that is,

$$T_x f \cdot v = \left. \frac{d}{dt} f(c(t)) \right|_{t=0}.$$

The vector $T_x f \cdot v$ does not depend on the curve c but only on the vector v. If M and N are manifolds and $f : M \to N$ is of class C^{r+1}, then $Tf : TM \to TN$ is a mapping of class C^r. Note that

$$\left. \frac{dc}{dt} \right|_{t=0} = T_0 c \cdot 1.$$

Vector Fields and Flows. Let us now interpret what we did with vector fields and flows in $\mathbb{R}^n$ in the context of manifolds. A ***vector field*** X on a manifold M is a map $X : M \to TM$ that assigns a vector $X(x)$ at the point $x \in M$; that is, $\tau_M \circ X =$ identity. An ***integral curve*** of X with initial condition x_0 at $t = 0$ is a (differentiable) map $c : \,]a, b[\, \to M$ such that $]a, b[$ is an open interval containing 0, $c(0) = x_0$, and

$$c'(t) = X(c(t))$$

for all $t \in \,]a, b[$. In formal presentations we usually suppress the domain of definition, even though this is technically important. The ***flow*** of X is the collection of maps

$$\varphi_t : M \to M$$

such that $t \mapsto \varphi_t(x)$ is the integral curve of X with initial condition x. Existence and uniqueness theorems from ordinary differential equations, as

[7] The tangent map is sometimes denoted by f_*.

reviewed in the last section, guarantee that φ is smooth in x and t (where defined) if X is. From uniqueness, we get the ***flow property***

$$\varphi_{t+s} = \varphi_t \circ \varphi_s$$

along with the initial condition $\varphi_0 =$ identity. The flow property generalizes the situation where $M = V$ is a *linear* space, $X(x) = Ax$ for a (bounded) *linear* operator A, and

$$\varphi_t(x) = e^{tA}x$$

to the *nonlinear* case.

Differentials. If $f : M \to \mathbb{R}$ is a smooth function, we can differentiate it at any point $x \in M$ to obtain a map $T_x f : T_x M \to T_{f(x)}\mathbb{R}$. Identifying the tangent space of $\mathbb{R}$ at any point with itself (a process we usually do in any vector space), we get a linear map $\mathbf{d}f(x) : T_x M \to \mathbb{R}$. That is, $\mathbf{d}f(x) \in T_x^* M$, the dual of the vector space $T_x M$.

In coordinates, the ***directional derivative*** $\mathbf{d}f(x) \cdot v$, where $v \in T_x M$, is given by

$$\mathbf{d}f(x) \cdot v = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} v^i.$$

We will employ the ***summation convention*** and drop the summation sign when there are repeated indices. We also call $\mathbf{d}f$ the ***differential*** of f.

One can show that specifying the directional derivatives completely determines a vector, and so we can identify a basis of $T_x M$ using the operators $\partial/\partial x^i$. We write

$$(e_1, \ldots, e_n) = \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right)$$

for this basis, so that $v = v^i \partial/\partial x^i$.

If we replace each vector space $T_x M$ with its dual $T_x^* M$, we obtain a new $2n$-manifold called the ***cotangent bundle*** and denoted by T^*M. The dual basis to $\partial/\partial x^i$ is denoted by dx^i. Thus, relative to a choice of local coordinates we get the basic formula

$$\mathbf{d}f(x) = \frac{\partial f}{\partial x^i} dx^i$$

for any smooth function $f : M \to \mathbb{R}$.

Degree of a Map. As we shall see in Chapter 4, an important notion for understanding stabilization is the notion of the ***degree*** of a map (see, e.g., Milnor [1965]). Let M and N be oriented n-dimensional manifolds without boundary, M compact and N connected. Let $f : M \to N$ be a smooth map. Let $x \in M$ be a regular point of the map and let $T_x f : T_x M \to T_{f(x)} N$ denote the corresponding tangent map, which is thus a linear isomorphism.

Define the ***sign*** of $T_x f$ to be $+1$ or -1 according to whether or not it reverses orientation. Then for any regular value $y \in N$ we define

$$\deg(f; y) = \sum_{x \in f^{-1}(y)} \operatorname{sign} T_x f \tag{2.2.2}$$

if $f^{-1}(y) \neq \varnothing$, 0 if $f^{-1}(y) = \varnothing$.

Now consider a smooth vector field X defined on an open set U of $\mathbb{R}^n$ with an isolated zero at $x \in U$. Consider the function

$$\frac{X(x)}{\|X(x)\|}, \tag{2.2.3}$$

which maps a small sphere centered at x into the unit sphere regarded as the oriented boundary of the corresponding ball. Note that this is just the unit direction vector of the vector field. The degree of this mapping is called the ***index*** of the vector field.

It is not hard to see, for example, that if x is a nondegenerate zero of X, then the index of the vector field X at x is either $+1$ or -1: If X is orientation-preserving, we can locally smoothly deform X to the identity without introducing any new zeros, and if it is orientation-reversing, to a reflection. (Details of this smooth isotopy may be found in Milnor [1965].)

In the plane the index of a zero of a vector field simply measures how many times the vector field rotates in the anticlockwise direction as one traverses a small loop around the zero in the anticlockwise direction. One can easily check that a source, sink, or center has index $+1$, while the index of a saddle is -1. Similarly, the index of a zero of the linear differential equation on $\mathbb{R}^n$, $\dot{x} = Ax$, A nonsingular, is Index $= \operatorname{sign}(\det A)$. For example, for the stable system $\dot{x} = -x$ on $\mathbb{R}^n$, the index of zero is $(-1)^n$.

Exercises

◇ **2.2-1.** Using the submersion criterion, show that the level set $x_1^2 + \cdots + x_n^2 - 1 = 0$ is a differentiable manifold of dimension $n - 1$.

◇ **2.2-2.** Show that the set $\{(x, y) \mid x^2(x+1) - y^2 = 0\}$ in $\mathbb{R}^2$ is *not* a differentiable manifold.

◇ **2.2-3.** Let $\mathcal{S} = \{X \in \mathrm{GL}(n, \mathbb{R}) \mid X^T A X = A\}$, as in the text. Note that the $n \times n$ identity matrix I is in $\mathcal{S}$, and show that for any pair of matrices $V_1, V_2 \in T_I\mathcal{S}$ we have $V_1 V_2 - V_2 V_1 \in T_I\mathcal{S}$.

◇ **2.2-4.** If $\varphi_t : S^2 \to S^2$ rotates points on S^2 about a fixed axis through an angle t, show that φ_t is the flow of a certain vector field on S^2.

◇ **2.2-5.** Let $f : S^2 \to \mathbb{R}$ be defined by $f(x, y, z) = z$. Compute $\mathbf{d}f$ relative to spherical coordinates (θ, φ).

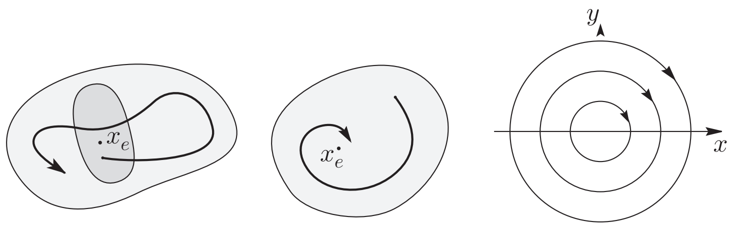
◇ **2.2-6.** One can show that the sum of the indices of the singular points of a vector field on a compact manifold without boundary is independent of the vector field and depends only on the manifold. This sum is called the ***Euler characteristic***. Use this fact to show that the Euler characteristic of the n-torus is zero and that of the n-sphere is $1+(-1)^n$. (Construct a vector field with no zeros on the torus and a vector field with the given index on the sphere.)

2.3 Stability

In this section we summarize some of the key notions of stability.

2.3.1 Definition. *Let x_0 be an equilibrium of the system of differential equations $\dot{x} = f(x)$. The point x_0 is said to be* ***nonlinearly*** *or* ***Lyapunov stable*** *if for any neighborhood U of x_0 there exists a neighborhood $V \subset U$ of x_0 such that any trajectory $x(t)$ of the system with initial point in V remains in U for all time. If in addition $x(t) \to x_0$ as $t \to \infty$, x_0 is said to be* ***asymptotically stable****.*

The basic notions of Lyapunov stability and asymptotic stability are illustrated in Figure 2.3.1. For the harmonic oscillator, the origin is Lyapunov stable, but not asymptotically stable.



(a) Stable (b) Asymptotically stable (c) Harmonic oscillator

FIGURE 2.3.1. In Lyapunov stability points near the equilibrium point stay near, while for asymptotic stability, they also converge to the equilibrium point as $t \to +\infty$.

The notion of stability can be attached to invariant sets other than equilibrium points via a similar definition. In particular, the notion of a stable periodic orbit is illustrated in Figure 2.3.2.

Spectral Stability. There are some specific criteria for stability. The most basic one is the classical spectral test of Lyapunov.

2.3.2 Definition. *Let x_0 be an equilibrium of the system of differential equations $\dot{x} = f(x)$. The point x_0 is said to be* ***spectrally stable*** *if all the*

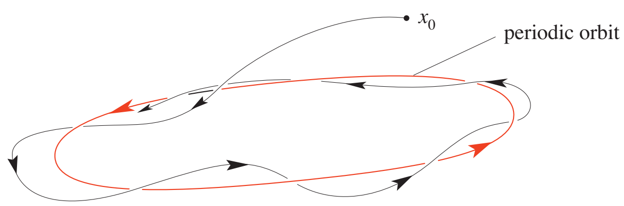


FIGURE 2.3.2. A periodic orbit is asymptotically stable when nearby orbits wind towards it.

eigenvalues of the linearization of f *at* x_0*, i.e., of the matrix*

$$A_{ij} = \frac{\partial f^i}{\partial x^j}(x_0),$$

lie in the left half-plane.

A basic result on stability is the following:

2.3.3 Theorem (Lyapunov)**.** *Spectral stability implies asymptotic stability.*

Invariant Manifolds. At a general equilibrium x_0, if one computes that spectrum of the linearization A and finds a number of eigenvalues in the left half-plane, then there is an invariant manifold (i.e., a manifold that is invariant under the flow and that is simply the graph of a mapping in this case) that is tangent to the corresponding (generalized) eigenspace; it is called the local ***stable manifold***. All trajectories on this stable manifold are asymptotic to the point x_0 as $t \to \infty$.

Similarly, associated with the eigenvalues in the right half-plane is an ***unstable manifold***. The basic notion of invariant manifolds is illustrated schematically in Figure 2.3.3.

If none of the eigenvalues associated with an equilibrium are on the imaginary axis, then the equilibrium is called ***hyperbolic***. In this case, the tangent spaces to the stable and unstable manifolds span the whole of $\mathbb{R}^n$. This is the situation shown in Figure 2.3.3. When there are eigenvalues on the imaginary axis, one introduces the notion of the ***center manifold*** as well; this is discussed in the next section.

One can also have invariant manifolds attached to other invariant sets, and in particular to periodic orbits. This is illustrated in Figure 2.3.4.

Much more on how to analyze and achieve stability through control will be discussed in later parts of this book.

The LaSalle Invariance Principle. A key ingredient in proving asymptotic stability of controlled or uncontrolled systems is the LaSalle invariance principle.

This main theorem may be stated as follows:

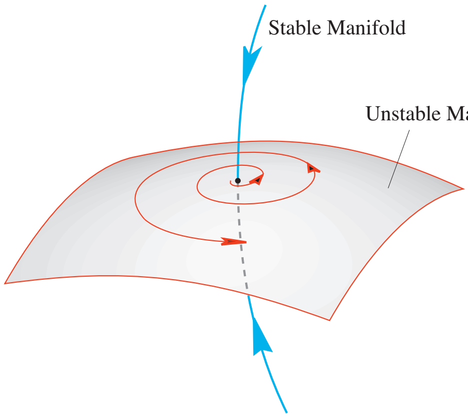


FIGURE 2.3.3. Invariant manifolds for an equilibrium point.

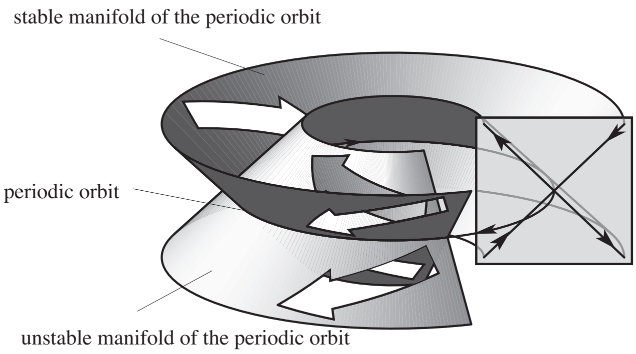


FIGURE 2.3.4. The stable and unstable manifolds of a periodic orbit.

2.3.4 Theorem. *Consider the smooth dynamical system on $\mathbb{R}^n$ given by $\dot{x} = f(x)$ and let Ω be a compact set in $\mathbb{R}^n$ that is (positively) invariant under the flow of f. Let $V : \Omega \to \mathbb{R}$, $V \geq 0$, be a C^1 function such that*

$$\dot{V}(x) = \frac{\partial V}{\partial x} \cdot f \leq 0$$

in Ω. Let M be the largest invariant set in Ω where $\dot{V}(x) = 0$. Then every solution with initial point in Ω tends asymptotically to M as $t \to \infty$. In particular, if M is an isolated equilibrium, it is asymptotically stable.

The invariance principle is due to Barbashin and Krasovskii [1952], Lasalle and Lefschetz [1961], and Krasovskii [1963]. The book of Khalil [1992] has a nice treatment.

Note that in the statement of the theorem, $V(x)$ need not be positive definite, but rather only semidefinite, and that if in particular M is an equilibrium, the theorem proves that the equilibrium is asymptotically stable. The set Ω in the LaSalle theorem also gives us an estimate of the region of attraction of an equilibrium. This is one of the reasons that this is a more attractive methodology than that of spectral stability tests, which could in principle give a very small region of attraction.

Exercises

◇ **2.3-1.** Derive the critical stability value $\omega_0 = \sqrt{g/R}$ for the particle in the rotating hoop.

◇ **2.3-2.** Consider the following vector field in $\mathbb{R}^3$:

$$\begin{aligned} \dot{x} &= -x + y + f\,, \\ \dot{y} &= -y + g\,, \\ \dot{z} &= z\,, \end{aligned}$$

where $f(x,y,z) = -\left(x + \frac{1}{2}y\right)^3$ and $g(x,y,z) = -\left(y + \frac{1}{2}x\right)^3$.

(a) Compute the linearized system at the origin and write it in the form $\dot{\mathbf{x}} = A\mathbf{x}$ for a suitable 3×3 matrix A and where $\mathbf{x}$ is the vector with components (x, y, z).

(b) Sketch the phase portrait of this linear system.

(c) To what extent is the phase portrait of the nonlinear system similar to that of the linear system in a neighborhood of the origin?

(d) Consider the function

$$V(x,y,z) = \frac{1}{2}\left[x^2 + y^2 + xy\right] .$$

Compute its time derivative along the flow of the given vector field.

(e) Show that the plane $z = 0$ is invariant.

(f) Is the origin globally attracting *within* the plane $z = 0$?

(g) Describe the invariant manifolds of the origin for this system.

(h) Can this vector field have any periodic orbits?

2.4 Center Manifolds

Here we discuss some results in center manifold theory and show how they relate to the Lyapunov–Malkin theorem, which plays an important role in the stability analysis of nonholonomic systems. The center manifold theorem provides useful insight into the existence of invariant manifolds. These invariant manifolds will play a crucial role in our analysis. Lyapunov's original proof of the Lyapunov–Malkin theorem used a different approach to proving the existence of local integrals, as we shall discuss below. Malkin extended the result to the nonautonomous case.

Center Manifold Theory in Stability Analysis. We consider firstly center manifold theory and its applications to the stability analysis of nonhyperbolic equilibria.

Consider a system of differential equations

$$\dot{x} = Ax + X(x, y), \tag{2.4.1}$$

$$\dot{y} = By + Y(x, y), \tag{2.4.2}$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, and A and B are constant matrices. It is supposed that all eigenvalues of A have nonzero real parts, and all eigenvalues of B have zero real parts. The functions X, Y are smooth, and satisfy the conditions $X(0,0) = 0$, $dX(0,0) = 0$, $Y(0,0) = 0$, $dY(0,0) = 0$. We now recall the following definition:

2.4.1 Definition. *A smooth invariant manifold of the form $x = h(y)$ where h satisfies $h(0) = 0$ and $dh(0) = 0$ is called a **center manifold**.*

We are going to use the following version of the center manifold theorem following the exposition of Carr [1981] (see also Chow and Hale [1982]).

2.4.2 Theorem (The center manifold theorem)**.** *Suppose that the functions $X(x, y)$, $Y(x, y)$ are C^k, $k \geq 2$. Then there exist a (local) center manifold for* (2.4.1), (2.4.2)*, $x = h(y)$, $|y| < \delta$, where h is C^k. The flow on the center manifold is governed by the system*

$$\dot{y} = By + Y(h(y), y). \tag{2.4.3}$$

The basic idea of realizing the center manifold as a graph over the linear center subspace is shown in Figure 2.4.1.

The next theorem explains that the reduced equation (2.4.3) contains information about stability of the zero solution of (2.4.1), (2.4.2).

2.4.3 Theorem. *Suppose that the zero solution of* (2.4.3) *is stable* (*resp. asymptotically stable*) *and that the eigenvalues of A are in the left half-plane. Then the zero solution of* (2.4.1), (2.4.2) *is stable* (*resp. asymptotically stable*)*. If either the zero solution of* (2.4.3) *is unstable, or if any eigenvalues of A are in the right half plane, then the zero solution of* (2.4.1), (2.4.2) *is also unstable.*

Let us now look at the special case of (2.4.2) in which the matrix B vanishes. Equations (2.4.1), (2.4.2) become

$$\dot{x} = Ax + X(x, y), \tag{2.4.4}$$

$$\dot{y} = Y(x, y). \tag{2.4.5}$$

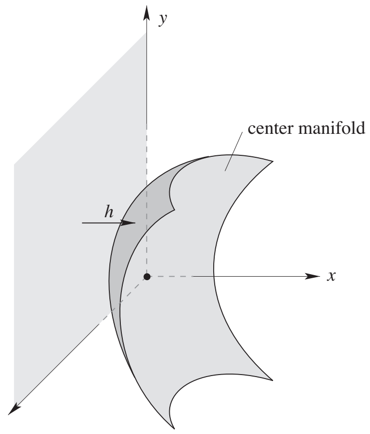


FIGURE 2.4.1. The center manifold realized as the graph of the function h.

2.4.4 Theorem. *Consider the system of equations* (2.4.4), (2.4.5). *If* $X(0, y) = 0$, $Y(0, y) = 0$, *and all of the eigenvalues of the matrix* A *have negative real parts, then the system* (2.4.4), (2.4.5) *has* n *local integrals in a neighborhood of* $x = 0$, $y = 0$.

Proof. The center manifold in this case is given by $x = 0$. Each point of the center manifold is an equilibrium of the system (2.4.4), (2.4.5). For each equilibrium point $(0, y_0)$ of our system, consider the associated m-dimensional stable manifold $S^{\mathrm{s}}(y_0)$. The center manifold and these manifolds $S^{\mathrm{s}}(y_0)$ can be used for a (local) substitution $(x, y) \mapsto (\bar{x}, \bar{y})$ such that in the new coordinates the system of differential equations becomes

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{X}(\bar{x}, \bar{y}), \quad \dot{\bar{y}} = 0.$$

The last system of equations has n integrals $\bar{y} = \text{const}$, so that the original equation has n smooth local integrals. Observe that the tangent spaces to the common level sets of these integrals at the equilibria are the planes $y = y_0$. Therefore, the integrals are of the form

$$y = f(x, k), \qquad \text{where } \partial_x f(0, k) = 0. \qquad \blacksquare$$

The Lyapunov–Malkin Theorem. The following theorem gives stability conditions for equilibria of the system (2.4.4), (2.4.5).

2.4.5 Theorem (Lyapunov–Malkin)**.** *Consider the system of differential equations* (2.4.4), (2.4.5), *where* $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, A *is an* $m \times m$ *matrix, and* $X(x, y)$, $Y(x, y)$ *represent nonlinear terms. If all eigenvalues of the matrix* A *have negative real parts, and* $X(x, y)$, $Y(x, y)$ *vanish when* $x = 0$, *then the solution* $x = 0$, $y = c$ *of the system* (2.4.4), (2.4.5) *is stable with respect to* x, y, *and asymptotically stable with respect to* x. *If a solution*

$x(t)$, $y(t)$ of (2.4.4), (2.4.5) *is close enough to the solution $x = 0$, $y = 0$, then*

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} y(t) = c.$$

Proof. (See Lyapunov [1907], Malkin [1938], and Zenkov, Bloch, and Marsden [1998].) From Theorem 2.4.4, the phase space of system (2.4.4), (2.4.5) is locally represented as a union of invariant leaves

$$Q_c = \{(x, y) \mid y = f(x, c)\}.$$

Use x as local coordinates on these leaves. On each leaf we have a reduced system that can be written as $\dot{x} = F(x)$, where $F(x) = Ax + X(x, f(x, c))$. Since $\det A \neq 0$, each reduced system has an isolated equilibrium $x = 0$ on a corresponding leaf. The equilibrium of the system reduced to a leaf passing through $x = 0, y = 0$ is asymptotically stable, since the matrix of the linearization of the reduced system is A. To finish the proof, we notice that the equilibria of systems on nearby leaves are asymptotically stable as well because the corresponding matrices A_c, by continuity, also have all eigenvalues in the left half-plane, since locally, A_c will be close to A. ■

Historical Note. The proof of the Lyapunov–Malkin theorem uses the fact that the system of differential equations has local integrals, as discussed above. To prove existence of these integrals, Lyapunov used a theorem of his own about the existence of solutions of PDEs. He did this assuming that the nonlinear terms on the right-hand sides are series in x and y with time-dependent bounded coefficients. Malkin generalized Lyapunov's result for systems for which the matrix A is time-dependent. We consider the nonanalytic case, and to prove existence of these local integrals, we use center manifold theory. This simplifies the arguments to some extent as well as showing how the results are related.

A Class Satisfying the Lyapunov–Malkin Theorem. The following lemma specifies a class of systems of differential equations that satisfy the conditions of the Lyapunov–Malkin theorem.

2.4.6 Lemma. *Consider a system of differential equations of the form*

$$\dot{w} = Aw + By + \mathcal{U}(w, y), \quad \dot{y} = \mathcal{Y}(w, y), \tag{2.4.6}$$

where $w \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $\det A \neq 0$, and where $\mathcal{U}$ and $\mathcal{Y}$ represent higher-order nonlinear terms. There is a change of variables of the form $w = x + \phi(y)$ such that:

(i) *In the new variables x, y, the system* (2.4.6) *has the form*

$$\dot{x} = Ax + X(x, y), \quad \dot{y} = Y(x, y).$$

(ii) *If $Y(0, y) = 0$, then $X(0, y) = 0$ as well.*

Proof. Put $w = x + \phi(y)$, where $\phi(y)$ is defined by

$$A\phi(y) + By + \mathcal{U}(\phi(y), y) = 0.$$

Then the system (2.4.6) written in terms of the variables x, y becomes

$$\dot{x} = Ax + X(x,y), \quad \dot{y} = Y(x,y),$$

where

$$\begin{aligned} X(x,y) &= A\phi(y) + By + \mathcal{U}(x + \phi(y), y) - \frac{\partial \phi}{\partial y} Y(x,y), \\ Y(x,y) &= \mathcal{Y}(x + \phi(y), y). \end{aligned}$$

Note that $Y(0,y) = 0$ implies $X(0,y) = 0$. ■

Exercise

◇ **2.4-1.** Sketch the phase portrait of the system

$$\begin{aligned} \dot{x} &= -x + xy\,, \\ \dot{y} &= xy\,. \end{aligned}$$

Verify the conclusions of the Lyapunov–Malkin theorem. See Figure 2.4.2.

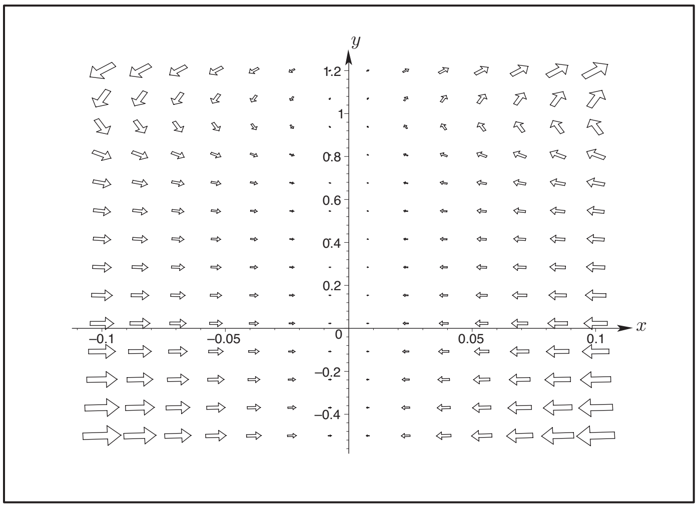


FIGURE 2.4.2. Phase portrait of the system $\dot{x} = -x + xy; \dot{y} = xy$.

2.5 Differential Forms

We next review some of the basic definitions, properties, and operations on differential forms, without proofs (see Abraham, Marsden, and Ratiu [1988] and references therein). *The main idea of differential forms is to provide a generalization of the basic operations of vector calculus,* div, grad*, and* curl,*of Green, Gauss, and Stokes to manifolds of arbitrary dimension.*

Basic Definitions. Let V be a vector space. A map $\alpha : V \times \cdots \times V$ (where there are k factors) $\to \mathbb{R}$ is ***multilinear*** when it is linear in each of its factors, that is,

$$\alpha(v_1, \ldots, av_j + bv_j', \ldots, v_k) \\ = a\alpha(v_1, \ldots, v_j, \ldots, v_k) + b\alpha(v_1, \ldots, v_j', \ldots, v_k)$$

for all j with $1 \leq j \leq k$. A k-multilinear map $\alpha : V \times \cdots \times V \to \mathbb{R}$ is ***skew*** (or ***alternating***) when it changes sign whenever two of its arguments are interchanged, that is, for all $v_1, \ldots, v_k \in V$,

$$\alpha(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\alpha(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k).$$

A ***2-form*** Ω on a manifold M is, for each point $x \in M$, a smooth skew-symmetric bilinear mapping $\Omega(x) : T_xM \times T_xM \to \mathbb{R}$. More generally, a ***k-form*** α (sometimes called a ***differential form of degree*** k) on a manifold M is a function $\alpha(x) : T_xM \times \cdots \times T_xM$ (there are k factors)$\to \mathbb{R}$ that assigns to each point $x \in M$ a smooth skew-symmetric k-multilinear map on the tangent space T_xM to M at x.

Without the skew-symmetry assumption, α would be referred to as a $(0, k)$-***tensor***.

Let $x^1, \ldots, x^n$ denote coordinates on M, let

$$\{e_1, \ldots, e_n\} = \left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right\}$$

be the corresponding basis for T_xM, and let $\{e^1, \ldots, e^n\} = \{dx^1, \ldots, dx^n\}$ be the dual basis for T_x^*M. Then at each $x \in M$, we can write a 2-form as

$$\Omega_x(v, w) = \Omega_{ij}(x) v^i w^j, \quad \text{where} \quad \Omega_{ij}(x) = \Omega_x\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right).$$

Here the ***summation convention*** is used; that is,

$$\Omega_x(v, w) = \Omega_{ij}(x) v^i w^j = \sum_{i,j=1}^{n} \Omega_{ij}(x) v^i w^j .$$

More generally, a k-form can be written

$$\alpha_x(v_1, \ldots, v_k) = \alpha_{i_1 \ldots i_k}(x) v_1^{i_1} \ldots v_k^{i_k},$$

where there is a sum on $i_1, \ldots, i_k$, where

$$\alpha_{i_1 \ldots i_k}(x) = \alpha_x \left(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}} \right),$$

and where $v_i = v_i^j \partial / \partial x^j$, with a sum on j.

Tensor and Wedge Products. If α is a $(0, k)$-tensor on a manifold M, and β is a $(0, l)$-tensor, their ***tensor product*** $\alpha \otimes \beta$ is the $(0, k+l)$-tensor on M defined by

$$(\alpha \otimes \beta)_x(v_1, \ldots, v_{k+l}) = \alpha_x(v_1, \ldots, v_k)\beta_x(v_{k+1}, \ldots, v_{k+l}) \tag{2.5.1}$$

at each point $x \in M$.

If t is a $(0, p)$-tensor, define the ***alternation operator*** $\mathbf{A}$ acting on t by

$$\mathbf{A}(t)(v_1, \ldots, v_p) = \frac{1}{p!} \sum_{\pi \in S_p} \operatorname{sgn}(\pi) t(v_{\pi(1)}, \ldots, v_{\pi(p)}), \tag{2.5.2}$$

where $\operatorname{sgn}(\pi)$ is the ***sign*** of the permutation π,

$$\operatorname{sgn}(\pi) = \begin{cases} +1 & \text{if } \pi \text{ is even}, \\ -1 & \text{if } \pi \text{ is odd}, \end{cases} \tag{2.5.3}$$

and S_p is the group of all permutations of the numbers $1, 2, \ldots, p$. A permutation is called ***odd*** if it can be written as the product of an odd number of transpositions (that is, a permutation that interchanges just two objects) and otherwise is ***even***. The operator $\mathbf{A}$ therefore skew-symmetrizes p-multilinear maps.

If α is a k-form and β is an l-form on M, their ***wedge product*** $\alpha \wedge \beta$ is the $(k + l)$-form on M defined by[8]

$$\alpha \wedge \beta = \frac{(k+l)!}{k!\, l!} \mathbf{A}(\alpha \otimes \beta). \tag{2.5.4}$$

For example, if α and β are one-forms, then

$$(\alpha \wedge \beta)(v_1, v_2) = \alpha(v_1)\beta(v_2) - \alpha(v_2)\beta(v_1),$$

while if α is a 2-form and β is a 1-form,

$$(\alpha \wedge \beta)(v_1, v_2, v_3) = \alpha(v_1, v_2)\beta(v_3) + \alpha(v_3, v_1)\beta(v_2) + \alpha(v_2, v_3)\beta(v_1).$$

We state the following without proof:

[8]The numerical factor in (2.5.4) agrees with the convention of Abraham and Marsden [1978], Abraham, Marsden, and Ratiu [1988], and Spivak [1979], but *not* that of Arnold [1989], Guillemin and Pollack [1974], or Kobayashi and Nomizu [1963]; it is the Bourbaki [1971] convention.

2.5.1 Proposition. *The wedge product has the following properties:*

(i) $\alpha \wedge \beta$ *is* ***associative***: $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$.

(ii) $\alpha \wedge \beta$ *is* ***bilinear*** *in* α, β :

$$\begin{aligned}(a\alpha_1 + b\alpha_2) \wedge \beta &= a(\alpha_1 \wedge \beta) + b(\alpha_2 \wedge \beta),\\ \alpha \wedge (c\beta_1 + d\beta_2) &= c(\alpha \wedge \beta_1) + d(\alpha \wedge \beta_2).\end{aligned}$$

(iii) $\alpha \wedge \beta$ *is* ***anticommutative***: $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$, *where* α *is a* k*-form and* β *is an* l*-form.*

In terms of the dual basis dx^i, any k-form can be written locally as

$$\alpha = \alpha_{i_1 \dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} ,$$

where the sum is over all i_j satisfying $i_1 < \cdots < i_k$.

Pull Back and Push Forward. Let $\varphi : M \to N$ be a C^∞ map from the manifold M to the manifold N and let α be a k-form on N. Define the ***pull back*** $\varphi^*\alpha$ of α by φ to be the k-form on M given by

$$(\varphi^*\alpha)_x(v_1, \dots, v_k) = \alpha_{\varphi(x)}(T_x\varphi \cdot v_1, \dots, T_x\varphi \cdot v_k). \tag{2.5.5}$$

If φ is a diffeomorphism, the ***push forward*** φ_* is defined by $\varphi_* = (\varphi^{-1})^*$.

Here is another basic property.

2.5.2 Proposition. *The pull back of a wedge product is the wedge product of the pull backs:*

$$\varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta. \tag{2.5.6}$$

Interior Products and Exterior Derivatives. Let α be a k-form on a manifold M, and X a vector field. The ***interior product*** $\mathbf{i}_X\alpha$ (sometimes called the contraction of X and α, and written $\mathbf{i}(X)\alpha$) is defined by

$$(\mathbf{i}_X\alpha)_x(v_2, \dots, v_k) = \alpha_x(X(x), v_2, \dots, v_k). \tag{2.5.7}$$

2.5.3 Proposition. *Let* α *be a* k*-form and* β *an* l*-form on a manifold* M*. Then*

$$\mathbf{i}_X(\alpha \wedge \beta) = (\mathbf{i}_X\alpha) \wedge \beta + (-1)^k \alpha \wedge (\mathbf{i}_X\beta). \tag{2.5.8}$$

The ***exterior derivative*** $\mathbf{d}\alpha$ of a k-form α on a manifold M is the $(k+1)$-form on M determined by the following proposition:

2.5.4 Proposition. *There is a unique mapping* $\mathbf{d}$ *from* k*-forms on* M *to* $(k+1)$*-forms on* M *such that:*

(i) *If* α *is a* 0*-form* ($k = 0$)*, that is,* $\alpha = f \in C^\infty(M)$*, then* $\mathbf{d}f$ *is the one-form that is the differential of* f*.*

(ii) $\mathbf{d}\alpha$ *is* ***linear*** *in* α*; that is, for all real numbers* c_1 *and* c_2*,*

$$\mathbf{d}(c_1\alpha_1 + c_2\alpha_2) = c_1\mathbf{d}\alpha_1 + c_2\mathbf{d}\alpha_2.$$

(iii) $\mathbf{d}\alpha$ *satisfies the* ***product rule****; that is,*

$$\mathbf{d}(\alpha \wedge \beta) = \mathbf{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathbf{d}\beta,$$

where α *is a* k*-form and* β *is an* l*-form.*

(iv) $\mathbf{d}^2 = 0$*; that is,* $\mathbf{d}(\mathbf{d}\alpha) = 0$ *for any* k*-form* α*.*

(v) $\mathbf{d}$ *is a* ***local operator****; that is,* $\mathbf{d}\alpha(x)$ *depends only on* α *restricted to any open neighborhood of* x*; in fact, if* U *is open in* M*, then*

$$\mathbf{d}(\alpha|U) = (\mathbf{d}\alpha)|U.$$

If α is a k-form given in coordinates by

$$\alpha = \alpha_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \quad (\text{sum on } i_1 < \cdots < i_k),$$

then the coordinate expression for the exterior derivative is

$$\mathbf{d}\alpha = \frac{\partial \alpha_{i_1 \ldots i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \quad (\text{sum on all } j \text{ and } i_1 < \cdots < i_k). \tag{2.5.9}$$

Formula (2.5.9) can be taken as the definition of the exterior derivative, provided one shows that (2.5.9) has the above-described properties and, correspondingly, is independent of the choice of coordinates.

Next is a useful proposition that, in essence, rests on the chain rule:

2.5.5 Proposition. *Exterior differentiation commutes with pull back:*

$$\mathbf{d}(\varphi^*\alpha) = \varphi^*(\mathbf{d}\alpha), \tag{2.5.10}$$

where α *is a* k*-form on a manifold* N *and* φ *is a smooth map from a manifold* M *to* N*.*

A k-form α is called ***closed*** if $\mathbf{d}\alpha = 0$ and ***exact*** if there is a $(k-1)$-form β such that $\alpha = \mathbf{d}\beta$. By Proposition 2.5.4 every exact form is closed. The exercises give an example of a closed nonexact one-form.

2.5.6 Proposition (Poincaré Lemma)**.** *A closed form is locally exact; that is, if* $\mathbf{d}\alpha = 0$*, there is a neighborhood about each point on which* $\alpha = \mathbf{d}\beta$*.*

The proof is given in the exercises.

Vector Calculus. The table below entitled "Vector Calculus and Differential Forms" summarizes how forms are related to the usual operations of vector calculus. We now elaborate on a few items in this table. In item 4, note that

$$\mathbf{d}f = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = (\text{grad} f)^\flat = (\nabla f)^\flat\,,$$

which is equivalent to $\nabla f = (\mathbf{d}f)^\sharp$. (The $\flat$ and $\sharp$ terminology are explained in the table below.)

The Hodge star operator on $\mathbb{R}^3$ maps k-forms to $(3-k)$-forms and is uniquely determined by linearity and the properties in item 2.[9]

In item 5, if we let $F = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$, so

$$F^\flat = F_1\,dx + F_2\,dy + F_3\,dz,$$

we obtain

$$\begin{aligned}\mathbf{d}(F^\flat) = {} & \mathbf{d}F_1 \wedge dx + F_1\mathbf{d}(dx) + \mathbf{d}F_2 \wedge dy + F_2\mathbf{d}(dy) + \mathbf{d}F_3 \wedge dz \\ & + F_3\mathbf{d}(dz)\,,\end{aligned}$$

which equals

$$\begin{aligned}&\left(\frac{\partial F_1}{\partial x}dx + \frac{\partial F_1}{\partial y}dy + \frac{\partial F_1}{\partial z}dz\right) \wedge dx \\ &\quad + \left(\frac{\partial F_2}{\partial x}dx + \frac{\partial F_2}{\partial y}dy + \frac{\partial F_2}{\partial z}dz\right) \wedge dy \\ &\quad + \left(\frac{\partial F_3}{\partial x}dx + \frac{\partial F_3}{\partial y}dy + \frac{\partial F_3}{\partial z}dz\right) \wedge dz\,.\end{aligned}$$

This becomes

$$\begin{aligned}&-\frac{\partial F_1}{\partial y}dx \wedge dy + \frac{\partial F_1}{\partial z}dz \wedge dx + \frac{\partial F_2}{\partial x}dx \wedge dy - \frac{\partial F_2}{\partial z}dy \wedge dz \\ &\quad - \frac{\partial F_3}{\partial x}dz \wedge dx + \frac{\partial F_3}{\partial y}dy \wedge dz \\ &\qquad = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx \wedge dy + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) dz \wedge dx \\ &\qquad\quad + \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dy \wedge dz\,.\end{aligned}$$

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[9] This operator can be defined on general Riemannian manifolds; see Abraham, Marsden, and Ratiu [1988].

Hence, using item 2,

$$\begin{aligned}
*(\mathbf{d}(F^\flat)) &= \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) dy \\
&\quad + \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dx, \\
(*(\mathbf{d}(F^\flat)))^\sharp &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) \mathbf{e}_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) \mathbf{e}_2 \\
&\quad + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \mathbf{e}_3 \\
&= \operatorname{curl} F = \nabla \times F.
\end{aligned}$$

With reference to item 6, let

$$F = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3,$$

so that

$$F^\flat = F_1\, dx + F_2\, dy + F_3\, dz.$$

Thus,

$$*(F^\flat) = F_1\, dy \wedge dz + F_2(-dx \wedge dz) + F_3\, dx \wedge dy,$$

and so

$$\begin{aligned}
\mathbf{d}(*(F^\flat)) &= \mathbf{d}F_1 \wedge dy \wedge dz - \mathbf{d}F_2 \wedge dx \wedge dz + \mathbf{d}F_3 \wedge dx \wedge dy \\
&= \left(\frac{\partial F_1}{\partial x}dx + \frac{\partial F_1}{\partial y}dy + \frac{\partial F_1}{\partial z}dz\right) \wedge dy \wedge dz \\
&\quad - \left(\frac{\partial F_2}{\partial x}dx + \frac{\partial F_2}{\partial y}dy + \frac{\partial F_2}{\partial z}dz\right) \wedge dx \wedge dz \\
&\quad + \left(\frac{\partial F_3}{\partial x}dx + \frac{\partial F_3}{\partial y}dy + \frac{\partial F_3}{\partial z}dz\right) \wedge dx \wedge dy \\
&= \frac{\partial F_1}{\partial x}dx \wedge dy \wedge dz + \frac{\partial F_2}{\partial y}dx \wedge dy \wedge dz + \frac{\partial F_3}{\partial z}dx \wedge dy \wedge dz \\
&= \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx \wedge dy \wedge dz \\
&= (\operatorname{div}\ F)\, dx \wedge dy \wedge dz.
\end{aligned}$$

Therefore, $*(\mathbf{d}(*(F^\flat))) = \operatorname{div}\ F = \nabla \cdot F$.

The definition and properties of vector-valued forms are direct extensions of these for usual forms on vector spaces and manifolds. One can think of a vector-valued form as an array of usual forms.

The following table should serve as a useful reference for future computations.

Vector Calculus and Differential Forms

1. **Sharp and Flat** (Using standard coordinates in $\mathbb{R}^3$)
   (a) $v^\flat = v^1\,dx + v^2\,dy + v^3\,dz =$ one-form corresponding to the vector $v = v^1\mathbf{e}_1 + v^2\mathbf{e}_2 + v^3\mathbf{e}_3$.
   (b) $\alpha^\sharp = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3 =$ vector corresponding to the one-form $\alpha = \alpha_1\,dx + \alpha_2\,dy + \alpha_3\,dz$.
2. **Hodge Star Operator**
   (a) $*1 = dx \wedge dy \wedge dz$.
   (b) $*dx = dy \wedge dz$, $*dy = -dx \wedge dz$, $*dz = dx \wedge dy$, $*(dy \wedge dz) = dx$, $*(dx \wedge dz) = -dy$, $*(dx \wedge dy) = dz$.
   (c) $*(dx \wedge dy \wedge dz) = 1$.
3. **Cross Product and Dot Product**
   (a) $v \times w = [*(v^\flat \wedge w^\flat)]^\sharp$.
   (b) $(v \cdot w)dx \wedge dy \wedge dz = v^\flat \wedge *(w^\flat)$.
4. **Gradient** $\nabla f = \mathrm{grad} f = (\mathbf{d}f)^\sharp$.
5. **Curl** $\nabla \times F = \mathrm{curl}\, F = [*(\mathbf{d}F^\flat)]^\sharp$.
6. **Divergence** $\nabla \cdot F = \mathrm{div}\ F = *\mathbf{d}(*F^\flat)$.

Exercises

◇ **2.5-1.** Let $\varphi : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $\varphi(x, y, z) = (x + z, xy)$. For

$$\alpha = e^v\,du + u\,dv \in \Omega^1(\mathbb{R}^2) \quad \text{and} \quad \beta = u\,du \wedge dv$$

compute $\alpha \wedge \beta, \varphi^*\alpha$, $\varphi^*\beta$, and $\varphi^*\alpha \wedge \varphi^*\beta$.

◇ **2.5-2.** Given

$$\alpha = y^2\,dx \wedge dz + \sin(xy)\,dx \wedge dy + e^x\,dy \wedge dz \in \Omega^2(\mathbb{R}^3)$$

and

$$X = 3\frac{\partial}{\partial x} + \cos z\frac{\partial}{\partial y} - x^2\frac{\partial}{\partial z} \in \mathfrak{X}(\mathbb{R}^3),$$

compute $\mathbf{d}\alpha$ and $\mathbf{i}_X\alpha$.

◇ **2.5-3.** (a) Denote by $\Lambda^k(\mathbb{R}^n)$ the vector space of all skew-symmetric k-linear maps on $\mathbb{R}^n$. Prove that this space has dimension $n!/k!\,(n-k)!$ by showing that a basis is given by $\{e^{i_1} \wedge \cdots \wedge e^{i_k} \mid i_1 < \cdots < i_k\}$, where $\{e_1, \ldots, e_n\}$ is a basis of $\mathbb{R}^n$ and $\{e^1, \ldots, e^n\}$ is its dual basis; that is, $e^i(e_j) = \delta^i_j$.

(b) If $\mu \in \Lambda^n(\mathbb{R}^n)$ is nonzero, prove that the map $v \in \mathbb{R}^n \mapsto \mathbf{i}_v\mu \in \Lambda^{n-1}(\mathbb{R}^n)$ is an isomorphism.

(c) If M is a smooth n-manifold and $\mu \in \Omega^n(M)$ is nowhere vanishing (in which case it is called a volume form), show that the map $X \in \mathfrak{X}(M) \mapsto \mathbf{i}_X\mu \in \Omega^{n-1}(M)$ is a module isomorphism over $\mathcal{F}(M)$.

◇ **2.5-4.** Let $\alpha = \alpha_i\, dx^i$ be a closed one-form in a ball around the origin in $\mathbb{R}^n$. Show that $\alpha = \mathbf{d}f$ for

$$f(x^1, \ldots, x^n) = \int_0^1 \alpha_j(tx^1, \ldots, tx^n)x^j\, dt.$$

◇ **2.5-5.** (a) Let U be an open ball around the origin in $\mathbb{R}^n$ and $\alpha \in \Omega^k(U)$ a closed form. Verify that $\alpha = \mathbf{d}\beta$, where

$$\begin{aligned}\beta(x^1, \ldots, x^n) \\ = \left(\int_0^1 t^{k-1}\alpha_{ji_1\ldots i_{k-1}}(tx^1, \ldots, tx^n)x^j\, dt\right) dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}},\end{aligned}$$

and where the sum is over $i_1 < \cdots < i_{k-1}$. Here,

$$\alpha = \alpha_{j_1\ldots j_k}\, dx^{j_1} \wedge \cdots \wedge dx^{j_k},$$

where $j_1 < \cdots < j_k$ and where α is extended to be skew-symmetric in its lower indices.

(b) Deduce the Poincaré lemma from **(a).**

2.6 Lie Derivatives

The *dynamic definition* of the Lie derivative is as follows. Let α be a k-form and let X be a vector field with flow φ_t. The ***Lie derivative*** of α along X is given by

$$\pounds_X\alpha = \lim_{t\to 0}\frac{1}{t}[(\varphi_t^*\alpha) - \alpha] = \left.\frac{d}{dt}\varphi_t^*\alpha\right|_{t=0}. \tag{2.6.1}$$

This definition together with properties of pull backs yields the following.

2.6.1 Theorem (Lie Derivative Theorem). *Using the above notation, we have*

$$\frac{d}{dt}\varphi_t^*\alpha = \varphi_t^* \pounds_X \alpha. \tag{2.6.2}$$

This fundamental formula holds also for *time-dependent* vector fields.

If f is a real-valued function on a manifold M and X is a vector field on M, the ***Lie derivative of*** f ***along*** X is the ***directional derivative***

$$\pounds_X f = X[f] := \mathbf{d}f \cdot X. \tag{2.6.3}$$

In coordinates on M,

$$\pounds_X f = X^i \frac{\partial f}{\partial x^i}. \tag{2.6.4}$$

If Y is a vector field on a manifold N and $\varphi : M \to N$ is a diffeomorphism, the ***pull back*** $\varphi^* Y$ is a vector field on M defined by

$$(\varphi^* Y)(x) = T_x\varphi^{-1} \circ Y \circ \varphi(x). \tag{2.6.5}$$

Two vector fields X on M and Y on N are said to be φ***-related*** if

$$T\varphi \circ X = Y \circ \varphi. \tag{2.6.6}$$

Clearly, if $\varphi : M \to N$ is a diffeomorphism and Y is a vector field on N, $\varphi^* Y$ and Y are φ-related. For a diffeomorphism φ, the ***push forward*** is defined, as for forms, by $\varphi_* = (\varphi^{-1})^*$.

Jacobi–Lie Brackets. In Section 1.8 we discussed the Jacobi–Lie bracket for vector fields in $\mathbb{R}^n$ and saw its importance for the analysis of control systems.

We now extend this operation to vector fields on manifolds. If M is a finite-dimensional (smooth) manifold, then the set of vector fields on M coincides with the set of derivations on $\mathcal{F}(M)$.[10] This identification is as follows. Given a vector field $X \in \mathfrak{X}(M)$ define the map $\theta_X : \mathcal{F}(M) \to \mathcal{F}(M)$ by $f \to X[f]$, where $X[f](x) = \mathbf{d}f(x) \cdot X(x)$, which in coordinates is just the directional derivative

$$X[f] = X^i \frac{\partial f}{\partial x^i}$$

with, as usual, a sum understood on the index i. This map θ_X is a derivation in that it is linear and satisfies the Leibniz rule for products. Conversely, any derivation is given in this fashion.

Given two vector fields X and Y on M, one can check that the map $f \mapsto X[Y[f]] - Y[X[f]]$ is a derivation; thus, it determines a unique vector field denoted by $[X, Y]$ and called the ***Jacobi–Lie bracket***. of X and Y.

[10]The same result is true for C^k manifolds and vector fields if $k \geq 2$. This property is false for infinite-dimensional manifolds; see Abraham, Marsden, and Ratiu [1988].

In coordinates,

$$[X, Y]^j = X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} = (X \cdot \nabla) Y^i - (Y \cdot \nabla) X^j, \tag{2.6.7}$$

and in general, where we identify X, Y with their local representatives,

$$[X, Y] = \mathbf{D}Y \cdot X - \mathbf{D}X \cdot Y. \tag{2.6.8}$$

There is an interesting link between the Jacobi–Lie bracket and the Lie derivative as follows. Defining $\pounds_X Y = [X, Y]$ gives the ***Lie derivative*** of Y along X. Then the Lie derivative theorem holds with α replaced by Y.

The Lie bracket of two vector fields has a geometric meaning in terms of successive applications of the flows of the two vector fields in the forward and reverse directions. We discussed this in Section 1.8. We invite the reader to generalize it to the context of manifolds.

If a set of vector fields X_i is such that there exist functions γ_{ijk} such that

$$[X_i, X_j] = \gamma_{ijk} X_k,$$

then the set is said to be ***involutive***. As we shall see later, it is in this case that one generates no new directions by bracketing, and so this is an impediment to showing controllability. This may be a good time to reread the Heisenberg example in Section 1.8.

Algebraic Definition of the Lie Derivative. The *algebraic approach* to the Lie derivative on forms or tensors proceeds as follows. Extend the definition of the Lie derivative from functions and vector fields to differential forms, by requiring that the Lie derivative be a derivation; for example, for one-forms α, write

$$\pounds_X \langle \alpha, Y \rangle = \langle \pounds_X \alpha, Y \rangle + \langle \alpha, \pounds_X Y \rangle, \tag{2.6.9}$$

where X, Y are vector fields and $\langle \alpha, Y \rangle = \alpha(Y)$. More generally,

$$\pounds_X (\alpha(Y_1, \ldots, Y_k)) = (\pounds_X \alpha)(Y_1, \ldots, Y_k) + \sum_{i=1}^{k} \alpha(Y_1, \ldots, \pounds_X Y_i, \ldots, Y_k), \tag{2.6.10}$$

where $X, Y_1, \ldots, Y_k$ are vector fields and α is a k-form.

2.6.2 Proposition. *The dynamic and algebraic definitions of the Lie derivative of a differential k-form are equivalent.*

Cartan's Magic Formula. A very important formula for the Lie derivative is given by the following.

2.6.3 Theorem. *For X a vector field and α a k-form on a manifold M, we have*

$$\pounds_X \alpha = \mathbf{d}\mathbf{i}_X \alpha + \mathbf{i}_X \mathbf{d}\alpha. \tag{2.6.11}$$

This is proved by a lengthy but straightforward calculation.

Another property of the Lie derivative is the following: If $\varphi : M \to N$ is a diffeomorphism, then

$$\varphi^* \pounds_Y \beta = \pounds_{\varphi^* Y} \varphi^* \beta$$

for $Y \in \mathfrak{X}(N)$, $\beta \in \Omega^k(N)$. More generally, if $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are ψ related, that is, $T\psi \circ X = Y \circ \psi$ for $\psi : M \to N$ a smooth map, then

$$\pounds_X \psi^* \beta = \psi^* \pounds_Y \beta \qquad \text{for all } \beta \in \Omega^k(N).$$

Volume Forms and Divergence. An n-manifold M is said to be ***orientable*** if there is a nowhere-vanishing n-form μ on it; μ is called a ***volume form***, and it is a basis of $\Omega^n(M)$ over $\mathcal{F}(M)$. Two volume forms μ_1 and μ_2 on M are said to define the same ***orientation*** if there is an $f \in \mathcal{F}(M)$ with $f > 0$ and such that $\mu_2 = f\mu_1$. Connected orientable manifolds admit precisely two orientations. A basis $\{v_1, \ldots, v_n\}$ of $T_m M$ is said to be ***positively oriented*** relative to the volume form μ on M if $\mu(m)(v_1, \ldots, v_n) > 0$. Note that the volume forms defining the same orientation form a convex cone in $\Omega^n(M)$; that is, if $a > 0$ and μ is a volume form, then $a\mu$ is again a volume form, and if $t \in [0, 1]$ and μ_1, μ_2 are volume forms, then $t\mu_1 + (1 - t)\mu_2$ is again a volume form. The first property is obvious. To prove the second, let $m \in M$ and let $\{v_1, \ldots, v_n\}$ be a positively oriented basis of $T_m M$ relative to the orientation defined by μ_1, or equivalently (by hypothesis) by μ_2. Then $\mu_1(m)(v_1, \ldots, v_n) > 0$, $\mu_2(m)(v_1, \ldots, v_n) > 0$, so that their convex combination is again strictly positive.

If $\mu \in \Omega^n(M)$ is a volume form, since $\pounds_X \mu \in \Omega^n(M)$, there is a function, called the ***divergence*** of X relative to μ and denoted $\operatorname{div}_\mu(X)$ or simply $\operatorname{div}(X)$, such that

$$\pounds_X \mu = \operatorname{div}_\mu(X)\mu. \tag{2.6.12}$$

From the dynamic approach to Lie derivatives it follows that $\operatorname{div}_\mu(X) = 0$ iff $F_t^* \mu = \mu$, where F_t is the flow of X. This condition says that F_t is ***volume-preserving***. If $\varphi : M \to M$, since $\varphi^* \mu \in \Omega^n(M)$, there is a function, called the ***Jacobian*** of φ and denoted by $J_\mu(\varphi)$ or simply $J(\varphi)$, such that

$$\varphi^* \mu = J_\mu(\varphi)\mu. \tag{2.6.13}$$

Thus, *φ is volume-preserving iff $J_\mu(\varphi) = 1$. The inverse function theorem shows that φ is a local diffeomorphism iff $J_\mu(\varphi) \neq 0$ on M.*

There are a number of valuable identities relating the Lie derivative, the exterior derivative, and the interior product. For example, if Θ is a one-form and X and Y are vector fields, identity 6 in the following table gives

$$\mathbf{d}\Theta(X, Y) = X[\Theta(Y)] - Y[\Theta(X)] - \Theta([X, Y]). \tag{2.6.14}$$

The following list of identities will be a useful reference for the remainder of the text.

Identities for Vector Fields and Forms

1. Vector fields on M with the bracket $[X, Y]$ form a ***Lie algebra***; that is, $[X, Y]$ is real bilinear and skew-symmetric, and ***Jacobi's identity*** holds:
$$[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0.$$
Locally,
$$[X, Y] = \mathbf{D}Y \cdot X - \mathbf{D}X \cdot Y = (X \cdot \nabla)Y - (Y \cdot \nabla)X\,,$$
and on functions,
$$[X, Y][f] = X[Y[f]] - Y[X[f]].$$

2. For diffeomorphisms φ and ψ,
$$\varphi_*[X, Y] = [\varphi_* X, \varphi_* Y] \quad \text{and} \quad (\varphi \circ \psi)_* X = \varphi_* \psi_* X.$$

3. The forms on a manifold constitute a real associative algebra with $\wedge$ as multiplication. Furthermore, $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$ for k- and l-forms α and β, respectively.

4. For maps φ and ψ,
$$\varphi^*(\alpha \wedge \beta) = \varphi^* \alpha \wedge \varphi^* \beta \quad \text{and} \quad (\varphi \circ \psi)^* \alpha = \psi^* \varphi^* \alpha.$$

5. $\mathbf{d}$ is a real linear map on forms, $\mathbf{dd}\alpha = 0$, and
$$\mathbf{d}(\alpha \wedge \beta) = \mathbf{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathbf{d}\beta$$
for α a k-form.

6. For α a k-form and $X_0, \ldots, X_k$ vector fields,
$$\begin{aligned}(\mathbf{d}\alpha)(X_0, \ldots, X_k) &= \sum_{i=0}^{k} (-1)^i X_i[\alpha(X_0, \ldots, \hat{X}_i, \ldots, X_k)] \\ &\quad + \sum_{0 \le i < j \le k} (-1)^{i+j} \alpha([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k)\,,\end{aligned}$$
where $\hat{X}_i$ means that X_i is omitted. Locally,
$$\mathbf{d}\alpha(x)(v_0, \ldots, v_k) = \sum_{i=0}^{k} (-1)^i \mathbf{D}\alpha(x) \cdot v_i(v_0, \ldots, \hat{v}_i, \ldots, v_k).$$

7. For a map $\varphi, \varphi^* \mathbf{d}\alpha = \mathbf{d}\varphi^*\alpha$.

8. **Poincaré Lemma.** If $\mathbf{d}\alpha = 0$, then α is locally exact; that is, there is a neighborhood U about each point on which $\alpha = \mathbf{d}\beta$. The same result holds globally on a contractible manifold.

9. $\mathbf{i}_X\alpha$ is real bilinear in x, α, and for $h : M \to \mathbb{R}$,
$$\mathbf{i}_{hX}\alpha = h\mathbf{i}_X\alpha = \mathbf{i}_X h\alpha.$$
Also, $\mathbf{i}_X\mathbf{i}_X\alpha = 0$ and
$$\mathbf{i}_X(\alpha \wedge \beta) = \mathbf{i}_X\alpha \wedge \beta + (-1)^k \alpha \wedge \mathbf{i}_X\beta$$
for α a k-form.

10. For a diffeomorphism φ,
$$\varphi^*(\mathbf{i}_X\alpha) = \mathbf{i}_{\varphi^*X}(\varphi^*\alpha);$$
if $f : M \to N$ is a mapping and Y is f-related to X, i.e., $Tf \circ X = Y \circ f$, then
$$\mathbf{i}_Y f^*\alpha = f^*\mathbf{i}_X\alpha.$$

11. $\pounds_X\alpha$ is real bilinear in x, α, and
$$\pounds_X(\alpha \wedge \beta) = \pounds_X\alpha \wedge \beta + \alpha \wedge \pounds_X\beta.$$

12. **Cartan's Magic Formula:** $\pounds_X\alpha = \mathbf{d}\mathbf{i}_X\alpha + \mathbf{i}_X\mathbf{d}\alpha$.

13. For a diffeomorphism φ,
$$\varphi^*\pounds_X\alpha = \pounds_{\varphi^*X}\varphi^*\alpha;$$
if $f : M \to N$ is a mapping and Y is f-related to X, then
$$\pounds_Y f^*\alpha = f^*\pounds_X\alpha.$$

14. For vector fields $X, X_1, \ldots, X_k$ and a k-form α,
$$\begin{aligned}(\pounds_X\alpha)(X_1, \ldots, X_k) =& X[\alpha(X_1, \ldots, X_k)] \\ & - \sum_{i=1}^{k} \alpha(X_1, \ldots, [X, X_i], \ldots, X_k).\end{aligned}$$
Locally,
$$\begin{aligned}(\pounds_X\alpha)(x) \cdot (v_1, \ldots, v_k) = & (\mathbf{D}\alpha_x \cdot X(x))(v_1, \ldots, v_k) \\ & + \sum_{i=0}^{k} \alpha_x(v_1, \ldots, \mathbf{D}X_x \cdot v_i, \ldots, v_k).\end{aligned}$$

15. The following identities hold:

    (a) $\pounds_{fX}\alpha = f\pounds_X\alpha$, $\pounds_X f\alpha = f\pounds_X\alpha + \mathbf{d}f \wedge \mathbf{i}_X\alpha$;

    (b) $\pounds_{[X,Y]}\alpha = \pounds_X\pounds_Y\alpha - \pounds_Y\pounds_X\alpha$;

    (c) $\mathbf{i}_{[X,Y]}\alpha = \pounds_X\mathbf{i}_Y\alpha - \mathbf{i}_Y\pounds_X\alpha$;

    (d) $\pounds_X\mathbf{d}\alpha = \mathbf{d}\pounds_X\alpha$;

    (e) $\pounds_X\mathbf{i}_X\alpha = \mathbf{i}_X\pounds_X\alpha$.

16. If M is a finite-dimensional manifold, $X = X^l\partial/\partial x^l$, and

$$\alpha = \alpha_{i_1\dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where $i_1 < \cdots < i_k$, then the following formulas hold:

$$\begin{aligned}
\mathbf{d}\alpha &= \left(\frac{\partial\alpha_{i_1\dots i_k}}{\partial x^l}\right) dx^l \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k},\\
\mathbf{i}_X\alpha &= X^l\alpha_{li_2\dots i_k} dx^{i_2} \wedge \cdots \wedge dx^{i_k},\\
\pounds_X\alpha &= X^l\left(\frac{\partial\alpha_{i_1\dots i_k}}{\partial x^l}\right) dx^{i_1} \wedge \cdots \wedge dx^{i_k}\\
&\qquad + \alpha_{li_2\dots i_k}\left(\frac{\partial X^l}{\partial x^{i_1}}\right) dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k} + \cdots .
\end{aligned}$$

Exercises

◇ **2.6-1.** Consider the two-form β on $\mathbb{R}^3$ given by

$$\beta = x\,dy \wedge\, dz + y\,dx \wedge\, dz + zdx \wedge\, dy$$

and the vector fields X, Y on $\mathbb{R}^3$ defined by

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}; \qquad Y = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

(a) Is β closed? exact?

(b) Compute $\mathbf{i}_X\beta$.

(c) Find the flow F_t of X.

(d) Compute $\left.\dfrac{d}{dt}\right|_{t=0} F_t^*\beta$ and $\left.\dfrac{d}{dt}\right|_{t=0} F_t^*Y$.

◇ **2.6-2.** Let M be an n-manifold, $\omega \in \Omega^n(M)$ a volume form, $X, Y \in \mathfrak{X}(M)$, and $f, g : M \to \mathbb{R}$ smooth functions such that $f(m) \neq 0$ for all m. Prove the following identities:

(a) $\operatorname{div}_{f\omega}(X) = \operatorname{div}_\omega(X) + X[f]/f$;

(b) $\operatorname{div}_\omega(gX) = g \operatorname{div}_\omega(X) + X[g]$;

(c) $\operatorname{div}_\omega([X, Y]) = X[\operatorname{div}_\omega(Y)] - Y[\operatorname{div}_\omega(X)]$.

◇ **2.6-3.** Show that the partial differential equation

$$\frac{\partial f}{\partial t} = \sum_{i=1}^{n} X^i(x^1, \ldots, x^n) \frac{\partial f}{\partial x^i}$$

with initial condition $f(x, 0) = g(x)$ has the solution $f(x, t) = g(F_t(x))$, where F_t is the flow of the vector field $(X^1, \ldots, X^n)$ in $\mathbb{R}^n$ whose flow is assumed to exist for all time. Show that the solution is *unique*. Generalize this exercise to the equation

$$\frac{\partial f}{\partial t} = X[f]$$

for X a vector field on a manifold M.

◇ **2.6-4.** Show that if M and N are orientable manifolds, so is $M \times N$.

2.7 Stokes's Theorem, Riemannian Manifolds, Distributions

The basic idea behind the definition of the integral of an n-form ω on an oriented n-manifold M is to pick a covering by coordinate charts and to sum up the ordinary integrals of $f(x^1, \ldots, x^n)\, dx^1 \cdots dx^n$ in these charts, where

$$\omega = f(x^1, \ldots, x^n)\, dx^1 \wedge \cdots \wedge dx^n$$

is the local representative of ω, being careful not to count overlaps twice. The change of variables formula guarantees that the result, denoted by $\int_M \omega$, is well-defined. Literally carrying this out as stated would involve some fairly serious combinatorial problems in keeping track of overlaps of coordinate charts. Thus, an alternative approach using a tool called partitions of unity (a bunch of positive functions that add up to one) is often used, since it makes the bookkeeping fairly easy. See Abraham, Marsden, and Ratiu [1988] for details.

If one has an oriented manifold with boundary, then the boundary ∂M inherits a compatible orientation. This proceeds in a way that generalizes the relation between the orientation of a surface and its boundary that one learns in the classical Stokes's theorem in $\mathbb{R}^3$.

2.7.1 Theorem (Stokes's Theorem)**.** *Suppose that M is a compact, oriented k-dimensional manifold with boundary ∂M. Let α be a smooth $(k-1)$-form on M. Then*

$$\int_M \mathbf{d}\alpha = \int_{\partial M} \alpha. \tag{2.7.1}$$

Special cases of Stokes's theorem are as follows:

The Integral Theorems of Calculus. Stokes's theorem generalizes and synthesizes the classical theorems:

(a) Fundamental Theorem of Calculus.

$$\int_b^a f'(x)\,dx = f(b) - f(a). \tag{2.7.2}$$

(b) Green's Theorem. For a region $\Omega \subset \mathbb{R}^2$,

$$\iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \int_{\partial\Omega} P\,dx + Q\,dy. \tag{2.7.3}$$

(c) Divergence Theorem. For a region $\Omega \subset \mathbb{R}^3$,

$$\iiint_\Omega \operatorname{div} \mathbf{F}\,dV = \iint_{\partial\Omega} \mathbf{F} \cdot \mathbf{n}\,dA. \tag{2.7.4}$$

(d) Classical Stokes's Theorem. For a surface $S \subset \mathbb{R}^3$,

$$\begin{aligned} \iint_S \Bigg\{ & \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \wedge dz \\ & + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy \Bigg\} \\ = & \iint_S \mathbf{n} \cdot \operatorname{curl} \mathbf{F}\,dA = \int_{\partial S} P\,dx + Q\,dy + R\,dz, \end{aligned} \tag{2.7.5}$$

where $\mathbf{F} = (P, Q, R)$.

Notice that the Poincaré lemma generalizes the vector calculus theorems in $\mathbb{R}^3$ saying that if $\operatorname{curl} \mathbf{F} = 0$, then $\mathbf{F} = \nabla f$, and if $\operatorname{div} \mathbf{F} = 0$, then $\mathbf{F} = \nabla \times \mathbf{G}$. Recall that it states, *If α is closed, then locally α is exact; that is, if $\mathbf{d}\alpha = 0$, then locally $\alpha = \mathbf{d}\beta$ for some β.*

Change of Variables. Another basic result in integration theory is the global change of variables formula.

2.7.2 Theorem (Change of Variables). *Suppose that M and N are oriented n-manifolds and $F : M \to N$ is an orientation-preserving diffeomorphism. If α is an n-form on N (with, say, compact support), then*

$$\int_M F^*\alpha = \int_N \alpha.$$

Riemannian Manifolds. A differentiable manifold with a positive definite symmetric quadratic form $\langle \cdot, \cdot \rangle$ on every tangent space TM_x is called a ***Riemannian manifold***. The quadratic form $\langle \cdot, \cdot \rangle$ itself, often denoted by $g(\ ,\ )$ is called a ***Riemannian metric***.

In local coordinates q^i on M and the associated tangent coordinates $\dot{q}^i$ the length of a vector $v = v^i e_i$ is then given by

$$g(v, v) = g_{ij}(q) v^i v^j\,, \qquad g_{ij} = g_{ji}\,,$$

where as above, the summation convention is in force.

Let f be a smooth function on M. The ***gradient vector field*** associated with f, which is denoted by $\operatorname{grad} f$ or ∇f, is defined by

$$\mathbf{d}f(v) = \langle \operatorname{grad} f, v \rangle$$

for any $v \in TM$. The flow of the vector field $\operatorname{grad} f$ is called the ***gradient flow*** of f.

Frobenius's Theorem. A basic result called ***Frobenius's theorem*** plays a critical role in control theory, and we shall have much to say about it later in the book. For now we just state it briefly, since it is normally regarded as part of the theory of differentiable manifolds. The theory of distributions plays a key role in both the theory of nonholonomic systems and nonlinear control theory. Two useful references (from the control-theoretic point of view) are Sussmann [1973] and Isidori [1995].

2.7.3 Definition. *A **smooth distribution** on a manifold M is the assignment to each point $x \in M$ of the subspace spanned by the values at x of a set of smooth vector fields on M; i.e., it is a "smooth" assignment of a subspace to the tangent space at each point, also called a **vector subbundle**. We denote the distribution by Δ and the subspace at $x \in M$ by $\Delta_x \subset T_x M$.*

A distribution is said to be ***involutive*** if for any two vector fields X, Y on M with values in Δ, $[X, Y]$ is also a vector field with values in Δ. The subbundle Δ is said to be ***integrable*** if for each point $x \in M$ there is a local submanifold of M containing x such that its tangent bundle equals Δ restricted to this submanifold. See Figure 2.7.1.

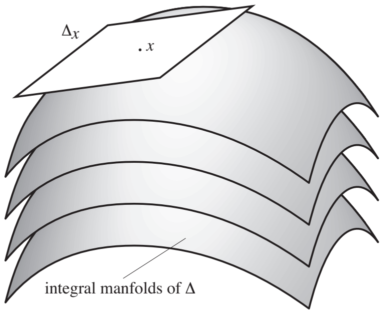

FIGURE 2.7.1. The integral manifolds of a distribution.

If Δ is integrable, the local integral manifolds can be extended to get, through each $x \in M$, a maximal integral manifold, which is an immersed submanifold of M. The collection of all maximal integral manifolds through all points of M forms a ***foliation***.

2.7.4 Theorem (Frobenius's Theorem)**.** *Involutivity of Δ is equivalent to the integrability of Δ, which in turn is equivalent to the existence of a foliation on M whose tangent bundle equals Δ.*

Given a set of smooth vector fields $X_1, \ldots, X_d$ on M we denote the distribution defined by their span by

$$\Delta = \operatorname{span}\{X_1, \ldots, X_d\}.$$

The distribution at any point is denoted by Δ_x. A distribution Δ on M is said to be ***nonsingular*** (or ***regular***) on M if there exists an integer d such that $\dim(\Delta_x) = d$ for all $x \in M$. A point $x \in M$ is said to be a regular point is there exists a neighborhood U of x such that Δ is nonsingular on U. Otherwise, the point is said to be ***singular***.

Note that the Frobenius theorem as stated above applies to nonsingular or regular distributions. For generalized distributions in the sense of Sussmann, see for example, Vaisman [1994].

Exercises

$\diamond$ **2.7-1.** Let Ω be a closed bounded region in $\mathbb{R}^2$. Use Green's theorem to show that the area of Ω equals the line integral

$$\frac{1}{2}\int_{\partial\Omega}(x\,dy - y\,dx).$$

$\diamond$ **2.7-2.** On $\mathbb{R}^2\backslash\{(0,0)\}$ consider the one-form

$$\alpha = (x\,dy - y\,dx)/(x^2 + y^2).$$

(a) Show that this form is closed.

(b) Using the angle θ as a variable on S^1, compute $i^*\alpha$, where $i : S^1 \to \mathbb{R}^2$ is the standard embedding.

(c) Show that α is not exact.

$\diamond$ **2.7-3.** Suppose that a set of linearly independent vector fields X_i has the property that there are functions γ_{ijk} such that

$$[X_i, X_j] = \gamma_{ijk} X_k.$$

Show that the span of these vector fields defines an integrable distribution.

$\diamond$ **2.7-4. The magnetic monopole.** Let $\mathbf{B} = g\mathbf{r}/r^3$ be a vector field on Euclidean three-space minus the origin, where $r = \|\mathbf{r}\|$. Show that $\mathbf{B}$ cannot be written as the curl of something.

$\diamond$ **2.7-5.** Let M be a manifold and ω a two-form on M.

(a) Consider the distribution D on M defined at $x \in M$ by

$$D_x = \{v_x \in T_x M \mid \mathbf{i}_{v_x}\omega = 0\}.$$

Develop a condition(s) that guarantees that this distribution is integrable.

(b) Let ω on $\mathbb{R}^4$, with coordinates (x, y, z, w), be given by

$$\omega = dx \wedge dy + dx \wedge dz + dx \wedge dw.$$

Compute the distribution D in this case. Does your condition hold?

(d) Find an explicit example of such a vector field X for the example in part (b).

2.8 Lie Groups

Lie groups arise in discussing conservation laws for mechanical and control systems and in the analysis of systems with some underlying symmetry. There is a huge literature on the subject. Useful references include Abraham and Marsden [1978], Marsden and Ratiu [1999], Sattinger and Weaver [1986], and Libermann and Marle [1987].

2.8.1 Definition. *A **Lie group** is a smooth manifold G that is a group and for which the group operations of multiplication, $(g, h) \mapsto gh$ for $g, h \in G$, and inversion, $g \mapsto g^{-1}$, are smooth.*

Before giving a brief description of some of the theory of Lie groups we mention an important example: the group of linear isomorphisms of $\mathbb{R}^n$ to itself. This is a Lie group of dimension n^2 called the general linear group and denoted by $\mathrm{GL}(n, \mathbb{R})$. The conditions for a Lie group are easily checked: This is a manifold, since it is an open subset of the linear space of all linear maps of $\mathbb{R}^n$ to itself; the group operations are smooth, since they are algebraic operations on the matrix entries.

2.8.2 Definition. *A **matrix Lie group** is a set of invertible $n \times n$ matrices that is closed under matrix multiplication and that is a submanifold of $\mathbb{R}^{n\times n}$.*

A theorem in Lie group theory shows that (although this is by no means obvious) one could equivalently define a matrix Lie group to be a (topologically) closed subgroup of $\mathrm{GL}(n, \mathbb{R})$. All of the Lie groups discussed in this book will be matrix Lie groups.

Lie Algebras. Lie groups are frequently studied in conjunction with *Lie algebras*, which are associated with the tangent spaces of Lie groups as we now describe. To begin with, we state a generalization of the result established in Exercise 2.2-3.

2.8.3 Proposition. *Let G be a matrix Lie group, and let $A, B \in T_I G$ (the tangent space to G at the identity element). Then $AB - BA \in T_I G$.*

Our proof makes use of the following lemma.

2.8.4 Lemma. *Let R be an arbitrary element of a matrix Lie group G, and let $B \in T_I G$. Then $RBR^{-1} \in T_I G$.*

Proof. Let $R_B(t)$ be a curve in G such that $R_B(0) = I$ and $R'_B(0) = B$. Then $S(t) = RR_B(t)R^{-1} \in G$ for all t, and $S(0) = I$. Hence $S'(0) \in T_I G$, proving the lemma. ▼

Proof of Proposition. Let $R_A(s)$ be a curve in G such that $R_A(0) = I$ and $R'_A(0) = A$. Thus, by the preceding lemma, $S(t) = R_A(t)BR_A(t)^{-1} \in T_I G$. Hence $S'(t) \in T_I G$, and in particular, $S'(0) = AB - BA \in T_I G$. ■

2.8.5 Definition. *For any pair of $n \times n$ matrices A, B we define the **matrix Lie bracket** $[A, B] = AB - BA$.*

2.8.6 Proposition. *The matrix Lie bracket operation has the following two properties:*

(i) *For any $n \times n$ matrices A and B, $[B, A] = -[A, B]$ (this is the property of **skew-symmetry**).*

(ii) *For any $n \times n$ matrices A, B, and C,*

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0.$$

(*This is known as the **Jacobi identity**.*)

The proof of this proposition involves a straightforward calculation and is left to the reader.

2.8.7 Definition. *A (matrix)* ***Lie algebra*** $\mathfrak{g}$ *is a set of* $n \times n$ *matrices that is a vector space with respect to the usual operations of matrix addition and multiplication by real numbers (scalars) and that is closed under the matrix Lie bracket operation* $[\cdot\,,\cdot]$.

2.8.8 Proposition. *For any matrix Lie group* G*, the tangent space at the identity* T_IG *is a Lie algebra.*

Proof. This is an immediate consequence of the fact that T_IG is a vector space and the preceding proposition. ■

One can also define a Lie algebra $\mathfrak{g}$ abstractly as a vector space over a field F on which a Lie bracket operation $[\cdot,\cdot]$ is defined such that $\mathfrak{g}$ is closed under this operation; $[A, \alpha B + \beta C] = \alpha[A, B] + \beta[A, C]$ for any $\alpha, \beta \in F$ and $A, B, C \in \mathfrak{g}$; and properties (i) and (ii) in theorem 2.8.6 hold.

For $A \in \mathfrak{g}$ we define the operator ad_X to be the operator that maps $B \in \mathfrak{g}$ to $[A, B]$. We write $\mathrm{ad}_A B = [A, B]$.

2.8.9 Definition. *A* ***representation*** *of a Lie algebra* $\mathfrak{g}$ *on a vector space* V *is a mapping* ρ *from* $\mathfrak{g}$ *to the linear transformations of* V *such that for* $A, B \in \mathfrak{g}$

(i) $\rho(\alpha A + \beta B) = \alpha\rho(A) + \beta\rho(B)$

(ii) $\rho([A, B]) = \rho(A)\rho(B) - \rho(B)\rho(A)$.

If the map ρ *is 1-1 the representation is said to be* ***faithful****.*

For a Lie algebra $\mathfrak{g}$ the map $A \to \mathrm{ad}_A$ is a representation of the Lie algebra $\mathfrak{g}$, with $\mathfrak{g}$ itself the vector space of the representation. This is called the ***adjoint representation***. The ad-action of the Lie algebra on itself is the infinitesimal action of the Adjoint action of the group—see later in this section and, for example, Arnold [1989].

The ***Killing form*** of a Lie algebra is the symmetric bilinear form defined by

$$\kappa(A, B) = \mathrm{Trace}(\mathrm{ad}_A \ \mathrm{ad}_B)\,. \tag{2.8.1}$$

One can show that a Lie algebra is ***semi-simple***, i.e. it contains no abelian ideals other than $\{0\}$, if and only if the Killing form is nondegenerate. Further, the group G corresponding to $\mathfrak{g}$ is compact if and only if the Killing form is negative definite. See, for example, Sattinger and Weaver [1986] for proofs.

A great deal of the structure of a Lie group may be inferred from studying the Lie algebra. Before discussing important general relationships between Lie groups and Lie algebras, we describe several examples that play an important role in mechanics and control.

The Special Orthogonal Group. The set of all elements of $\mathrm{O}(n)$ having determinant 1 is a subgroup called the ***special orthogonal group***, denoted by $\mathrm{SO}(n)$. Because any $X \in \mathrm{O}(n)$ satisfies $XX^T = I$, it follows that $\det X = \pm 1$. We could also characterize $\mathrm{SO}(n)$ as the connected component of the identity element in $\mathrm{O}(n)$. Thus, $T_I \,\mathrm{SO}(n) = T_I \,\mathrm{O}(n)$. From this observation and the calculation carried out for $\mathrm{GL}(n, \mathbb{R})$ in Section 2.2, $T_I \,\mathrm{SO}(n)$ is just the set of $n \times n$ skew-symmetric matrices, which we denote by $\mathfrak{so}(n)$.

The Symplectic Group. Suppose $n = 2l$ (that is, n is even) and consider the nonsingular skew-symmetric matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where I is the $l \times l$ identity matrix. It is an exercise left to the reader to verify that

$$\mathrm{Sp}(l) = \{X \in \mathrm{GL}(2l) \mid XJX^T = J\}$$

is a group. It is called the ***symplectic group***. Again referring to the example of $\mathrm{GL}(n, \mathbb{R})$ in Section 2.2, we find that this matrix Lie algebra $T_I \,\mathrm{Sp}(l)$ is the set of $n \times n$ matrices satisfying $JY^T + YJ = 0$. We denote this Lie algebra by $\mathrm{sp}(l)$.

The Heisenberg Group. Consider the set of all 3×3 matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

where x, y, and z are real numbers. It is straightforward to show that this is a group, and since it is a submanifold of the set of all 3×3 matrices, it is a Lie group. Call it H. The corresponding Lie algebra may be written down from the definition. Specifically,

$$X_1(t) = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad X_2(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix},$$

$$X_3(t) = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are three curves in H that pass through the identity when $t = 0$. The derivatives $X_i'(0)$ are elements of the Lie algebra:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

respectively. Since three independent parameters are used to specify H, H is three-dimensional. The matrices A, B, and C are linearly independent and span the Lie algebra. The commutation relations for the Lie brackets of these three basis elements are $[A,B]=C$, $[A,C]=0$, and $[B,C]=0$. This Lie algebra is called the ***Heisenberg algebra***.

Recall that we encountered this algebra in Section 1.8 when we analyzed the Heisenberg system, and we shall encounter it several times again.

The Euclidean Group. Consider the Lie group of all 4×4 matrices of the form

$$\begin{pmatrix} R & v \\ 0 & 1 \end{pmatrix},$$

where $R \in \mathrm{SO}(3)$ and $v \in \mathbb{R}^3$. This group is usually denoted by SE(3) and is called the ***special Euclidean group***. Let the associated matrix be denoted by

$$E(R,v) = \begin{pmatrix} R & v \\ 0 & 1 \end{pmatrix}.$$

The corresponding Lie algebra, se(3), is six-dimensional and is spanned by

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The special Euclidean group is of central interest in mechanics since it describes the set of rigid motions and coordinate transformations of 3-space. More specifically, suppose there are two coordinate frames A and B located in space such that the origin of the B-frame has A-frame coordinates $v=(v_1,v_2,v_3)^T$ and such that the unit vectors in the principal B-frame coordinate directions are $(r_{11},r_{21},r_{31})^T$, $(r_{12},r_{22},r_{32})^T$, and $(r_{13},r_{23},r_{33})^T$ with respect to A-frame coordinates. The ***rigid motion*** that moves the A-frame into coincidence with the B-frame is specified by the rotation

$$\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

followed by the translation $v=(v_1,v_2,v_3)^T$. Thus a point with A-frame coordinates $x=(x_1,x_2,x_3)$ is moved under the rigid motion to a new location whose A-frame coordinates are $Rx+v$.

Remark. The above discussion is nice and concrete, but gives the impression that one needs coordinate frames to define the Euclidean group. More intrinsically, the Euclidean group SE(3) can also be defined simply as the set of all isometries of $\mathbb{R}^3$ to itself. (It is a famous theorem of Mazur and Ulam that such isometries are, in fact, affine maps.)

Resuming the previous discussion, we observe that the group SE(3) is also associated with the set of *rigid coordinate transformations* of $\mathbb{R}^3$ as follows. Suppose a point Q is located in space and has A-frame coordinates $(x_1^A, x_2^A, x_3^A)^T$ and B-frame coordinates $(x_1^B, x_2^B, x_3^B)^T$. The relationship between these coordinate descriptions is given by

$$x^A = Rx^B + v.$$

Let G be a matrix Lie group and let $\mathfrak{g} = T_I G$ be the corresponding Lie algebra. The dimensions of the differentiable manifold G and the vector space $\mathfrak{g}$ are of course the same, and there must be a one-to-one local correspondence between a neighborhood of 0 in $\mathfrak{g}$ and a neighborhood of the identity element I in G. One explicit local correspondence is provided by the exponential mapping $\exp : \mathfrak{g} \to G$, which we now describe.

Let $A \in \mathbb{R}^{n\times n}$ (the space of $n \times n$ matrices). We define $\exp(A)$ by the series

$$I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \cdots . \tag{2.8.2}$$

2.8.10 Proposition. *The series* (2.8.2) *is absolutely convergent.*

Proof. The ijth entry in the nth term of this matrix series is bounded in absolute value by $(n-1)\bar{a}^n/n!$, where $\bar{a} = \max_{ij}\{|a_{ij}|\}$. Hence, the ijth element in each term in the series is bounded in absolute value by the corresponding term in the absolutely convergent series $e^{\bar{a}n} = 1 + \bar{a}n + \frac{1}{2}\bar{a}^2n^2 + \cdots$. Hence each entry in the series of matrices converges absolutely, proving the proposition. ■

2.8.11 Proposition. *Let G be a matrix Lie group with corresponding Lie algebra $\mathfrak{g}$. If $A \in \mathfrak{g}$, then* $\exp(At) \in G$ *for all real numbers t.*

Group Actions. We now define the action of a Lie group G on a manifold M. Roughly speaking, a group action is a group of transformations of M indexed by elements of the group G and whose composition in M is compatible with group multiplication in G.

2.8.12 Definition. *Let M be a manifold and let G be a Lie group. A* ***left action*** *of a Lie group G on M is a smooth mapping $\Phi : G \times M \to M$ such that* (i) $\Phi(e, x) = x$ *for all $x \in M$,* (ii) $\Phi(g, \Phi(h, x)) = \Phi(gh, x)$ *for all $g, h \in G$ and $x \in M$, and* (iii) $\Phi(g, \cdot)$ *is a diffeomorphism for each $g \in G$.*

We often use the convenient notation gx for $\Phi(g, x)$ and think of the group element g *acting* on the point $x \in M$. The condition above then simply reads $(gh)x = g(hx)$.

Similarly, one can define a ***right action***, which is a map $\Psi : M \times G \to M$ satisfying $\Psi(x,e) = x$ and $\Psi(\Psi(x,g),h) = \Psi(x,gh)$.

Orbits. Given a group action of G on M, for a given point $x \in M$, we let

$$\operatorname{Orb} x = \{gx \mid g \in G\} \ ,$$

called the ***group orbit*** through x. It can be shown that orbits are always smooth (possibly immersed) manifolds. This notion generalizes the notion of an orbit of a dynamical system, for the flow of a vector field on M can be thought of as an action of $\mathbb{R}$ on M, and in this case the general notion of orbit reduces to the familiar notion of orbit.

A simple example is the action of SO(3) on $\mathbb{R}^3$ given by matrix multiplication: The action of $A \in \mathrm{SO}(3)$ on a point $x \in \mathbb{R}^3$ is simply the product Ax. In this case, the orbit of the origin is a single point (the origin itself), while the orbit of another point is the sphere through that point.

Infinitesimal Generator. An important concept for mechanics is that of the infinitesimal generator of the group action:

2.8.13 Definition. *Suppose $\Phi : G \times M \to M$ is an action. For $\xi \in \mathfrak{g}$, the map $\Phi^\xi : \mathbb{R} \times M \to M$ defined by $\Phi^\xi(t,x) = \Phi(\exp(t\xi), x)$ is an $\mathbb{R}$-action—that is, a flow—on M. The vector field on M that generates this flow, namely*

$$\xi_M(x) = \left.\frac{d}{dt}\right|_{t=0} \Phi^\xi(t,x) \,. \tag{2.8.3}$$

*is called the **infinitesimal generator** of the action corresponding to ξ.*

A basic important identity relating the Jacobi–Lie bracket of generators to the Lie algebra bracket is as follows (see, for example, Marsden and Ratiu [1999] for the proof):

$$[\xi_Q, \eta_Q] = -[\xi, \eta]_Q \,. \tag{2.8.4}$$

Left- and Right-Invariant Vector Fields. A Lie group acts on its tangent bundle by the tangent map. Given $\xi \in \mathfrak{g}$ we can consider the action of G on ξ either on the left or the right: $T_eL_g\xi$ or $T_eR_g\xi$, where L_g and R_g denote left and right translations, respectively; for example, $L_g : G \to G$ is the map given by $g' \mapsto gg'$. We can abbreviate these expressions and write $g\xi$ and ξg, respectively. For matrix Lie groups this action is just multiplication on the left or right.

Allowing $g \in G$ to vary over the group, the vectors $T_eL_g\xi$, $T_eR_g\xi$ define left- and right-invariant vector fields, that is, vector fields satisfying

$$(T_hL_g)\, X(h) = X(gh) \text{ or } (T_hR_g)\, X(h) = X(hg)\,, \tag{2.8.5}$$

respectively. If we let $\xi_L(g) = T_eL_g\xi$, then the Jacobi–Lie bracket of two such left-invariant vector fields in fact gives the Lie algebra bracket:

$$[\xi_L, \eta_L](g) = [\xi, \eta]_L(g).$$

For the right-invariant case, one inserts a minus sign.

Spatial and Body Velocities. There are two ways to pull back a tangent vector to a group to the identity. One can think of these as "body" or "spatial" velocities denoted by

$$\xi^b = \left(T_g L_{g^{-1}}\right) \dot{g} \quad \text{and} \quad \xi^s = \left(T_g R_{g^{-1}}\right) \dot{g}\,, \tag{2.8.6}$$

respectively.

Adjoint and Coadjoint Actions. We also define the ***adjoint action*** of G on its Lie algebra to be given by

$$\mathrm{Ad}_g\, \xi = T_{g^{-1}} L_g \left(T_e R_{g^{-1}} \xi\right) \tag{2.8.7}$$

for $\xi \in \mathfrak{g}$.

For matrix groups this is simply conjugation by the matrix g: $g\xi g^{-1}$. Thus $\xi^s = \mathrm{Ad}_g\, \xi^b$.

The dual action $\mathrm{Ad}^*_{g^{-1}}$ is called the ***coadjoint action***.

Quotient Spaces and Equivariance. If we have an action of a group G on M and the action is free (that is, if $gx = x$ for any x implies that g is the identity) and if the action is also proper (that is, the map $(g, x) \mapsto (g, gx)$ is a proper map: inverse images of compact sets are compact), then it can be shown (see, for example, Abraham and Marsden [1978]) that the space of orbits, denoted by M/G, is a smooth manifold and the natural projection $\pi : M \to M/G$ taking a point to its orbit is a smooth submersion.

If G acts on two manifolds M and N and if $f : M \to N$ is ***equivariant***, that is, $f(gx) = gf(x)$, then f induces, in a natural way, a map of the quotients: $f_G : M/G \to N/G$.

There are similar statements for other equivariant objects. For example, let X be an ***equivariant vector field*** on M; that is, fixing g and denoting the map $x \mapsto gx$ by Φ_g, we have $\Phi_g^* X = X$. Then X induces, in a natural way, a vector field $X_{M/G}$ on M/G.

Exercises

◇ **2.8-1.** A point P in $\mathbb{R}^3$ undergoes a rigid motion associated with $E(R_1, v_1)$ followed by a rigid motion associated with $E(R_2, v_2)$. What matrix element of SE(3) is associated with the composite of these motions in the given order?

◇ **2.8-2.** A coordinate frame B is located with respect to a coordinate frame A as follows. B is initially coincident with A, but is displaced by the rigid motion associated with $E(R_1, v_1)$ and is then subsequently further displaced by $E(R_2, v_2)$. What matrix element of SE(3) is associated with the coordinate transformation from the A-frame to the B-frame? (That is, what matrix element of SE(3) is used to describe A-frame coordinates of a point in terms of the B-frame coordinates of the same point?)

◇ **2.8-3.** Let $Y \in \mathfrak{sp}(l)$ be partitioned into $l \times l$ blocks,

$$Y = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Write down a complete set of equations involving A, B, C, and D that must be satisfied if $Y \in \mathfrak{sp}(l)$. Deduce that the dimension of $\mathfrak{sp}(l)$ as a real vector space is $2l^2 + l = n(n+1)/2$, and consequently, $\dim \mathrm{Sp}(l) = 2l^2 + l$.

◇ **2.8-4.** Suppose the $n \times n$ matrices A and M satisfy $AM + MA^T = 0$. Show that $\exp(At)M\exp(A^T t) = M$ for all t. This direct calculation shows that for $A \in \mathfrak{so}(n)$ or $A \in \mathfrak{sp}(l)$, we have $\exp(At) \in \mathrm{SO}(n)$ or $\exp(At) \in \mathrm{Sp}(l)$, respectively.

2.9 Fiber Bundles and Connections

In this section we give a somewhat brief but, we hope, instructive treatment of fiber bundles and related concepts. We describe both theory and some illustrative examples. Our exposition is somewhat more explicit than is usual. References are given to more comprehensive treatments.

Fiber Bundles. Fiber bundles provide a basic geometric structure for the understanding of many mechanical and control problems, in particular for nonholonomic problems. References include Abraham, Marsden, and Ratiu [1988], Steenrod [1951], and Schutz [1980].

A fiber bundle essentially consists of a given space (the base) together with another space (the fiber) attached at each point, plus some compatibility conditions. More formally, we have the following:

2.9.1 Definition. *A* ***fiber bundle*** *is a space Q for which the following are given: a space B called the* ***base space****, a* ***projection*** *$\pi : Q \to B$ with fibers $\pi^{-1}(b), b \in B$, homeomorphic to a space F, a* ***structure group*** *G of homeomorphisms of F into itself, and a* ***covering*** *of B by open sets U_j, satisfying*

(i) *the bundle is locally trivial, i.e., $\pi^{-1}(U_j)$ is homeomorphic to the product space $U_j \times F$ and*

(ii) *if h_j is the map giving the homeomorphism on the fibers above the set U_j, for any $x \in U_j \cap U_k$, $h_j(h_k^{-1})$ is an element of the structure group G.*

If the fibers of the bundle are homeomorphic to the structure group, we call the bundle a ***principal bundle***.

If the fibers of the bundle are homeomorphic to a vector space, we call the bundle a ***vector bundle***.

2.9.2 Example. A basic example of a vector bundle is TS^1, the tangent bundle of the circle. The base is S^1, the fibers are homeomorphic to $\mathbb{R}$, and since the tangent space can be represented by any nonzero real number, the structure group is ratios of nonzero real numbers and may be identified with $\mathbb{R}\backslash\{0\}$.

The frame bundle of a manifold has the same structure group as TM, but the fibers are the set of all ***bases*** for the tangent space. Hence for TS^1 the fibers of the frame bundle are homeomorphic to its structure group $\mathbb{R}\backslash\{0\}$, and hence the frame bundle is a principal bundle. In fact, all frame bundles are principal. ♦

Connections. An important additional structure on a bundle is a ***connection*** or ***Ehresmann connection***; see, for example, Kobayashi and Nomizu [1963], Marsden, Montgomery, and Ratiu [1990], or Bloch, Krishnaprasad, Marsden, and Ratiu [1996]. We follow the treatment in the last of these here.

However, before we give the precise mathematical definitions we will give a somewhat intuitive discussion of the nature of and need for connections. A nice reference in this regard is the book by Burke [1985].

Suppose we have a bundle and consider (locally) a section of this bundle, i.e., a choice of a point in the fiber over each point in the base. We call such a choice a "field."

The idea is to single out fields that are "constant." For vector fields on the plane, for example, it is clear what we want such fields to be—they should be *literally* constant. For vector fields on a manifold or an arbitrary bundle, we have to specify this notion. Such fields are called "horizontal" and are also key to defining a notion of derivative, or rate of change of a vector field along a curve.[11] A connection is used to single out horizontal fields, and is chosen to have other desirable properties, such as linearity. For example, the sum of two constant fields should still be constant. As we shall see below, we can specify horizontality by taking a class of fields that are the kernel of a suitable form. Note that we do not in general have a metric; given one, there is a natural choice of connection and horizontality on the tangent bundle, as we shall see below.

More formally, we consider a bundle with projection map π and as usual let $T_q\pi$ denote its tangent map at any point. We call the kernel of $T_q\pi$ at any point the ***vertical space*** and denote it by V_q.

2.9.3 Definition. *An **Ehresmann connection** A is a vector-valued one-form on Q that satisfies:*

(i) *A is **vertical valued**: $A_q : T_qQ \to V_q$ is a linear map for each point $q \in Q$.*

[11] Recall that we already have a notion of derivative, namely the Lie derivative. However, Lie derivatives do not give one a way of differentiating vector fields along curves.

(ii) *A is a* ***projection****:* $A(v_q) = v_q$ *for all* $v_q \in V_q$.

The key property of the connection is the following: If we denote by H_q or hor_q the kernel of A_q and call it the ***horizontal space***, the tangent space to Q is the direct sum of the V_q and H_q; i.e., we can split the tangent space to Q into horizontal and vertical parts. For example, we can project a tangent vector onto its vertical part using the connection. Note that the vertical space at Q is tangent to the fiber over q.

Later on when we discuss nonholonomic systems we shall choose the connection so that the constraint distribution is the horizontal space of the connection.

Now define the bundle coordinates $q^i = (r^\alpha, s^a)$ for the base and fiber. The coordinate representation of the projection π is just projection onto the factor r, and the connection A can be represented locally by a vector-valued differential form ω^a:

$$A = \omega^a \frac{\partial}{\partial s^a}, \quad \text{where} \quad \omega^a(q) = ds^a + A^a_\alpha(r, s) dr^\alpha.$$

We can see this as follows: Let

$$v_q = \sum_\beta \dot{r}^\beta \frac{\partial}{\partial r^\beta} + \sum_b \dot{s}^b \frac{\partial}{\partial s^b}$$

be an element of T_qQ. Then $\omega^a(v_q) = \dot{s}^a + A^a_\alpha \dot{r}^\alpha$ and

$$A(v_q) = (\dot{s}^a + A^a_\alpha \dot{r}^\alpha) \frac{\partial}{\partial s^a}.$$

This clearly demonstrates that A is a projection, since when A acts again only ds^a results in a nonzero term, and this has coefficient unity.

2.9.4 Example. It may be helpful to the reader to keep in mind here the physical example of the vertical rolling disk from Chapter 1. There it is natural to choose $r^1 = \theta$, $r^2 = \varphi$, $s^1 = x$, $s^2 = y$. Then the connection given by the constraints gives $\omega_1 = dx - \cos\varphi \, d\theta$ and $\omega_2 = dy - \sin\varphi \, d\theta$.

Note that we use a different notation, namely ω^a, for the local coordinate representation of the connection A for three reasons. First, it is common in the literature to use ω to stand for constraint one-forms. Second, in the preceding formula, it is standard to define the components of the connection A by A^a_α as shown, reflecting the fact that the connection is a projection; to distinguish this use of indices on A from the use of indices on the constraint one-forms, it is convenient to use a different letter. Third, we want to regard ω^a as (coordinate-dependent) differential forms, as opposed to A, which is a vertical-valued form; again, a different letter emphasizes this fact. Note in particular that the exterior derivative of A is not defined, but we can (locally) take the exterior derivative of ω^a. In fact, this will give an easy way to compute the curvature, as we shall see. ♦

Horizontal Lift. Given an Ehresmann connection A, a point $q \in Q$, and a vector $v_r \in T_rB$ tangent to the base at a point $r = \pi(q) \in B$, we can define the ***horizontal lift*** of v_r to be the unique vector v_r^h in H_q that projects to v_r under $T_q\pi$. If we have a vector $X_q \in T_qQ$, we shall also write its ***horizontal part*** as

$$\operatorname{hor} X_q = X_q - A(q) \cdot X_q.$$

In coordinates, the vertical projection is the map

$$(\dot{r}^\alpha, \dot{s}^a) \mapsto (0, \dot{s}^a + A^a_\alpha(r,s)\dot{r}^\alpha)\,,$$

while the horizontal projection is the map

$$(\dot{r}^\alpha, \dot{s}^a) \mapsto (\dot{r}^\alpha, -A^a_\alpha(r,s)\dot{r}^\alpha).$$

Curvature. Next, we give the basic notion of curvature.

2.9.5 Definition. *The **curvature** of A is the vertical-vector-valued two-form B on Q defined by its action on two vector fields X and Y on Q by*

$$B(X,Y) = -A([\operatorname{hor} X, \operatorname{hor} Y]),$$

where the bracket on the right hand side is the Jacobi–Lie bracket of vector fields obtained by extending the stated vectors to vector fields.

One can show that curvature is independent of the extension of the vector fields.

Notice that this definition shows that the curvature exactly measures the failure of the horizontal distribution to be integrable.

Recall from equation (2.6.14) that we have the following useful identity for the exterior derivative $\mathbf{d}\alpha$ of a one-form α (which could be vector-space valued) on a manifold M acting on two vector fields X, Y:

$$(\mathbf{d}\alpha)(X,Y) = X[\alpha(Y)] - Y[\alpha(X)] - \alpha([X,Y]).$$

This identity shows that *in coordinates*, one can evaluate the curvature by writing the connection as a form ω^a in coordinates, computing its exterior derivative (component by component), and restricting the result to horizontal vectors, that is, to the constraint distribution. In other words,

$$B(X,Y) = d\omega^a(\operatorname{hor} X, \operatorname{hor} Y)\frac{\partial}{\partial s^a},$$

so that the local expression for curvature is given by

$$B(X,Y)^a = B^a_{\alpha\beta} X^\alpha Y^\beta\,, \tag{2.9.1}$$

where the coefficients $B^a_{\alpha\beta}$ are given by

$$B^b_{\alpha\beta} = \left(\frac{\partial A^b_\alpha}{\partial r^\beta} - \frac{\partial A^b_\beta}{\partial r^\alpha} + A^a_\alpha \frac{\partial A^b_\beta}{\partial s^a} - A^a_\beta \frac{\partial A^b_\alpha}{\partial s^a}\right). \tag{2.9.2}$$

2.9.6 Example (Connections on $T\mathbb{R}^1$). The idea of a connection can be illustrated by considering the simplest possible example: a connection on the bundle $TQ = T\mathbb{R}^1$ with coordinates $(x, \dot{x})$. We may define the horizontal space to be the kernel of the form

$$d\dot{x} + A_1^1(x, \dot{x})dx,$$

where A_1^1 is a smooth function of x and $\dot{x}$. More specifically, we can choose a connection that is linear in the velocities:

$$d\dot{x} + a(x)\dot{x}dx.$$

Here A is the $\mathbb{R}$-valued form

$$(d\dot{x} + a(x)\dot{x}dx)\frac{\partial}{\partial \dot{x}}.$$

Elements of T_qQ are of the form

$$v_q = \dot{x}\frac{\partial}{\partial x} + \ddot{x}\frac{\partial}{\partial \dot{x}},$$

and their projection onto the vertical space is

$$A(v_q) = (\ddot{x} + a(x)\dot{x}^2)\frac{\partial}{\partial \dot{x}}\,.$$

The kernel of A, i.e., the horizontal vectors, is the span of

$$\frac{\partial}{\partial x} - a(x)\dot{x}\frac{\partial}{\partial \dot{x}}\,.$$

Note that the standard choice is $a(x) = 0$; i.e., the standard horizontal space is the span of the vectors $\partial/\partial x$. ♦

Linear Connections, Affine Connections, and Geodesics. Here we consider how Ehresmann connections specialize to linear connections and affine connections defined in the tangent bundle, and we shall derive the geodesic equations. (As above, a good related reference for some of these ideas, but with a rather different approach, is Burke [1985].)

As above, we use bundle coordinates (r^α, s^a), and we specify the connection by the one-forms

$$\omega^a(q) = ds^a + A_\alpha^a(r, s)dr^\alpha\,,$$

and the action of A on a tangent vector $v_q = (\dot{r}^\alpha, \dot{s}^a)$ is given by

$$A(v_q) = (\dot{s}^a + A_\alpha^a\dot{r}^\alpha)\frac{\partial}{\partial s^a}\,. \tag{2.9.3}$$

For linear connections we require that the sum of two (local) horizontal sections be horizontal; i.e., if $(\dot{r}^\alpha, \dot{s}^a(r))$ and $(\dot{r}^\alpha, \dot{\hat{s}}^a(r))$ are horizontal, then so should be $\left(\dot{r}^\alpha, \dot{s}^a(r) + \dot{\hat{s}}^a(r)\right)$. Thus if we have

$$\dot{s}^a + A^a_\alpha(r,s)\dot{r}^\alpha = \dot{\hat{s}}^a + A^a_\alpha(r,\hat{s})\dot{r}^\alpha = 0,$$

then we require

$$\dot{s}^a + \dot{\hat{s}}^a + A^a_\alpha(r, s+\hat{s})\dot{r}^\alpha = 0\,.$$

Hence we take the connection coefficients be of the form

$$A^a_\alpha(r,s) = \Gamma^a_{\alpha b}(r)s^b\,. \tag{2.9.4}$$

If the bundle is the tangent bundle, these are called the ***components*** of the ***affine connection*** in the tangent bundle.

In the tangent bundle we have $s^a = \dot{r}^a$. We define ***geodesic motion*** along a curve $r(t)$ as being one for which the tangent vector is ***parallel transported*** along the curve; i.e., v_q along the curve is always horizontal, or $A(v_q)$ is zero. Making use of (2.9.3), this condition is

$$\ddot{r}^a + \Gamma^a_{bc}\dot{r}^b\dot{r}^b = 0\,. \tag{2.9.5}$$

This is the equation of geodesic motion. We can also determine this equation by another method, developed in what follows.

2.9.7 Example (Connections on $T\mathbb{R}^1$ continued). Returning to our system on $T\mathbb{R}^1$ suppose now that we have a curve $x(t)$ such that its tangent vector is parallel transported along the curve. In this case v_q along the curve being horizontal, or having $A(v_q)$ equal to zero, gives

$$\ddot{x} + a(x)\dot{x}^2 = 0\,.$$

For $a(x) = 0$ this reduces to $\ddot{x} = 0$, the equation of motion for a free particle on the line. Our example gives the generalization of this equation for arbitrary connections. ♦

Affine Connections and the Covariant Derivative. In the tangent bundle we can specify a linear connection by its action on vector fields, or by a map from vector fields (X, Y) to the vector field $\nabla_X Y$ that satisfies for smooth functions f and g and a vector fields X, Y, Z:

(i) $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$.

(ii) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$.

(iii) $\nabla_X(fY) = f\nabla_X Y + (\mathbf{d}f\cdot X)Y$, where $\mathbf{d}f \cdot X$ is the directional derivative of f along X, or Lie derivative.

Given a basis of vector fields $\frac{\partial}{\partial r_j}$ we can represent ∇ by

$$\nabla_{\partial/\partial r_i} \frac{\partial}{\partial r_j} = \Gamma^k_{ij} \frac{\partial}{\partial r_k}\,. \tag{2.9.6}$$

For X, Y vector fields given locally by $X = X^i(\partial/\partial r_i),\ Y = Y^i(\partial/\partial r_i)$, (i) and (iii) imply

$$\nabla_X Y = \left(X^j \frac{\partial Y^i}{\partial r^j} + X^k Y^j \Gamma^i_{kj} \right) \frac{\partial}{\partial r^i}. \tag{2.9.7}$$

The geodesic equations above then may be written

$$\nabla_{\dot r} \dot r = 0\,. \tag{2.9.8}$$

We can see this directly by a simple computation, again using (i) and (iii):

$$\begin{aligned}
\nabla_{\dot r^i(\partial/\partial r_i)} \dot r^j \frac{\partial}{\partial r_j} &= \dot r^i \nabla_{\partial/\partial r_i} \dot r^j \frac{\partial}{\partial r_j} \\
&= \dot r^i \frac{\partial}{\partial r_i} \dot r^j \frac{\partial}{\partial r_j} + \dot r^i \dot r^j \Gamma^k_{ij} \frac{\partial}{\partial r_k} \\
&= (\ddot r^j + \Gamma^j_{ik} \dot r^i \dot r^k) \frac{\partial}{\partial r_j} \quad \text{(by the chain rule).}
\end{aligned}$$

Sometimes we will write

$$\nabla_{\dot r} \dot r = \frac{D^2 r}{dt^2}, \qquad \frac{DX}{dt} = \nabla_{\dot r(t)} X. \tag{2.9.9}$$

We define DX/dt to be the ***covariant derivative***.

By (2.9.7), in local coordinates

$$\frac{DX}{dt} = \nabla_{\dot r} X = \left(\dot r^j \frac{\partial X^i}{\partial r^j} + \Gamma^i_{kj} X^k \dot r^j \right) \frac{\partial}{\partial r^i} = \left(\dot X^i + \Gamma^i_{kj} X^k \dot r^j \right) \frac{\partial}{\partial r^i}, \tag{2.9.10}$$

where $\dot r(t) = \dot r^i(\partial/\partial r^i)$. For $X = \dot r$ we of course recover the geodesic equations.

Curvature and Torsion. For an affine connection we define the curvature and torsion as follows. For X, Y, and Z arbitrary vector fields on M, the ***curvature tensor*** R and the ***torsion tensor*** are defined by

$$R(X,Y)(Z) = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}(Z)$$

and

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]\,.$$

Riemannian Connections. Now suppose M is endowed with a Riemannian metric g. This means that we can define orthonormal bases of $T_p(M)$ at each $p \in M$, and can define a subbundle P' of the frame bundle P whose fibers are orthonormal bases, and P' has structure group $\mathrm{O}(n)$. This subbundle is said to be a ***reduced bundle*** of P.

There exists a unique affine connection on M, called the ***Riemannian connection*** or ***Levi-Civita connection***, such that $\nabla g = 0$ and the torsion tensor T vanishes. An affine connection is called a metric connection if $\nabla g = 0$.

If the metric is given by $g = \sum g_{ij} dx^i dx^j$, the connection coefficients, which are called ***Christoffel symbols***, are given by

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left\{ \frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right\},$$

where, as usual, there is a sum over the index l understood.

Principal Connections. We now consider the special case of principal connections. We start with a free and proper group action of a Lie group G with Lie algebra $\mathfrak{g}$ on a manifold Q and construct the projection map $\pi : Q \to Q/G$; this setup is also referred to as a ***principal bundle***. The kernel $\ker T_q\pi$ (the tangent space to the group orbit through q) is called the vertical space of the bundle at the point q and is denoted by ver_q.

2.9.8 Definition. *A **principal connection** on the principal bundle $\pi : Q \to Q/G$ is a map (referred to as the **connection form**) $\mathcal{A} : TQ \to \mathfrak{g}$ that is linear on each tangent space (i.e., $\mathcal{A}$ is a $\mathfrak{g}$-valued one-form) and is such that*

(i) $\mathcal{A}(\xi_Q(q)) = \xi$ *for all* $\xi \in \mathfrak{g}$ *and* $q \in Q$, *and*

(ii) $\mathcal{A}$ *is equivariant:*

$$\mathcal{A}(T_q\Phi_g(v_q)) = \mathrm{Ad}_g \, \mathcal{A}(v_q)$$

for all $v_q \in T_qQ$ *and* $g \in G$, *where* Φ_g *denotes the given action of* G *on* Q *and where* Ad *denotes the adjoint action of* G *on* $\mathfrak{g}$.

The ***horizontal space*** of the connection at $q \in Q$ is the linear space

$$\mathrm{hor}_q = \{ v_q \in T_qQ \mid \mathcal{A}(v_q) = 0 \}.$$

Thus, at any point, we have the decomposition

$$T_qQ = \mathrm{hor}_q \oplus \mathrm{ver}_q.$$

Often one finds connections defined by specifying the horizontal spaces (complementary to the vertical spaces) at each point and requiring that they transform correctly under the group action. In particular, notice that a connection is uniquely determined by the specification of its horizontal spaces, a fact that we will use later on. We will denote the projections onto the horizontal and vertical spaces relative to the above decomposition using the same notation; thus, for $v_q \in T_qQ$, we write

$$v_q = \operatorname{hor}_q v_q + \operatorname{ver}_q v_q.$$

The projection onto the vertical part is given by

$$\operatorname{ver}_q v_q = (\mathcal{A}(v_q))_Q(q)\,,$$

and the projection to the horizontal part is thus

$$\operatorname{hor}_q v_q = v_q - (\mathcal{A}(v_q))_Q(q).$$

The projection map at each point defines an isomorphism from the horizontal space to the tangent space to the base; its inverse is called the ***horizontal lift***. Using the uniqueness theory of ODEs one finds that a curve in the base passing through a point $\pi(q)$ can be lifted uniquely to a horizontal curve through q in Q (i.e., a curve whose tangent vector at any point is a horizontal vector).

Since we have a splitting, we can also regard a principal connection as a special type of Ehresmann connection. However, Ehresmann connections are regarded as vertical-valued forms, whereas principal connections are regarded as Lie-algebra-valued. Thus, the Ehresmann connection A and the connection one-form $\mathcal{A}$ are different, and we will distinguish them; they are related in this case by

$$A(v_q) = (\mathcal{A}(v_q))_Q(q).$$

The general notions of curvature and other properties that hold for general Ehresmann connections specialize to the case of principal connections. As in the general case, given any vector field X on the base space, using the horizontal lift, there is a unique vector field X^h that is horizontal and that is π-related to X; that is, at each point q, we have

$$T_q\pi \cdot X^h(q) = X(\pi(q))\,,$$

and the vertical part is zero:

$$(\mathcal{A}(X^h_q))_Q(q) = 0.$$

It is well known (see, for example, Abraham, Marsden, and Ratiu [1988]) that the relation of being π-related is bracket-preserving; in our case, this means that

$$\operatorname{hor}\,[X^h, Y^h] = [X, Y]^h,$$

where X and Y are vector fields on the base.

2.9.9 Definition. *The* ***covariant exterior derivative*** $\mathbf{D}$ *of a Lie-algebra-valued one-form* α *is defined by applying the ordinary exterior derivative* $\mathbf{d}$ *to the horizontal parts of vectors:*

$$\mathbf{D}\alpha(X, Y) = \mathbf{d}\alpha(\mathrm{hor}\, X, \mathrm{hor}\, Y).$$

The ***curvature*** *of a connection* $\mathcal{A}$ *is its covariant exterior derivative, and it is denoted by* $\mathcal{B}$*.*

Thus, $\mathcal{B}$ is the Lie-algebra-valued two-form given by

$$\mathcal{B}(X, Y) = \mathbf{d}\mathcal{A}(\mathrm{hor}\, X, \mathrm{hor}\, Y).$$

Using the identity

$$(\mathbf{d}\alpha)(X, Y) = X[\alpha(Y)] - Y[\alpha(X)] - \alpha([X, Y])$$

together with the definition of horizontal shows that for two vector fields X and Y on Q, we have

$$\mathcal{B}(X, Y) = -\mathcal{A}([\mathrm{hor}\, X, \mathrm{hor}\, Y]),$$

where the bracket on the right-hand side is the Jacobi–Lie bracket of vector fields. The ***Cartan structure equations*** say that if X and Y are vector fields that are invariant under the group action, then

$$\mathcal{B}(X, Y) = \mathbf{d}\mathcal{A}(X, Y) - [\mathcal{A}(X), \mathcal{A}(Y)],$$

where the bracket on the right-hand side is the Lie algebra bracket. This follows readily from the definitions, the fact that $[\xi_Q, \eta_Q] = -[\xi, \eta]_Q$ (see equation (2.8.4)), the first property in the definition of a connection, and writing $\mathrm{hor}\, X = X - \mathrm{ver}\, X$, and similarly for Y, in the preceding formula for the curvature. The proof of the structure equations is given in the Internet supplement.

Remark. Given a general distribution $\mathcal{D} \subset TQ$ on a manifold Q one can also define its curvature in an analogous way directly in terms of its lack of integrability. Define vertical vectors at $q \in Q$ to be the quotient space $T_qQ/\mathcal{D}_q$ and define the curvature acting on two horizontal vector fields u, v (that is, two vector fields that take their values in the distribution) to be the projection onto the quotient of their Jacobi–Lie bracket. One can check that this operation depends only on the point values of the vector fields, so indeed defines a two-form on horizontal vectors.

The Maurer–Cartan Equations. A consequence of the structure equations relates curvature to the process of left and right trivialization.

2.9.10 Theorem (Maurer–Cartan Equations)**.** *Let* G *be a Lie group and let* $\rho : TG \to \mathfrak{g}$ *be the map that right translates vectors to the identity:*

$$\rho(v_g) = T_g R_{g^{-1}} \cdot v_g.$$

Then

$$\mathbf{d}\rho - [\rho, \rho] = 0.$$

Proof. Note that ρ is literally a connection on G for the left action. In considering this, keep in mind that for the action by left multiplication we have $\xi_Q(q) = T_e R_g \cdot \xi$. On the other hand, the curvature of this connection must be zero, since the shape space G/G is a point. Thus, the result follows from the structure equations. ∎

Of course, there is a similar result for the left trivialization λ, and we get the identity

$$\mathbf{d}\lambda + [\lambda, \lambda] = 0.$$

Bianchi Identities. The Bianchi identities are a famous set of identities for the Riemann curvature tensor of a given Riemannian metric. We defined the Riemann curvature tensor above for general affine connections. The relation between the Riemannian connection and the present formalism is to use the frame bundle as the bundle Q and think of it as a principal bundle over the underlying manifold M and the group $\mathrm{SO}(n)$ as the structure group. Then the curvature as defined here coincides with the Riemann curvature tensor. We will not go into this in detail here, since it is not needed for our present purposes, and instead we refer to Spivak [1979] or Kobayashi and Nomizu [1963] for an exposition of this. It is interesting that in the context of principal connections, the general proof is rather easy.

2.9.11 Theorem (Bianchi Identities). *We have the identity $D\mathcal{B} = 0$, that is, for any vector fields u, v, w on Q,*

$$\mathbf{d}\mathcal{B}(\mathrm{hor}(u), \mathrm{hor}(v), \mathrm{hor}(w)) = 0.$$

Proof. From the structure equations and the fact that $\mathbf{d}^2\mathcal{A} = 0$ we find that $\mathbf{d}\mathcal{B} = \mathbf{d}[\mathcal{A}, \mathcal{A}]$. Using the identity relating the exterior derivative and the Jacobi–Lie bracket of vector fields, we get

$$\begin{aligned}(\mathbf{d}[\mathcal{A}, \mathcal{A}])(\mathrm{hor}(u), \mathrm{hor}(v), \mathrm{hor}(w)) = &\,\mathrm{hor}(u)[\,[\mathcal{A}, \mathcal{A}](\mathrm{hor}(v), \mathrm{hor}(w))] + \text{cyclic} \\ &- ([\mathcal{A}, \mathcal{A}])([\mathrm{hor}(u), \mathrm{hor}(v)], \mathrm{hor}(w)) \\ &- \text{cyclic}.\end{aligned}$$

But all the terms in this expression are zero, since $\mathcal{A}$ vanishes on horizontal vectors. ∎

Local Formulas for the Connection. Pick a local trivialization of the bundle; that is, locally in the base, we write $Q = Q/G \times G$, where the action of G is given by left translation on the second factor. We choose coordinates r^α on the first factor and a basis e_a of the Lie algebra $\mathfrak{g}$ of G. We write coordinates of an element ξ relative to this basis as ξ^a. Let tangent vectors in this local trivialization at the point (r, g) be denoted by (u, w). We will write the action of $\mathcal{A}$ on this vector simply as $\mathcal{A}(u, w)$. Using this notation, we can write the connection form in this local trivialization as

$$\mathcal{A}(u, w) = \mathrm{Ad}_g(w_b + \mathcal{A}_{\mathrm{loc}}(r) \cdot u), \tag{2.9.11}$$

where w_b is the left translation of w to the identity (that is, the expression of w in "body coordinates"). The preceding equation defines the expression $\mathcal{A}_{\mathrm{loc}}(r)$. We define the connection components by writing

$$\mathcal{A}_{\mathrm{loc}}(r) \cdot u = \mathcal{A}^a_\alpha u^\alpha e_a.$$

We can also phrase this local representation in the following way:

2.9.12 Proposition. *In local coordinates $q = (g, r)$ a principal connection one-form can be written as*

$$\mathcal{A} = \mathrm{Ad}_g \left(g^{-1} dg + A(r) dr \right) , \tag{2.9.12}$$

so that

$$\mathcal{A} \cdot \dot{q} = \mathrm{Ad}_g \left(g^{-1} \dot{g} + A(r) \dot{r} \right) , \tag{2.9.13}$$

where $g^{-1}\dot{g}$ denotes the lifted action of g^{-1} on the tangent vector $\dot{g}$.

Proof. The infinitesimal generator of the action of the group on itself by the left action is of the form $\xi_G(g) = \xi g$, where again we are using shorthand for the lifted action. (Note that this is a push forward of ξ by the right action!)

Condition (i) of definition 2.9.8 then implies that

$$\mathcal{A}(r, g) \cdot (0, \xi g) = \xi \,.$$

Writing

$$(0, \xi g) = \xi g \frac{\partial}{\partial g}$$

we see that this holds if

$$\mathcal{A} = \mathrm{Ad}_g \left(g^{-1} dg \right) + A(r, g) dr \,.$$

Thus $\mathcal{A} \cdot \dot{q}$ must be of the form

$$\mathcal{A}(q) \cdot (\dot{r}, \dot{g}) = \dot{g} g^{-1} + A(r, g) \dot{r} \,.$$

Now the equivariance condition (ii) of definition 2.9.8 implies

$$\mathrm{Ad}_h A(g, r) = A(hg, r) \,.$$

Setting $h = g^{-1}$ this gives $\mathrm{Ad}_{g^{-1}} A(r, g) = A(r, e)$ or $A(r, g) = \mathrm{Ad}_g A(r, e) \equiv \mathrm{Ad}_g A(r)$, giving the result. ■

Local Formulas for the Curvature. Similarly, the curvature can be written in a local representation as

$$\mathcal{B}((u_1, w_1), (u_2, w_2)) = \mathrm{Ad}_g(\mathcal{B}_{\mathrm{loc}}(r) \cdot (u_1, u_2)),$$

which again serves to define the expression $\mathcal{B}_{\mathrm{loc}}(r)$. We can also define the coordinate form for the local expression of the curvature by writing

$$\mathcal{B}_{\mathrm{loc}}(r) \cdot (u_1, u_2) = \mathcal{B}^a_{\alpha\beta} u_1^\alpha u_2^\beta e_a.$$

Then one has the formula

$$\mathcal{B}^b_{\alpha\beta} = \left(\frac{\partial \mathcal{A}^b_\beta}{\partial r^\alpha} - \frac{\partial \mathcal{A}^b_\alpha}{\partial r^\beta} - C^b_{ac} \mathcal{A}^a_\alpha \mathcal{A}^c_\beta \right),$$

where C^b_{ac} are the structure constants of the Lie algebra defined by

$$[e_a, e_c] = C^b_{ac} e_b.$$

Parallel Translation and Holonomy Groups. Let P be a principal bundle with a connection and C a piecewise differentiable curve in its base space M with beginning point p and endpoint q. Suppose x is a point on the fiber over p. Then there is a unique curve C^*_x in P starting at x such that $\pi(C^*_x) = C$ and each tangent vector to C^*_x is horizontal. The curve C^*_x is said to be a *lift* of C that starts at x, and the map that takes x to the lift of q, the endpoint of the lifted curve, is said to be ***parallel translation***.

Now suppose C is a closed curve starting at p. Parallel translation then maps the point x to a point in the same fiber over p, xa say, $a \in G$. Thus each closed curve at p and fiber point x determines an element of G, and the set of all such elements forms a subgroup of G called the ***holonomy group*** of the connection with reference point x.

Holonomy for the Heisenberg Control System. A nice example of holonomy in action is for the Heisenberg control system:

$$\begin{aligned} \dot{x} &= u_1\,, \\ \dot{y} &= u_2\,, \\ \dot{z} &= xu_2 - yu_1\,. \end{aligned} \tag{2.9.14}$$

Here we consider the bundle $\mathbb{R}^3$ with base the xy-plane, fiber z, and connection

$$A = (dz - xdy + ydx)\frac{\partial}{\partial z}\,. \tag{2.9.15}$$

A horizontal curve has tangent vectors $(\dot{x}, \dot{y}, x\dot{y} - y\dot{x})$.

Now suppose we consider a loop in the base with z starting at the point z_0. Then the final position in the fiber, z_f, is given by

$$z_f - z_0 = \oint x\,dy - y\,dx\,. \tag{2.9.16}$$

By Green's theorem the right-hand side is just $2A$, where A is the area of the loop! Hence the term "nonholonomic integrator." This is also sometimes referred to as the ***area rule***. Recalling also our analysis of Lie brackets, note that that if the loop is a square with sides of length ϵ, then $A = \epsilon^2$.

We will see more analysis of this in Chapter 6, where we analyze the control of nonholonomic systems. For more on holonomy and phases, see Chapter 3.

Exercises

$\diamond$ **2.9-1.** Consider the trivial bundle $\mathbb{R}^4$ with base $\mathbb{R}^2$ parametrized by coordinates (θ, φ) and fibers $\mathbb{R}^2$ parametrized by coordinates (x, y). Compute the curvature of the connection given by the vertical rolling disk constraints $\omega_1 = dx - \cos\varphi \, d\theta$ and $\omega_2 = dy - \sin\varphi \, d\theta$.

$\diamond$ **2.9-2.** Consider the same space as above but with connection given by the integrable constraints $\omega_1 = dx - \cos\theta \, d\theta$ and $\omega_2 = dy - \sin\theta \, d\theta$. Show that the curvature of this connection is zero.

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