1
Introduction

1.1 The classical partial differential equations

In this introductory chapter, we give a brief survey of three main types of partial differential equations that occur in classical physics. We begin by establishing some convenient notation.

Let Ω be a domain (an open and connected set) in three-dimensional space $\mathbf{R}^3$, and let T be an open interval on the time axis. By $C^k(\Omega)$, resp. $C^k(\Omega \times T)$, we mean the set of all real-valued functions $u(x, y, z)$, resp. $u(x, y, z, t)$, with all their partial derivatives of order up to and including k defined and continuous in the respective regions. It is often practical to collect the three spatial coordinates (x, y, z) in a vector $\mathbf{x}$ and describe the functions as $u(\mathbf{x})$, resp. $u(\mathbf{x}, t)$. By Δ we mean the LAPLACE operator

$$\Delta = \nabla^2 := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} .$$

Partial derivatives will mostly be indicated by subscripts, e.g.,

$$u_t = \frac{\partial u}{\partial t} , \qquad u_{yx} = \frac{\partial^2 u}{\partial x \partial y} .$$

The first equation to be considered is called the *heat equation* or the *diffusion equation:*

$$\Delta u = \frac{1}{a^2} \frac{\partial u}{\partial t} , \qquad (\mathbf{x}, t) \in \Omega \times T.$$

As the name indicates, this equation describes conduction of heat in a homogeneous medium. The temperature at the point $\mathbf{x}$ at time t is given by $u(\mathbf{x}, t)$, and a is a constant that depends on the conducting properties of the medium. The equation can also be used to describe various processes of diffusion, e.g., the diffusion of a dissolved substance in the solvent liquid, neutrons in a nuclear reactor, BROWNian motion, etc.

The equation represents a category of second-order partial differential equations that is traditionally categorized as *parabolic*. Characteristically, these equations describe *non-reversible* processes, and their solutions are highly regular functions (of class C^∞).

In this book, we shall solve some special problems for the heat equation. We shall be dealing with situations where the spatial variable can be regarded as one-dimensional: heat conduction in a homogeneous rod, completely isolated from the exterior (except possibly at the ends of the rod). In this case, the equation reduces to

$$u_{xx} = \frac{1}{a^2}\, u_t \,.$$

The *wave equation* has the form

$$\Delta u = \frac{1}{c^2}\,\frac{\partial^2 u}{\partial t^2}\,, \qquad (\mathbf{x}, t) \in \Omega \times T.$$

where c is a constant. This equation describes vibrations in a homogeneous medium. The value $u(\mathbf{x}, t)$ is interpreted as the deviation at time t from the position at rest of the point with rest position given by $\mathbf{x}$.

The equation is a case of *hyperbolic* equations. Equations of this category typically describe reversible processes (the past can be deduced from the present and future by "reversion of time"). Sometimes it is even suitable to allow solutions for which the partial derivatives involved in the equation do not exist in the usual sense. (Think of shock waves such as the sonic bangs that occur when an aeroplane goes supersonic.) We shall be studying the *one-dimensional* wave equation later on in the book. This case can, for instance, describe the motion of a vibrating string.

Finally we consider an equation that does not involve time. It is called the *Laplace* equation and it looks simply like this:

$$\Delta u = 0.$$

It occurs in a number of physical situations: as a special case of the heat equation, when one considers a stationary situation, a *steady state*, that does not depend on time (so that $u_t = 0$); as an equation satisfied by the potential of a conservative force; and as an object of considerable purely mathematical interest. Together with the closely related POISSON equation, $\Delta u(\mathbf{x}) = F(\mathbf{x})$, where F is a known function, it is typical of equations classified as *elliptic.* The solutions of the Laplace equation are very regular functions: not only do they have derivatives of all orders, there are even certain possibilities to reconstruct the whole function from its local behaviour near a single point. (If the reader is familiar with analytic functions, this should come as no news in the two-dimensional case: then the solutions are harmonic functions that can be interpreted (locally) as real parts of analytic functions.)

The names *elliptic, parabolic,* and *hyperbolic* are due to superficial similarities in the appearance of the differential equations and the equations of conics in the plane. The precise definitions of the different types are as follows: The unknown function is $u = u(\mathbf{x}) = u(x_1, x_2, \ldots, x_m)$. The equations considered are *linear;* i.e., they can be written as a sum of terms equal to a known function (which can be identically zero), where each term in the sum consists of a coefficient (constant or variable) times some derivative of u, or u itself. The derivatives are of degree at most 2. By changing variables (possibly locally around each point in the domain), one can then write the equation so that no mixed derivatives occur (this is analogous to the diagonalization of quadratic forms). It then reduces to the form

$$a_1 u_{11} + a_2 u_{22} + \cdots + a_m u_{mm} + \{\text{terms containing } u_j \text{ and } u\} = f(\mathbf{x}),$$

where $u_j = \partial u/\partial x_j$ etc. If all the a_j have the same sign, the equation is elliptic; if at least one of them is zero, the equation is parabolic; and if there exist a_j's of opposite signs, it is hyperbolic.

An equation can belong to different categories in different parts of the domain, as, for example, the TRICOMI equation $u_{xx} + x u_{yy} = 0$ (where $u = u(x, y)$), which is elliptic in the right-hand half-plane and hyperbolic in the left-hand half-plane. Another example occurs in the study of the so-called velocity potential $u(x, y)$ for planar laminary fluid flow. Consider, for instance, an aeroplane wing in a streaming medium. In the case of *ideal* flow one has $\Delta u = 0$. Otherwise, when there is friction (air resistance), the equation looks something like $(1-M^2)u_{xx} + u_{yy} = 0$, with $M = v/v_0$, where v is the speed of the flowing medium and v_0 is the velocity of sound in the medium. This equation is elliptic, with nice solutions, as long as $v < v_0$, while it is hyperbolic if $v > v_0$ and then has solutions that represent shock waves (sonic bangs). Something quite complicated happens when the speed of sound is surpassed.

1.2 Well-posed problems

A *problem* for a differential equation consists of the equation together with some further conditions such as initial or boundary conditions of some form. In order that a problem be "nice" to handle it is often desirable that it have certain properties:

1. There *exists* a solution to the problem.
2. There exists *only one* solution (i.e., the solution is uniquely determined).
3. The solution is *stable*, i.e., small changes in the given data give rise to small changes in the appearance of the solution.

A problem having these properties (the third condition must be made precise in some way or other) is traditionally said to be *well posed*. It is, however, far from true that all physically relevant problems are well posed. The third condition, in particular, has caught the attention of mathematicians in recent years, since it has become apparent that it is often very hard to satisfy it. The study of these matters is part of what is popularly labeled chaos research.

To satisfy the reader's curiosity, we shall give some examples to illuminate the concept of well-posedness.

Example 1.1. It can be shown that for suitably chosen functions $f \in C^\infty$, the equation $u_x + u_y + (x + 2iy)u_t = f$ has no solution $u = u(x, y, t)$ at all (in the class of complex-valued functions) (Hans Lewy, 1957). Thus, in this case, condition 1 fails. □

Example 1.2. A natural problem for the heat equation (in one spatial dimension) is this one:

$$u_{xx}(x,t) = u_t(x,t),\ x > 0,\ t > 0; \quad u(x,0) = 0,\ x > 0; \quad u(0,t) = 0,\ t > 0.$$

This is a mathematical model for the temperature in a semi-infinite rod, represented by the positive x-axis, in the situation when at time 0 the rod is at temperature 0, and the end point $x = 0$ is kept at temperature 0 the whole time $t > 0$. The obvious and intuitive solution is, of course, that the rod will remain at temperature 0, i.e., $u(x,t) = 0$ for all $x > 0$, $t > 0$. But the mathematical problem has additional solutions: let

$$u(x,t) = \frac{x}{t^{3/2}}\, e^{-x^2/(4t)}, \quad x > 0,\ t > 0.$$

It is a simple exercise in partial differentiation to show that this function satisfies the heat equation; it is obvious that $u(0,t) = 0$, and it is an easy exercise in limits to check that $\lim\limits_{t \searrow 0} u(x,t) = 0$. The function must be considered a solution of the problem, as the formulation stands. Thus, the problem fails to have property 2.

The disturbing solution has a rather peculiar feature: it could be said to represent a certain (finite) amount of heat, located at the end point of the rod at time 0. The value of $u(\sqrt{2t}, t)$ is $\sqrt{(2/e)}/t$, which tends to $+\infty$ as $t \searrow 0$. One way of excluding it as a solution is adding some condition to the formulation of the problem; as an example it is actually sufficient to demand that a solution must be bounded. (We do not prove here that this does solve the dilemma.) □

Example 1.3. A simple example of instability is exhibited by an ordinary differential equation such as $y''(t) + y(t) = f(t)$ with initial conditions $y(0) = 1$, $y'(0) = 0$. If, for example, we take $f(t) = 1$, the solution is $y(t) = 1$. If we introduce a small perturbation in the right-hand member by taking $f(t) = 1 + \varepsilon \cos t$, where $\varepsilon \neq 0$, the solution is given by $y(t) = 1 + \frac{1}{2}\,\varepsilon t \sin t$. As time goes by, this expression will oscillate with increasing amplitude and "explode". The phenomenon is called *resonance*. □

1.3 The one-dimensional wave equation

We shall attempt to find *all* solutions of class C^2 of the one-dimensional wave equation

$$c^2\, u_{xx} = u_{tt}.$$

Initially, we consider solutions defined in the open half-plane $t > 0$.

Introduce new coordinates (ξ, η), defined by

$$\xi = x - ct, \qquad \eta = x + ct.$$

It is an easy exercise in applying the chain rule to show that

$$\begin{aligned} u_{xx} &= \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi\,\partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \\ u_{tt} &= \frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial \xi^2} - 2\,\frac{\partial^2 u}{\partial \xi\,\partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \right). \end{aligned}$$

Inserting these expressions in the equation and simplifying we obtain

$$c^2 \cdot 4\,\frac{\partial^2 u}{\partial \xi\,\partial \eta} = 0 \qquad \Longleftrightarrow \qquad \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \eta}\right) = 0.$$

Now we can integrate step by step. First we see that $\partial u/\partial \eta$ must be a function of only η, say, $\partial u/\partial \eta = h(\eta)$. If ψ is an antiderivative of h, another integration yields $u = \varphi(\xi) + \psi(\eta)$, where φ is a new arbitrary function. Returning to the original variables (x, t), we have found that

$$u(x,t) = \varphi(x - ct) + \psi(x + ct). \tag{1.1}$$

In this expression, φ and ψ are more-or-less arbitrary functions of one variable. If the solution u really is supposed to be of class C^2, we must demand that φ and ψ have continuous second derivatives.

It is illuminating to take a closer look at the significance of the two terms in the solution. First, assume that $\psi(s) = 0$ for all s, so that $u(x,t) =$

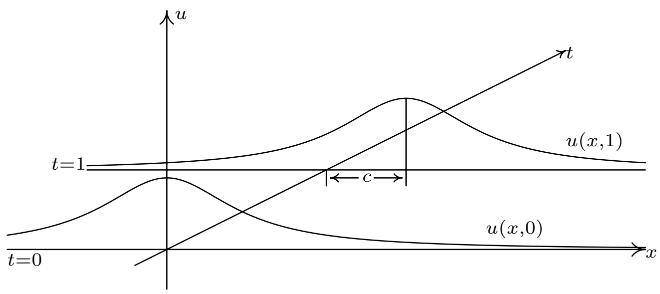

FIGURE 1.1.

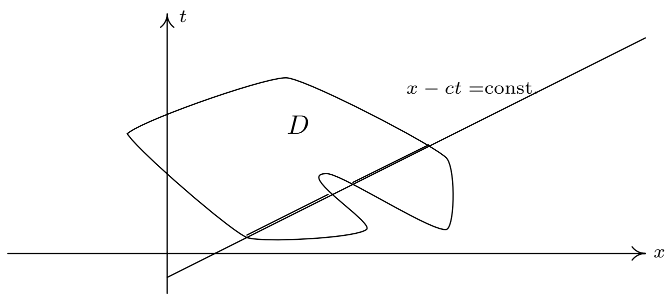

FIGURE 1.2.

$\varphi(x-ct)$. For $t=0$, the graph of the function $x \mapsto u(x,0)$ looks just like the graph of φ itself. At a later moment, the graph of $x \mapsto u(x,t)$ will have the same shape as that of φ, but it is pushed ct units of length to the right. Thus, the term $\varphi(x-ct)$ represents a *wave moving to the right along the x-axis* with constant speed equal to c. See Figure 1.1! In an analogous manner, the term $\psi(x+ct)$ describes a wave moving to the left with the same speed. The general solution of the one-dimensional wave equation thus consists of a superposition of two waves, moving along the x-axis in opposite directions.

The lines $x \pm ct =$ constant, passing through the half-plane $t>0$, constitute a net of level curves for the two terms in the solution. These lines are called the *characteristic curves* or simply *characteristics* of the equation. If, instead of the half-plane, we study solutions in some other region D, the derivation of the general solution works in the same way as above, as long as the characteristics run unbroken through D. In a region such as that shown in Figure 1.2, the function φ need not take on the same value on the two indicated sections that do lie on the same line but are not connected inside D. In such a case, the general solution must be described in a more complicated way. But if the region is *convex*, the formula (1.1) gives the general solution.

Remark. In a way, the general behavior of the solution is similar also in higher spatial dimensions. For example, the two-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

has solutions that represent wave-shapes passing the plane in all directions, and the general solution can be seen as a sort of superposition of such solutions. But here the directions are infinite in number, and there are both planar and circular wave-fronts to consider. The superposition cannot be realized as a sum — one has to use integrals. It is, however, usually of little interest to exhibit the general solution of the equation. It is much more valuable to be able to pick out some particular solution that is of importance for a concrete situation. □

Let us now solve a natural *initial value problem* for the wave equation in one spatial dimension. Let $f(x)$ and $g(x)$ be given functions on $\mathbf{R}$. We want to find all functions $u(x,t)$ that satisfy

$$\text{(P)} \qquad \begin{cases} c^2\, u_{xx} = u_{tt}\,, & -\infty < x < \infty, \quad t > 0; \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), & -\infty < x < \infty. \end{cases}$$

(The initial conditions assert that we know the shape of the solution at $t = 0$, and also its rate of change at the same time.) By our previous calculations, we know that the solution must have the form (1.1), and so our task is to determine the functions φ and ψ so that

$$f(x) = u(x,0) = \varphi(x)+\psi(x), \quad g(x) = u_t(x,0) = -c\,\varphi'(x)+c\,\psi'(x). \tag{1.2}$$

An antiderivative of g is given by $G(x) = \int_0^x g(y)\,dy$, and the second formula can then be integrated to

$$-\varphi(x) + \psi(x) = \frac{1}{c}\,G(x) + K,$$

where K is the integration constant. Combining this with the first formula of (1.2), we can solve for φ and ψ:

$$\varphi(x) = \frac{1}{2}\left(f(x) - \frac{1}{c}\,G(x) - K\right), \qquad \psi(x) = \frac{1}{2}\left(f(x) + \frac{1}{c}\,G(x) + K\right).$$

Substitution now gives

$$\begin{aligned} u(x,t) &= \varphi(x-ct) + \psi(x+ct) \\ &= \frac{1}{2}\left(f(x-ct) - \frac{1}{c}\,G(x-ct) - K + f(x+ct) + \frac{1}{c}\,G(x+ct) + K\right) \\ &= \frac{f(x-ct)+f(x+ct)}{2} + \frac{G(x+ct)-G(x-ct)}{2c} \\ &= \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy. \end{aligned} \tag{1.3}$$

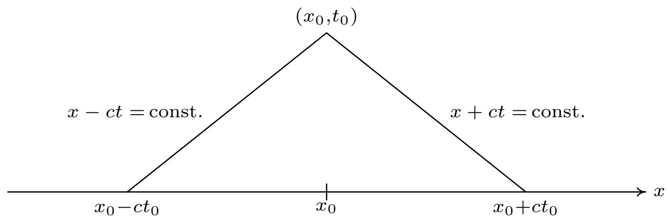

FIGURE 1.3.

The final result is called D'ALEMBERT'S formula. It is something as rare as an explicit (and unique) solution of a problem for a partial differential equation.

Remark. If we want to compute the value of the solution $u(x,t)$ at a particular point (x_0,t_0), d'Alembert's formula tells us that it is sufficient to know the initial values on the interval $[x_0-ct_0, x_0+ct_0]$: this is again a manifestation of the fact that the "waves" propagate with speed c. Conversely, the initial values taken on $[x_0-ct_0, x_0+ct_0]$ are sufficient to determine the solution in the isosceles triangle with base equal to this interval and having its other sides along characteristics. See Figure 1.3. □

In a similar way one can solve suitably formulated problems in other regions. We give an example for a semi-infinite spatial interval.

Example 1.4. Find all solutions $u(x,t)$ of $u_{xx}=u_{tt}$ for $x>0$, $t>0$, that satisfy $u(x,0)=2x$ and $u_t(x,0)=1$ for $x>0$ and, in addition, $u(0,t)=2t$ for $t>0$.

Solution. Since the first quadrant of the xt-plane is convex, all solutions of the equation must have the appearance

$$u(x,t)=\varphi(x-t)+\psi(x+t), \qquad x>0,\ t>0.$$

Our task is to determine what the functions φ and ψ look like. We need information about $\psi(s)$ when s is a positive number, and we must find out what $\varphi(s)$ is for all real s.

If $t=0$ we get $2x=u(x,0)=\varphi(x)+\psi(x)$ and $1=u_t(x,0)=-\varphi'(x)+\psi'(x)$; and for $x=0$ we must have $2t=\varphi(-t)+\psi(t)$. To liberate ourselves from the magic of letters, we neutralize the name of the variable and call it s. The three conditions then look like this, collected together:

$$\begin{cases} 2s=\varphi(s)+\psi(s) \\ \ \ 1=-\varphi'(s)+\psi'(s) \qquad\qquad s>0. \\ 2s=\varphi(-s)+\psi(s) \end{cases}$$

The second condition can be integrated to $-\varphi(s) + \psi(s) = s + C$, and combining this with the first condition we get

$$\varphi(s) = \tfrac{1}{2}\,s - \tfrac{1}{2}\,C, \quad \psi(s) = \tfrac{3}{2}\,s + \tfrac{1}{2}\,C \quad \text{for } s > 0.$$

The third condition then yields $\varphi(-s) = 2s - \psi(s) = \frac{1}{2}\,s - \frac{1}{2}\,C, \quad s > 0$, where we switch the sign of s to get

$$\varphi(s) = -\tfrac{1}{2}\,s - \tfrac{1}{2}\,C \quad \text{for } s < 0.$$

Now we put the solution together:

$$u(x,t) = \varphi(x-t) + \psi(x+t) = \begin{cases} \frac{1}{2}(x-t) + \frac{3}{2}(x+t) = 2x + t, x > t > 0, \\ \frac{1}{2}(t-x) + \frac{3}{2}(x+t) = x + 2t, 0 < x < t. \end{cases}$$

Evidently, there is just one solution of the given problem.

A closer look shows that this function is continuous along the line $x = t$, but it is in fact not differentiable there. It represents an "angular" wave. It seems a trifle fastidious to reject it as a solution of the wave equation, just because it is not of class C^2. One way to solve this conflict is furnished by the theory of *distributions*, which generalizes the notion of functions in such a way that even "angular" functions are assigned a sort of derivative. □

Exercise

1.1 Find the solution of the problem (P), when $f(x) = e^{-x^2}$, $g(x) = \dfrac{1}{1+x^2}$.

1.4 Fourier's method

We shall give a sketch of an idea that was tried by Jean-Baptiste Joseph Fourier in his famous treatise of 1822, *Théorie analytique de la chaleur.* It constitutes an attempt at solving a problem for the one-dimensional heat equation. If the physical units for heat conductivity, etc., are suitably chosen, this equation can be written as

$$u_{xx} = u_t\,,$$

where $u = u(x,t)$ is the temperature at the point x on a thin rod at time t. We assume the rod to be isolated from its surroundings, so that no exchange of heat takes place, except possibly at the ends of the rod. Let us now assume the length of the rod to be π, so that it can be identified with the interval $[0, \pi]$ of the x-axis. In the situation considered by Fourier, both ends of the rod are kept at temperature 0 from the moment when $t = 0$, and the temperature of the rod at the initial moment is assumed to be equal to a known function $f(x)$. It is then physically reasonable that we should be able to find the temperature $u(x,t)$ at any point x and at any time $t > 0$. The problem can be summarized thus:

$$\begin{cases} \text{(E)} \;\; u_{xx} = u_t, & 0 < x < \pi, \quad t > 0; \\ \text{(B)} \;\; u(0,t) = u(\pi,t) = 0, & t > 0; \\ \;\text{(I)} \;\; u(x,0) = f(x), & 0 < x < \pi. \end{cases} \tag{1.4}$$

The letters on the left stand for *equation, boundary conditions,* and *initial condition*, respectively. The conditions (E) and (B) share a specific property: if they are satisfied by two functions u and v, then all linear combinations $\alpha u + \beta v$ of them also satisfy the same conditions. This property is traditionally expressed by saying that the conditions (E) and (B) are *homogeneous.* Fourier's idea was to try to find solutions to the partial problem consisting of just these conditions, disregarding (I) for a while.

It is evident that the function $u(x,t) = 0$ for all (x,t) is a solution of the homogeneous conditions. It is regarded as a trivial and uninteresting solution. Let us instead look for solutions that are not identically zero. Fourier chose, possibly for no other reason than the fact that it turned out to be fruitful, to look for solutions having the particular form $u(x,t) = X(x)\,T(t)$, where the functions $X(x)$ and $T(t)$ depend each on just one of the variables.

Substituting this expression for u into the equation (E), we get

$$X''(x)\,T(t) = X(x)\,T'(t), \qquad 0 < x < \pi, \quad t > 0.$$

If we divide this by the product $X(x)\,T(t)$ (consciously ignoring the risk that the denominator might be zero somewhere), we get

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)}, \qquad 0 < x < \pi, \quad t > 0. \tag{1.5}$$

This equality has a peculiar property. If we change the value of the variable t, this does not affect the left-hand member, which implies that the right-hand member must also be unchanged. But this member is a function of only t; it must then be constant. Similarly, if x is changed, this does not affect the right-hand member and thus not the left-hand member, either. Indeed, we get that both sides of the equality are constant for all the values of x and t that are being considered. This constant value we denote (by tradition) by $-\lambda$. This means that we can split the formula (1.5) into two formulae, each being an ordinary differential equation:

$$X''(x) + \lambda X(x) = 0, \quad 0 < x < \pi; \qquad\qquad T'(t) + \lambda T(t) = 0, \quad t > 0.$$

One usually says that one has *separated the variables*, and the whole method is also called the method of *separation of variables*.

We shall also include the boundary condition (B). Inserting the expression $u(x,t) = X(x)\,T(t)$, we get

$$X(0)\,T(t) = X(\pi)\,T(t) = 0, \quad t > 0.$$

Now if, for example, $X(0) \neq 0$, this would force us to have $T(t) = 0$ for $t > 0$, which would give us the trivial solution $u(x,t) \equiv 0$. If we want to find interesting solutions we must thus demand that $X(0) = 0$; for the same reason we must have $X(\pi) = 0$. This gives rise to the following *boundary value problem* for X:

$$X''(x) + \lambda X(x) = 0, \quad 0 < x < \pi; \qquad X(0) = X(\pi) = 0. \tag{1.6}$$

In order to find nontrivial solutions of this, we consider the different possible cases, depending on the value of λ.

$\lambda < 0$: Then we can write $\lambda = -\alpha^2$, where we can just as well assume that $\alpha > 0$. The general solution of the differential equation is then $X(x) = Ae^{\alpha x} + Be^{-\alpha x}$. The boundary conditions become

$$\begin{cases} 0 = X(0) = A + B, \\ 0 = X(\pi) = Ae^{\alpha\pi} + Be^{-\alpha\pi}. \end{cases}$$

This can be seen as a homogeneous linear system of equations with A and B as unknowns and determinant $e^{-\alpha\pi} - e^{\alpha\pi} = -2\sinh\alpha\pi \neq 0$. It has thus a unique solution $A = B = 0$, but this leads to an uninteresting function X.

$\lambda = 0$: In this case the differential equation reduces to $X''(x) = 0$ with solutions $X(x) = Ax + B$, and the boundary conditions imply, as in the previous case, that $A = B = 0$, and we find no interesting solution.

$\lambda > 0$: Now let $\lambda = \omega^2$, where we can assume that $\omega > 0$. The general solution is given by $X(x) = A\cos\omega x + B\sin\omega x$. The first boundary condition gives $0 = X(0) = A$, which leaves us with $X(x) = B\sin\omega x$. The second boundary condition then gives

$$0 = X(\pi) = B\sin\omega\pi. \tag{1.7}$$

If here $B = 0$, we are yet again left with an uninteresting solution. But, happily, (1.7) can hold without B having to be zero. Instead, we can arrange it so that ω is chosen such that $\sin\omega\pi = 0$, and this happens precisely if ω is an integer. Since we assumed that $\omega > 0$ this means that ω is one of the numbers $1, 2, 3, \ldots$.

Thus we have found that the problem (1.6) has a nontrivial solution exactly if λ has the form $\lambda = n^2$, where n is a positive integer, and then the solution is of the form $X(x) = X_n(x) = B_n \sin nx$, where B_n is a constant.

For these values of λ, let us also solve the problem $T'(t) + \lambda T(t) = 0$ or $T'(t) = -n^2 T(t)$, which has the general solution $T(t) = T_n(t) = C_n e^{-n^2 t}$.

If we let $B_n C_n = b_n$, we have thus arrived at the following result: *The homogeneous problem* (E)+(B) *has the solutions*

$$u(x,t) = u_n(x,t) = b_n\, e^{-n^2 t} \sin nx, \quad n = 1, 2, 3, \ldots.$$

Because of the homogeneity, all sums of such expressions are also solutions of the same problem. Thus, the homogeneous sub-problem of the original problem (1.4) certainly has the solutions

$$u(x,t) = \sum_{n=1}^{N} b_n\, e^{-n^2 t} \sin nx, \tag{1.8}$$

where N is any positive integer and the b_n are arbitrary real numbers. The great question now is the following: among all these functions, can we find one that satisfies the *non-homogeneous* condition (I): $u(x,0) = f(x) =$ a known function?

Substitution in (1.8) gives the relation

$$f(x) = u(x,0) = \sum_{n=1}^{N} b_n \sin nx, \quad 0 < x < \pi. \tag{1.9}$$

If the function f happens to be a linear combination of sine functions of this kind, we can consider the problem as solved. Otherwise, it is rather natural to pose a couple of questions:

1. Can we permit the sum in (1.8) to consist of an *infinity* of terms?
2. Is it possible to approximate a (more or less) arbitrary function f using sums like the one in (1.9)?

The first of these questions can be given a partial answer using the theory of *uniform convergence*. The second question will be answered (in a rather positive way) later on in this book. We shall return to our heat conduction problem in Chapter 6.

Exercise

1.2 Find a solution of the problem treated in the text if the initial condition (I) is $u(x,0) = \sin 2x + 2\sin 5x$.

Historical notes

The partial differential equations mentioned in this section evolved during the eighteenth century for the description of various physical phenomena. The Laplace operator occurs, as its name indicates, in the works of Pierre Simon de Laplace, French astronomer and mathematician (1749–1827). In the theory of analytic functions, however, it had surely been known to Euler before it was given its name.

The wave equation was established in the middle of the eighteenth century and studied by several famous mathematicans, such as J. L. R. d'Alembert (1717–83), Leonhard Euler (1707–83) and Daniel Bernoulli (1700–82).

The heat equation came into focus at the beginning of the following century. The most important name in its early history is Joseph Fourier (1768–1830). Much of the contents of this book has its origins in the treatise *Théorie analytique de la chaleur.* We shall return to Fourier in the historical notes to Chapter 4.

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