3

Laplace and Z transforms

3.1 The Laplace transform

Let f be a function defined on the interval $\mathbf{R}_+ = [0, \infty[$. Alternatively, we can think of $f(t)$ as being defined for all real t, but satisfying $f(t) = 0$ for all $t < 0$. This can be expressed by writing

$$f(t) = f(t)H(t),$$

where H is the Heaviside function. Now let s be a real (or complex, if you like) number. If the integral

$$\widetilde{f}(s) = \int_0^\infty f(t)\, e^{-st}\, dt \tag{3.1}$$

exists (with a finite value), we say that it is the *Laplace transform of f, evaluated at the point s*. We shall write, interchangeably, $\widetilde{f}(s)$ or $\mathcal{L}[f](s)$. In applications, one also often uses the notation $F(s)$ (capital letter for the transform of the corresponding lower-case letter).

Example 3.1. Let $f(t) = e^{at}$, $t \geq 0$. Then,

$$\int_0^\infty f(t)\, e^{-st}\, dt = \int_0^\infty e^{at-st}\, dt = \left[\frac{e^{(a-s)t}}{a-s}\right]_{t=0}^{\infty} = \frac{1}{s-a},$$

provided that $a - s < 0$ so that the evaluation at infinity yields zero. Thus we have $\widetilde{f}(s) = 1/(s-a)$ for $s > a$, or

$$\mathcal{L}[e^{at}](s) = \frac{1}{s-a}, \quad s > a.$$

In particular, if $a = 0$, we have the Laplace transform of the constant function 1: it is equal to $1/s$ for $s > 0$. $\square$

Example 3.2. Let $f(t) = t$, $t > 0$. Then, integrating by parts, we get

$$\begin{aligned}\widetilde{f}(s) &= \int_0^\infty t\, e^{-st}\, dt = \left[t \cdot \frac{e^{-st}}{-s} \right]_{t=0}^{\infty} + \frac{1}{s} \int_0^\infty 1 \cdot e^{-st}\, dt \\ &= 0 + \frac{1}{s} \mathcal{L}[1](s) = \frac{1}{s^2}.\end{aligned}$$

This works for $s > 0$. $\square$

It may happen that the Laplace transform does not exist for any real value of s. Examples of this are given by $f(t) = 1/t$, $f(t) = e^{t^2}$.

A profound understanding of the workings of the Laplace transform requires considering it to be a so-called analytic function of a complex variable, but in most of this book we shall assume that the variable s is real. We shall, however, permit the function f to take complex values: it is practical to be allowed to work with functions such as $f(t) = e^{i\alpha t}$.

Furthermore, we shall assume that the integral (3.1) is not merely convergent, but that it actually converges *absolutely.* This enables us to estimate integrals, using the inequality $|\int f\,| \le \int |f|$.

Example 3.3. Let $f(t) = e^{ibt}$. Then we can imitate Example 3.1 above and write

$$\begin{aligned}\int_0^\infty f(t)\, e^{-st}\, dt = \int_0^\infty e^{(ib-s)t}\, dt &= \left[\frac{e^{(ib-s)t}}{ib-s} \right]_{t=0}^{\infty} \\ &= \frac{1}{ib-s} \big[e^{-st}(\cos bt + i \sin bt) \big]_{t=0}^{\infty}.\end{aligned}$$

For $s > 0$ the substitution as $t \to \infty$ will tend to zero, because the factor e^{-st} tends to zero and the rest of the expression is bounded. The result is thus that $\mathcal{L}[e^{ibt}](s) = 1/(s - ib)$, which means that the formula that we proved in Example 3.1 holds true also when a is purely imaginary. It is left to the reader to check that the same formula holds if a is an arbitrary complex number and $s > \operatorname{Re} a$. $\square$

It would be convenient to have some simple set of conditions on a function f that ensure that the Laplace transform is absolutely convergent for some value of s. Such a set of conditions is given in the following definition.

Definition 3.1 *Let k be a positive number. Assume that f has the following properties:*

(i) *f is continuous on $[0, \infty[$ except possibly for a finite number of jump discontinuities in every finite subinterval;*

(ii) *there is a positive number M such that $|f(t)| \le M e^{kt}$ for all $t \ge 0$.*

Then we say that f belongs to the class $\mathcal{E}_k$. If $f \in \mathcal{E}_k$ for some *value of k, we say that $f \in \mathcal{E}$.*

Using set notation we can say that $\mathcal{E} = \bigcup_{k>0} \mathcal{E}_k$. Condition (ii) means that f grows at most *exponentially;* this word lies behind the use of the letter $\mathcal{E}$. If $f \in \mathcal{E}_k$ for one value of k, then also $f \in \mathcal{E}_k$ for all *larger* k.

Theorem 3.1 *If $f \in \mathcal{E}_k$, then $\widetilde{f}(s)$ exists for all $s > k$.*

Proof. We begin by observing that condition (i) for the class $\mathcal{E}_k$ implies that the integral

$$\int_0^T f(t)\, e^{-st}\, dt$$

exists finitely for all s and all $T > 0$. Now assume $s > k$. Thus there exists a number M and a number t_0 so that $f(t)e^{-kt} \le M$ for $t > t_0$. Then we can estimate as follows:

$$\begin{aligned}\int_{t_0}^T |f(t)|\, e^{-st}\, dt &= \int_{t_0}^T |f(t)|\, e^{-kt}\, e^{-(s-k)t}\, dt \le \int_{t_0}^T M e^{-(s-k)t}\, dt \\ &\le M \int_{t_0}^\infty e^{-(s-k)t}\, dt \le M \int_0^\infty e^{-(s-k)t}\, dt = \frac{M}{s-k} < \infty.\end{aligned}$$

This means that the generalized integral over $[t_0, \infty[$ converges absolutely, and then this is equally true for the integral over $[0, \infty[$. $\square$

The result of the theorem can be "bootstrapped" in the following way. If $\sigma_0 = \inf\{k : f \in \mathcal{E}_k\}$, then the Laplace transform exists for all $s > \sigma_0$. Indeed, let $k = (s + \sigma_0)/2$, so that $\sigma_0 < k < s$; then $f \in \mathcal{E}_k$ (why?), and the theorem can be applied. The number σ_0 is a reasonably exact measure of the rate of growth of the function f. In what follows we shall sometimes use the notation σ_0 or $\sigma_0(f)$ for this measure.

As a consequence of the theorem we now know that a large set of common functions do have Laplace transforms. Among them are, e.g., polynomials, trigonometric functions such as sin and cos and ordinary exponential functions; also sums and products of such functions. If you have studied simple differential equations you may recall that these functions are precisely the possible solutions of homogeneous linear differential equations with constant coefficients, such as, for example,

$$y^{(\mathrm{v})} + 4y^{(\mathrm{iv})} - 8y''' + 15y'' - 24y' = 0.$$

We shall soon see that Laplace transforms give us a new technique for solving these equations. We shall also be able to solve more general problems, like integral equations of this kind:

$$\int_0^t f(t-x)\, f(x)\, dx + 3\int_0^t f(x)\, dx + 2t = 0, \quad t > 0. \tag{3.2}$$

Another consequence of the theorem is worth emphasizing: if a Laplace transform exists for *one* value of s, then it is also defined for all *larger* values of s. If we are dealing with several different transforms having various domains, we can always be sure that they are all defined at least in one common semi-infinite interval. It is customary to be rather sloppy about specifying the domains of definition for Laplace transforms: we make a tacit agreement that s is large enough so that all transforms occuring in a given situation are defined.

Exercises

3.1 Let $f(t) = e^{t^2}$, $g(t) = e^{-t^2}$. Show that $f \notin \mathcal{E}$, whereas $g \in \mathcal{E}_k$ for all k.

3.2 Compute the Laplace transform of $f(t) = e^{at}$, where $a = \alpha + i\beta$ is a complex constant.

3.3 Let $f(t) = \sin t$ for $0 \le t \le \pi$, $f(t) = 0$ otherwise. Find $\widetilde{f}(s)$.

3.2 Operations

The Laplace transformation obeys some simple rules of computation and also some less simple rules. The simplest ones are collected in the following table. Everywhere we assume that s takes sufficiently large values, as discussed at the end of the preceding section.

1. $\mathcal{L}[\alpha f + \beta g](s) = \alpha \widetilde{f}(s) + \beta \widetilde{g}(s)$, if α and β are constants.
2. $\mathcal{L}[e^{at} f(t)](s) = \widetilde{f}(s-a)$, if a is a constant (damping rule).
3. If we define $f(t) = 0$ for $t < 0$ and if $a > 0$, then
$$\mathcal{L}[f(t-a)](s) = e^{-as}\widetilde{f}(s) \quad \text{(delaying rule).}$$
4. $\mathcal{L}[f(at)](s) = \dfrac{1}{a}\widetilde{f}(s/a)$, if $a > 0$.

The proofs of these rules are easy. As an example we give the computations that yield rules 3 and 4:

$$\begin{aligned}
\mathcal{L}[f(t-a)](s) &= \int_0^\infty f(t-a)\, e^{-st}\, dt \left\{ \begin{gathered} u = t-a \\ du = dt \\ t = 0 \Leftrightarrow u = -a \end{gathered} \right\} \\
&= \int_{-a}^\infty f(u)\, e^{-s(u+a)}\, du = e^{-as} \int_{-a}^\infty f(u)\, e^{-su}\, du \\
&= e^{-as} \int_0^\infty f(u)\, e^{-su}\, du = e^{-as}\widetilde{f}(s);
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}[f(at)](s) &= \int_0^\infty f(at)\, e^{-st}\, dt \left\{ \begin{gathered} u = at \\ du = a\, dt \end{gathered} \right\} = \int_0^\infty f(u)\, e^{-s\cdot u/a}\, \frac{du}{a} \\
&= \frac{1}{a}\int_0^\infty f(u) \exp\left(-\frac{s}{a}\cdot u\right) du = \frac{1}{a}\widetilde{f}\left(\frac{s}{a}\right).
\end{aligned}$$

Example 3.4. Using rule 1 and the result of Example 3.3 in the preceding section, we can find the Laplace transforms of cos and sin:

$$\mathcal{L}[\cos bt](s) = \tfrac{1}{2}\mathcal{L}[e^{ibt} + e^{-ibt}](s) = \tfrac{1}{2}\left(\frac{1}{s-ib} + \frac{1}{s+ib}\right) = \frac{s}{s^2+b^2},$$
$$\mathcal{L}[\sin bt](s) = \frac{1}{2i}\mathcal{L}[e^{ibt} - e^{-ibt}](s) = \frac{1}{2i}\left(\frac{1}{s-ib} - \frac{1}{s+ib}\right) = \frac{b}{s^2+b^2}.$$

□

Example 3.5. Applying rule 2 to the result of Example 3.4 we get

$$\mathcal{L}[e^{at}\cos bt](s) = \frac{s-a}{(s-a)^2+b^2}, \qquad \mathcal{L}[e^{at}\sin bt](s) = \frac{b}{(s-a)^2+b^2}.$$

□

A couple of deeper rules are given in the following theorems.

Theorem 3.2 *If $f \in \mathcal{E}_{k_0}$, then $(t \mapsto tf(t)) \in \mathcal{E}_{k_1}$ for $k_1 > k_0$ and*

$$\mathcal{L}[tf(t)](s) = -\frac{d}{ds}\widetilde{f}(s).$$

Proof. We shall use a theorem on differentiation of integrals. In order to keep it lucid, we assume that f is continuous on the whole of $\mathbf{R}_+$; otherwise we would have to split into integrals over subintervals where f *is* continuous, and this introduces certain purely technical complications. Since $f \in \mathcal{E}_{k_0}$, we know that $|f(t)| \le Me^{k_0 t}$ for some number M and all sufficiently large t, say $t > t_1$. Let $\delta > 0$. Then there is a t_2 such that $|t| < e^{\delta t}$ for $t > t_2$. If $t > t_0 = \max(t_1, t_2)$ we have

$$|tf(t)| \le e^{\delta t} \cdot Me^{k_0 t} = Me^{(k_0+\delta)t} = Me^{k_1 t},$$

which means that $tf(t)$ belongs to $\mathcal{E}_{k_1}$ and has a Laplace transform.

If we differentiate the formula $\widetilde{f}(s) = \int_0^\infty f(t)\,e^{-st}\,dt$ formally with respect to s, we get $\left(\widetilde{f}\right)'(s) = \int_0^\infty (-t)f(t)\,e^{-st}\,dt$. According to the theorem concerning differentiation of integrals, this maneuver is permitted if we can find a "dominating" function g (that may depend on t but not on s) such that the integrand in the differentiated formula can be estimated by g for all $t \ge 0$ and all values of s that we consider, and which is such that $\int_0^\infty g$ is convergent. Let a be a number greater than the constant k_1 and put $g(t) = |tf(t)\,e^{-at}|$. For all $s \ge a$ we have then $|(-t)\,f(t)\,e^{-st}| \le g(t)$, and

$$\int_0^\infty g(t)\,dt = \int_0^\infty |tf(t)|e^{-at}\,dt \le M\int_0^\infty e^{k_1 t}\cdot e^{-at}\,dt$$
$$= M\int_0^\infty e^{-(a-k_1)t}\,dt = \frac{M}{a-k_1} < \infty.$$

This shows that the conditions for differentiating formally are fulfilled, and the theorem is proved. $\square$

Example 3.6. We know that $\mathcal{L}[1](s) = 1/s$ for $s > 0$. Then we can say that

$$\mathcal{L}[t](s) = \mathcal{L}[t \cdot 1](s) = -\frac{d}{ds}\frac{1}{s} = -\left(-\frac{1}{s^2}\right) = \frac{1}{s^2}, \quad s > 0.$$

Repeating this argument (do it!) we find that

$$\mathcal{L}[t^n](s) = \frac{n!}{s^{n+1}}, \quad s > 0.$$

$\square$

Example 3.7. Also, rule 2 allows us to conclude that

$$\mathcal{L}[t^n\, e^{at}](s) = \frac{n!}{(s-a)^{n+1}}, \quad s > 0.$$

$\square$

A sort of reverse of Theorem 3.2 is the following. The notation $f(0+)$ stands for the right-hand limit $\lim_{t \to 0+} f(t) = \lim_{t \searrow 0} f(t)$.

Theorem 3.3 *Assume that $f \in \mathcal{E}$ is continuous on $\mathbf{R}_+$. Also assume that the derivative $f'(t)$ exists for all $t \geq 0$ (with $f'(0)$ interpreted as the right-hand derivative) and that $f' \in \mathcal{E}$. Then*

$$\mathcal{L}[f'](s) = s\,\widetilde{f}(s) - f(0+).$$

Proof. Suppose that $f \in \mathcal{E}_{k_0}$ and $f' \in \mathcal{E}_{k_1}$, and take s to be larger than both k_0 and k_1. Let T be a positive number. Integration by parts gives

$$\int_0^T f'(t)\, e^{-st}\, dt = f(T)\, e^{-sT} - f(0+)\, e^0 + s \int_0^T f(t)\, e^{-st}\, dt.$$

When $T \to \infty$, the first term in the right-hand member tends to zero, and the result is the desired formula. $\square$

Theorem 3.3 will be used for solving differential equations.

The following theorem states a few additional properties of the Laplace transform.

Theorem 3.4 (a) *If $f \in \mathcal{E}$, then*

$$\lim_{s \to \infty} \widetilde{f}(s) = 0. \tag{3.3}$$

(b) The initial value rule: *If $f(0+)$ exists, then*

$$\lim_{s \to \infty} s\widetilde{f}(s) = f(0+). \tag{3.4}$$

(c) The final value rule: *If $f(t)$ has a limit as $t \to +\infty$, then*

$$\lim_{s \searrow 0+} s\widetilde{f}(s) = f(+\infty) = \lim_{t \to \infty} f(t). \tag{3.5}$$

In applications, the rule (3.5) is useful for deciding the ultimate or "steady-state" behavior of a function or a signal.

Proof. (a) Let $\varepsilon > 0$ be given and choose $\delta > 0$ so small that

$$\int_0^\delta |f(t)|\, dt < \varepsilon.$$

Let $k > 0$ be such that $f \in \mathcal{E}_k$ and let $s_0 > k$. Then for $s > s_0$ we get

$$\begin{aligned}
|\widetilde{f}(s)| &\le \int_0^\delta |f(t)|\, e^{-st}\, dt + \int_\delta^\infty |f(t)|\, e^{-st}\, dt \\
&\le \int_0^\delta |f(t)|\, dt + \int_\delta^\infty |f(t)|\, e^{-s_0 t} e^{-(s-s_0)t}\, dt \\
&\le \varepsilon + e^{-(s-s_0)\delta} \int_\delta^\infty |f(t)| e^{-s_0 t}\, dt \le \varepsilon + C e^{-(s-s_0)\delta} = \varepsilon + C e^{\delta s_0} \cdot e^{-\delta s}.
\end{aligned}$$

The last term tends to zero as $s \to \infty$ and thus it is less than ε if s is large enough. This proves that $|\widetilde{f}(s)| < 2\varepsilon$ for all sufficiently large s, and since ε can be arbitrarily small, we have proved (3.3).

(b) The idea of proof is similar to the preceding. $\varepsilon > 0$ is arbitrary, but now we choose $\delta > 0$ so small that $|f(t) - f(0+)| < \varepsilon$ for $0 < t < \delta$. With s_0 as above we get, for $s > s_0$,

$$\begin{aligned}
&s\widetilde{f}(s) \\
&= s \int_0^\delta (f(t) - f(0+))\, e^{-st}\, dt + s f(0+) \int_0^\delta e^{-st}\, dt + s \int_\delta^\infty f(t)\, e^{-st}\, dt.
\end{aligned}$$

The modulus of the first term is

$$\le s\varepsilon \int_0^\delta e^{-st}\, dt \le s\varepsilon \int_0^\infty e^{-st}\, dt = s\varepsilon \cdot \frac{1}{s} = \varepsilon, \quad \text{if } s > 0.$$

The second term can be computed:

$$= s f(0+) \frac{1 - e^{-s\delta}}{s} = f(0+)(1 - e^{-s\delta}) \to f(0+) \quad \text{as } s \to \infty.$$

Finally, the modulus of the third term can be estimated:

$$\le s \int_\delta^\infty |f(t)| e^{-s_0 t} e^{-(s-s_0)t}\, dt \le s e^{-s\delta} \cdot e^{s_0 \delta} \int_\delta^\infty |f(t)|\, e^{-s_0 t}\, dt = C s e^{-\delta s},$$

which tends to zero as $s \to \infty$. Just as in the proof of (3.3) we can draw the conclusion (3.4).

(c) This proof also runs along similar paths. We begin by writing

$$s\widetilde{f}(s) = s\int_0^T f(t)\,e^{-st}\,dt + s\int_T^\infty (f(t)-f(\infty))e^{-st}\,dt + f(\infty)e^{-sT}.$$

Choose T so large that $|f(t)-f(\infty)| < \varepsilon$ for $t \geq T$. The modulus of the first term can be estimated by $s\int_0^T |f| \to 0$ as $s \to 0+$, and the modulus of the second one is

$$\leq s\int_T^\infty \varepsilon\cdot e^{-st}\,dt = \varepsilon\,e^{-sT} \leq \varepsilon.$$

The proof is finished in an analogous way to the others. $\square$

We round off this section by a generalization of the rule for Laplace transformation of a power t (cf. Example 3.6). To this end we need a generalization of factorials to non-integers. This is provided by Euler's *Gamma function*, whis is defined by

$$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}\,du, \quad x > 0.$$

It is easy to see that this integral converges for positive x. It is also easy to see that $\Gamma(1) = 1$. Integrating by parts we find

$$\Gamma(x+1) = \int_0^\infty u^x e^{-u}\,du = \Big[-u^x e^{-u}\Big]_0^\infty + x\int_0^\infty u^{x-1}e^{-u}\,du = x\Gamma(x).$$

From this we deduce that $\Gamma(2) = 1\cdot\Gamma(1) = 1$, $\Gamma(3) = 2$, and, by induction, $\Gamma(n+1) = n!$ for integral n. Thus, this function can be viewed as an interpolation of the factorial.

Now we let $f(t) = t^a$, where $a > -1$. It is then clear that f has a Laplace transform, and we find, for $s > 0$,

$$\begin{aligned}\widetilde{f}(s) &= \int_0^\infty t^a e^{-st}\,dt \left\{\begin{array}{l} st = u \\ dt = du/s \end{array}\right\} = \int_0^\infty \left(\frac{u}{s}\right)^a e^{-u}\,\frac{du}{s} \\ &= \frac{1}{s^{a+1}}\int_0^\infty u^a e^{-u}\,du = \frac{\Gamma(a+1)}{s^{a+1}}\,.\end{aligned}$$

If a is an integer, this reduces to the formula of Example 3.6.

Exercises

3.4 Find the Laplace transforms of (a) $2t^2 - e^{-t}$
(b) $(t^2+1)^2$ (c) $(\sin t - \cos t)^2$ (d) $\cosh^2 4t$ (e) $e^{2t}\sin 3t$ (f) $t^3 \sin 3t$.

3.5 Compute the Laplace transform of $f(t) = \begin{cases} 1/\varepsilon & \text{for } 0 < t < \varepsilon, \\ 0 & \text{otherwise.} \end{cases}$

3.6 Find the transform of $f(t) = \begin{cases} (t-1)^2 & \text{for } t > 1, \\ 0 & \text{otherwise.} \end{cases}$

3.7 Solve the same problem for $f(t) = \displaystyle\int_0^t \frac{1 - e^{-u}}{u}\, du$.

3.8 Compute $\displaystyle\int_0^\infty t\, e^{-3t} \sin t\, dt$. (Hint: $\widetilde{f}(3)$!)

3.9 Find the Laplace transform of f, if we define $f(t) = t \sin t$ for $0 \le t \le \pi$, $f(t) = 0$ otherwise. (Hint: use the result of Exercise 3.3, p. 42.)

3.10 Find the Laplace transform of the function f defined by

$$f(t) = na \quad \text{for} \quad n - 1 \le x < n, \qquad n = 1, 2, 3, \ldots.$$

3.11 Compute $\mathcal{L}[te^{-t} \sin t](s)$.

3.12 Explain why the function $\dfrac{s^2}{s^2+1}$ cannot be the Laplace transform of any $f \in \mathcal{E}$.

3.13 Show that if f is periodic with period a, then

$$\widetilde{f}(s) = \frac{1}{1 - e^{-as}} \int_0^a f(t)\, e^{-st}\, dt.$$

(Hint: $\int_0^\infty = \sum_0^\infty \int_{ak}^{a(k+1)}$. Let $u = t - ak$, use the formula for the sum of a geometric series.)

3.14 Find the Laplace transform of the function with period 1 that is described by $f(t) = t$ for $0 < t < 1$.

3.15 Verify the final value rule (3.5) for $\widetilde{f}(s) = 1/(s(s+1))$ by comparing $f(t)$ and $\lim\limits_{s \to 0+} s\widetilde{f}(s)$.

3.16 Prove that $\Gamma\big(\frac{1}{2}\big) = \sqrt{\pi}$. What are the values of $\Gamma\big(\frac{3}{2}\big)$ and $\Gamma\big(\frac{5}{2}\big)$?

3.3 Applications to differential equations

Example 3.8. Let us try to solve the initial value problem

$$y'' - 4y' + 3y = t, \quad t > 0; \qquad y(0) = 3, \quad y'(0) = 2. \tag{3.6}$$

We assume that $y = y(t)$ is a solution such that y, as well as y' and y'', has a Laplace transform. By Theorem 3.3 we have then

$$\begin{aligned}
&\mathcal{L}[y'](s) = s\widetilde{y} - y(0) = s\widetilde{y} - 3, \\
&\mathcal{L}[y''](s) = s\mathcal{L}[y'](s) - y'(0) = s(s\widetilde{y} - 3) - 2 = s^2\widetilde{y} - 3s - 2.
\end{aligned}$$

Due to linearity, we can transform the left-hand side of the equation to get

$$(s^2\widetilde{y} - 3s - 2) - 4(s\widetilde{y} - 3) + 3\widetilde{y} = (s^2 - 4s + 3)\widetilde{y} - 3s + 10,$$

and this must be equal to the transform of the right-hand side, which is $1/s^2$. The result is an algebraic equation, which we can solve for $\widetilde{y}$:

$$(s^2-4s+3)\widetilde{y}-3s+10=\frac{1}{s^2} \iff \widetilde{y}=\frac{3s^3-10s^2+1}{s^2(s^2-4s+3)}=\frac{3s^3-10s^2+1}{s^2(s-1)(s-3)}.$$

The last expression can be expanded into partial fractions. Assume that

$$\frac{3s^3-10s^2+1}{s^2(s-1)(s-3)}=\frac{A}{s^2}+\frac{B}{s}+\frac{C}{s-1}+\frac{D}{s-3}.$$

Multiplying by the common denominator and identifying coefficients we find that $A=\frac{1}{3}$, $B=\frac{4}{9}$, $C=3$, and $D=-\frac{4}{9}$. Thus we have

$$\widetilde{y}=\tfrac{1}{3}\cdot\frac{1}{s^2}+\tfrac{4}{9}\cdot\frac{1}{s}+3\cdot\frac{1}{s-1}-\tfrac{4}{9}\cdot\frac{1}{s-3}.$$

It so happens that there exists a function with precisely this Laplace transform, namely, the function

$$z=\tfrac{1}{3}t+\tfrac{4}{9}+3e^t-\tfrac{4}{9}e^{3t}.$$

Could it be the case that $y=z$? One way of finding this out is by differentiating and investigating if indeed z does satisfy the equation and initial conditions. And it does (check for yourself)! By the general theory of differential equations, the problem (3.6) has a unique solution, and it follows that z must be the solution we are looking for. □

The example demonstrates a very useful method for treating linear intitial value problems. There is one difficulty that is revealed at the end of the example: could it be possible that two *different* functions might have the same Laplace transform? This question is answered by the following theorem.

Theorem 3.5 (Uniqueness for Laplace transforms) *If f and g both belong to $\mathcal{E}$, and $\widetilde{f}(s)=\widetilde{g}(s)$ for all (sufficiently) large values of s, then $f(t)=g(t)$ for all values of t where f and g are continuous.*

We omit the proof of this at this point. It is given in Sec. 7.10. In that section we also prove a formula for the reconstruction of $f(t)$ when $\widetilde{f}(s)$ is known — a so-called *inversion formula* for the Laplace transform. The present theorem, however, gives us the possibility to invert Laplace transforms by *recognizing* functions, just as we did in the example.

This requires that we have access to a table of Laplace transforms of such functions that can be expected to occur. Such a table is found at the end of the book (p. 247 ff), and similar tables are included in all decent handbooks on the subject. Several of the entries in such tables have already been proved in the examples of this chapter; others can be done as exercises by the interested student.

We point out that the uniqueness result as such does not rule out the possibility that a differential equation (or other problem) may have solutions that have no Laplace transforms, e.g., solutions that grow faster than exponentially. To preclude such solutions one must look into the theory of differential equations. For linear equations there is a result on unique solutions for initial value problems, which may serve the purpose. If the coefficients are constants and the equation is homogeneous, one actually knows that all solutions have at most exponential growth.

The Laplace transform method is ideally adapted to solving initial value problems. Strictly speaking, the method takes into consideration only what goes on for $t \geq 0$. Very often, however, the expressions obtained for the solutions are also valid for $t < 0$.

We include some examples on using a table of Laplace transforms in a few more complicated situations. The technique may remind the reader of the integration of rational functions.

Example 3.9. Find $f(t)$, when $\widetilde{f}(s) = \dfrac{2s+3}{s^2+4s+13}$.

Solution. Complete the square in the denominator: $s^2+4s+13 = (s+2)^2+9$. Then split the numerator to enable us to recognize transforms of cosines and sines:

$$\frac{2s+3}{s^2+4s+13} = \frac{2(s+2)-1}{(s+2)^2+3^2} = 2 \cdot \frac{s+2}{(s+2)^2+3^2} - \tfrac{1}{3} \cdot \frac{3}{(s+2)^2+3^2},$$

and now we can see that this is the transform of $f(t) = 2e^{-2t}\cos 3t - \frac{1}{3}e^{-2t}\sin 3t$. □

Example 3.10. Find $g(t)$, if $\widetilde{g}(s) = \dfrac{2s}{(s^2+1)^2}$.

Solution. We recognize the transform as a derivative:

$$\widetilde{g}(s) = -\frac{d}{ds}\,\frac{1}{s^2+1}.$$

By Theorem 3.2 and the known transform of the sine we get $g(t) = t\sin t$. □

Example 3.11. Solve the initial value problem

$$y'' + 4y' + 13y = 13, \quad y(0) = y'(0) = 0.$$

Solution. Transformation gives

$$(s^2+4s+13)\widetilde{y} = \frac{13}{s} \iff \widetilde{y} = \frac{13}{s\big((s+2)^2+9\big)}.$$

Expand into partial fractions:

$$\widetilde{y} = \frac{1}{s} - \frac{s+4}{(s+2)^2+9} = \frac{1}{s} - \frac{s+2}{(s+2)^2+9} - \tfrac{2}{3} \cdot \frac{3}{(s+2)^2+9}\,.$$

The solution is found to be

$$y(t) = \big(1 - e^{-2t}(\cos 3t + \tfrac{2}{3}\sin 3t)\big)H(t).$$

(Here we have multiplied the result by a Heaviside factor, to indicate that we are considering the solution only for $t \geq 0$. This factor is often omitted. Whether or not it should be there is often a matter of dispute among users of the transform.) □

We can also treat *systems* of differential equations.

Example 3.12. Solve the initial value problem

$$\begin{cases} x' = x + 3y, \\ y' = 3x + y; \end{cases} \qquad x(0) = 5, \quad y(0) = 1.$$

Solution. Laplace transformation gives

$$\begin{cases} s\widetilde{x} - 5 = \widetilde{x} + 3\widetilde{y} \\ s\widetilde{y} - 1 = 3\widetilde{x} + \widetilde{y} \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} (1-s)\widetilde{x} + 3\widetilde{y} = -5 \\ 3\widetilde{x} + (1-s)\widetilde{y} = -1 \end{cases}$$

We can, for example, solve the second equation for $\widetilde{x} = \frac{1}{3}(s-1)\widetilde{y} - \frac{1}{3}$ and substitute this into the first, whereupon simplification yields $(s^2-2s-8)\widetilde{y} = s + 14$ and

$$\widetilde{y} = \frac{s+14}{(s-4)(s+2)} = \frac{3}{s-4} - \frac{2}{s+2}.$$

We see that $y = 3e^{4t} - 2e^{-2t}$, and then we deduce, in one way or another, that $x = 3e^{4t} + 2e^{-2t}$. (Think of at least three different ways of performing this last step!) □

Finally, we demonstrate how even a *partial* differential equation can be treated by Laplace transforms. The trick is to transform with respect to one of the independent variables and let the others stand. Using this technique often involves taking rather bold chances in the hope that rules of computation be valid. One way of regarding this is to view it precisely as taking chances – if we arrive at a tentative solution, it can always be checked by substitution in the original problem.

Example 3.13. Find a solution of the problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < 1,\ t > 0; \qquad \begin{aligned} &u(0,t) = 1,\ u(1,t) = 1, \quad t > 0; \\ &u(x,0) = 1 + \sin \pi x, \quad 0 < x < 1. \end{aligned}$$

Solution. We introduce the Laplace transform $U(x, s)$ of $u(x, t)$, i.e.,

$$U(x, s) = \mathcal{L}[t \mapsto u(x, t)](s) = \int_0^\infty u(x, t)\, e^{-st}\, dt.$$

Here, x is thought of as a constant. Then we change our attitude and assume that this integral can be differentiated with respect to x, indeed twice, so that

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial^2}{\partial x^2} \int_0^\infty u(x, t)\, e^{-st}\, dt = \int_0^\infty \frac{\partial^2}{\partial x^2} u(x, t)\, e^{-st}\, dt.$$

The differential equation is then transformed into

$$\frac{\partial^2 U}{\partial x^2} = sU - (1 + \sin \pi x), \qquad 0 < x < 1,$$

and the boundary conditions into

$$U(0, s) = \frac{1}{s}, \qquad U(1, s) = \frac{1}{s}.$$

Now we switch attitudes again: think of s as a constant and solve the boundary value problem. Just to feel comfortable we could write the equation as

$$U'' - sU = -1 - \sin \pi x. \tag{3.7}$$

The homogeneous equation has a characteristic equation $r^2 - s = 0$ and its solution is $U_H = Ae^{x\sqrt{s}} + Be^{-x\sqrt{s}}$. (Here, the "constants" A and B are in general functions of s.) A particular solution to the inhomogeneous equation could have the form $U_P = a + b \sin \pi x + c \cos \pi x$, and insertion and identification gives $a = 1/s$, $b = 1/(s + \pi^2)$, $c = 0$. Thus the general solution of (3.7) is

$$U(x, s) = A(s)e^{x\sqrt{s}} + B(s)e^{-x\sqrt{s}} + \frac{1}{s} + \frac{\sin \pi x}{s + \pi^2}.$$

The boundary conditions force us to take $A(s) = B(s) = 0$, so we are left with $U(x, s) = \dfrac{1}{s} + \dfrac{\sin \pi x}{s + \pi^2}$. Now we again consider x as a constant and recognize that U is the Laplace transform of $u(x, t) = 1 + e^{-\pi^2 t} \sin \pi x$. The fact that this function really does solve the original problem must be checked directly (since we have made an assumption on differentiability of an integral, which might have been too bold). □

Remark. This problem can also be attacked by other methods developed in later parts of the book (Chapter 6). □

Exercises

3.17 Invert the following Laplace transforms: (a) $\dfrac{1}{s(s+1)}$ (b) $\dfrac{3}{(s-1)^2}$
(c) $\dfrac{1}{s(s+2)^2}$ (d) $\dfrac{5}{s^2(s-5)^2}$ (e) $\dfrac{1}{(s-a)(s-b)}$ (f) $\dfrac{1}{s^2+4s+29}$.

3.18 Use partial fractions to find f when $\widetilde{f}(s)$ is given by
(a) $s^{-2}(s+1)^{-1}$, (b) $b^2 s^{-1}(s^2+b^2)^{-1}$, (c) $s(s-3)^{-5}$,
(d) $(s^2+2)s^{-1}(s+1)^{-1}(s+2)^{-1}$.

3.19 Invert the following Laplace transforms: (a) $\dfrac{1+e^{-s}}{s}$ (b) $\dfrac{e^{-s}}{(s-1)(s-2)}$
(c) $\ln\dfrac{s+3}{s+2}$ (d) $\ln\dfrac{s^2+1}{s(s+3)}$ (e) $\dfrac{s+1}{s^{4/3}}$ (f) $\dfrac{\sqrt{s}-1}{s}$.

3.20 Solve the initial value problem $y''+y=2e^t$, $t>0$, $y(0)=y'(0)=2$.

3.21 Solve the initial value problem $\begin{cases} y''(t)-2y'(t)+y(t)=t\,e^t\,\sin t, \\ y(0)=0,\ y'(0)=0. \end{cases}$

3.22 Solve $\begin{cases} y^{(3)}(t)-y''(t)+4y'(t)-4y(t)=-3e^t+4e^{2t}, \\ y(0)=0,\ y'(0)=5,\ y''(0)=3. \end{cases}$

3.23 Solve the system $\begin{cases} x'(t)+y'(t)=t, \\ x''(t)-y(t)=e^{-t}, \\ x(0)=3,\ x'(0)=-2,\ y(0)=0. \end{cases}$

3.24 Solve the system $\begin{cases} x'(t)-y'(t)-2x(t)+2y(t)=\sin t, \\ x''(t)+2y'(t)+x(t)=0, \\ x(0)=x'(0)=y(0)=0. \end{cases}$

3.25 Solve the problem

$$y''(t)-3y'(t)+2y(t)=\begin{cases} 1,\ t>2 \\ 0,\ t<2 \end{cases} ;\quad y(0)=1,\ y'(0)=0.$$

3.26 Solve the system

$$\begin{cases} \dfrac{dy}{dt}=2z-2y+e^{-t} \\ \dfrac{dz}{dt}=y-3z \end{cases} \qquad t>0; \qquad y(0)=1, \quad z(0)=2.$$

3.27 Solve the differential equation

$$2y^{(\mathrm{iv})}+y'''-y''-y'-y=t+2, \quad t>0,$$

with initial conditions $y(0)=y'(0)=0$, $y''(0)=y'''(0)=1$.

3.28 Solve the differential equation

$$y''+3y'+2y=e^{-t}\sin t, \quad t>0; \qquad y(0)=1, \quad y'(0)=-3.$$

3.4 Convolution

In control theory, for example, one studies the effect on an incoming signal by a "black box" that transforms it into an "outsignal":

$$\xrightarrow{\text{insignal}} \boxed{\begin{matrix}\text{black}\\ \text{box}\end{matrix}} \xrightarrow{\text{outsignal}}$$

Let the insignal be represented by the function $t \mapsto x(t)$, $t \geq 0$, and the outsignal by $t \mapsto y(t)$, $t \geq 0$. We assume that the system has four important properties:

(a) it is *linear*, which means that a linear combination of inputs results in the corresponding linear combination of outputs;

(b) it is *translation invariant*, which means, loosely, that the black box operates in the same way at all points in time;

(c) it is *continuous* in the sense that "small" changes in the input generate "small" changes in the output (which should be formulated more precisely when necessary);

(d) it is *causal*, i.e., the outsignal at a certain moment t does not depend on the insignal at moments *later* than t.

It can then be shown (see Appendix A) that there exists a function $t \mapsto g(t)$, $t \geq 0$, such that

$$y(t) = \int_0^t x(u)g(t-u)\,du = \int_0^t x(t-u)g(u)\,du. \tag{3.8}$$

The function g can be said to contain all information about the system.

The formula (3.8) is an example of a notion called the *convolution* of the two functions x and g. (We shall encounter other versions of convolution in other parts of this book.) We shall now study this notion from a mathematical point of view.

Thus, we assume that f and g are two functions, both belonging to $\mathcal{E}$. The *convolution* $f * g$ is a new function defined by the formula

$$(f*g)(t) = f*g(t) = \int_0^t f(u)\,g(t-u)\,du, \qquad t \geq 0.$$

It is not hard to see that this function is continuous on $[0, \infty[$, and it might possibly belong to $\mathcal{E}$. Indeed, it is not very difficult to show directly that if $f \in \mathcal{E}_{k_1}$ and $g \in \mathcal{E}_{k_2}$, then $f * g \in \mathcal{E}_k$ for all $k > \max(k_1, k_2)$. (See Exercise 3.38.) Using the notation $\sigma_0(f)$, introduced after Theorem 3.1, we could express this as $\sigma_0(f*g) \leq \max\bigl(\sigma_0(f), \sigma_0(g)\bigr)$.)

Convolution can be regarded as an operation for functions, a sort of "multiplication." For this operation a few simple rules hold; the reader is invited to check them out:

$$\begin{aligned} f * g &= g * f && \text{(commutative law)} \\ f * (g * h) &= (f * g) * h && \text{(associative law)} \\ f * (g + h) &= f * g + f * h && \text{(distributive law)} \end{aligned}$$

Example 3.14. Let $f(t) = e^t$, $g(t) = e^{-2t}$. Then

$$\begin{aligned} f * g(t) &= \int_0^t e^u \, e^{-2(t-u)} \, du = \int_0^t e^{u-2t+2u} \, du = e^{-2t} \int_0^t e^{3u} \, du \\ &= e^{-2t} \left[\tfrac{1}{3} \, e^{3u} \right]_{u=0}^{u=t} = \tfrac{1}{3} e^{-2t} \big(e^{3t} - 1 \big) = \frac{e^t - e^{-2t}}{3}. \end{aligned}$$

□

Example 3.15. If $g(t) = 1$, then $f * g(t) = \int_0^t f(u) \, du$. Thus, "integration" can be considered to be convolution with the function 1. □

When dealing with convolutions, the Laplace transform is useful because of the following theorem.

Theorem 3.6 *The Laplace transform of a convolution is the product of the Laplace transforms of the two convolution factors:*

$$\mathcal{L}[f * g](s) = \widetilde{f}(s) \, \widetilde{g}(s).$$

Proof. Let s be so large that both $\widetilde{f}(s)$ and $\widetilde{g}(s)$ exist. We have agreed in section 3.1 that this means that the corresponding integrals converge absolutely. Now consider the improper double integral

$$\iint_Q |f(u)g(v)| e^{-s(u+v)} \, du \, dv,$$

where Q is the first quadrant in the uv plane. The integrated function being positive, the integral can be calculated just as we choose. For example, we can write

$$\begin{aligned} \iint_Q |f(u)g(v)| e^{-s(u+v)} \, du \, dv &= \int_0^\infty du \int_0^\infty |f(u)||g(v)| e^{-su} e^{-sv} \, dv \\ &= \int_0^\infty |f(u)| e^{-su} \, du \int_0^\infty |g(v)| e^{-sv} \, dv. \end{aligned}$$

The two one-dimensional integrals here are assumed to be convergent, which means that the double integral also converges. But this in turn means that the improper double integral *without* modulus signs,

$$\Phi(s) = \iint_Q f(u)g(v) e^{-s(u+v)} \, du \, dv$$

is absolutely convergent. It can then also be computed in any manner, and we do it in two ways. One way is imitating the previous calculation:

$$\begin{aligned}\Phi(s) &= \int_0^\infty du \int_0^\infty f(u)g(v)e^{-su}e^{-sv}\,dv \\ &= \int_0^\infty f(u)e^{-su}\,du \int_0^\infty g(v)e^{-sv}\,dv = \widetilde{f}(s)\,\widetilde{g}(s).\end{aligned}$$

Another way is integrating on triangles $D_T : u \geq 0,\ v \geq 0,\ u+v \leq T$. But

$$\begin{aligned}&\int_0^T f*g(t)e^{-st}\,dt = \int_0^T \left(\int_0^t f(u)g(t-u)\,du\right)e^{-su}\,dt \\ &= \int_0^T dt \int_0^t f(u)e^{-su}\,g(t-u)e^{-s(t-u)}\,du \\ &= \int_0^T f(u)e^{-su}\,du \int_u^T g(t-u)e^{-s(t-u)}\,dt = \begin{Bmatrix} t-u=v \\ dt=dv \end{Bmatrix} \\ &= \int_0^T f(u)e^{-su}\,du \int_0^{T-u} g(v)e^{-sv}\,dv \\ &= \iint_{D_T} f(u)g(v)e^{-su}e^{-sv}\,du\,dv \to \Phi(s)\end{aligned}$$

as $T \to \infty$. This proves the formula in the theorem. □

Example 3.16. As an illustration of the theorem we can take the situation in Example 3.14. There we have

$$\begin{aligned}\widetilde{f}(s) &= \frac{1}{s-1}, \qquad \widetilde{g}(s) = \frac{1}{s+2}, \\ \widetilde{f}(s)\widetilde{g}(s) &= \frac{1}{(s-1)(s+2)} = \frac{\frac{1}{3}}{s-1} - \frac{\frac{1}{3}}{s+2} = \mathcal{L}[f*g](s).\end{aligned}$$

□

Example 3.17. Find a function f that satisfies the integral equation

$$f(t) = 1 + \int_0^\infty f(t-u)\sin u\,du, \qquad t \geq 0.$$

Solution. Suppose that $f \in \mathcal{E}$. Then we can transform the equation to get

$$\widetilde{f}(s) = \frac{1}{s} + \widetilde{f}(s)\cdot\frac{1}{s^2+1},$$

from which we solve

$$\widetilde{f}(s) = \frac{s^2+1}{s^2}\cdot\frac{1}{s} = \frac{s^2+1}{s^3} = \frac{1}{s} + \frac{1}{s^3}\,,$$

and we see that $f(t) = 1 + \frac{1}{2}t^2$ ought to be a solution. Indeed it is, because this function belongs to $\mathcal{E}$, and then our successive steps make up a sequence of equivalent statements. (It is also possible to check the solution by substitution in the given integral equation. This should be done, if time permits.) □

Exercises

3.29 Calculate directly the convolution of e^{at} and e^{bt} (consider separately the cases $a \neq b$ and $a = b$). Check the result by taking Laplace transforms.

3.30 Use the convolution formula to determine f if $\widetilde{f}(s)$ is given by (a) $s^{-1}(s+1)^{-1}$, (b) $s^{-1}(s^2+a^2)^{-1}$.

3.31 Find a function with the Laplace transform $\dfrac{s^2}{(s^2+1)^2}$.

3.32 Find a function f such that

$$\int_0^x e^{-y} \cos y \, f(x-y)\, dy = x^2 e^{-x}, \quad x \geq 0.$$

3.33 Find a solution of the integral equation

$$\int_0^t (t-u)^2 f(u)\, du = t^3, \quad t \geq 0.$$

3.34 Find two solutions of the integral equation (3.2) on page 41.

3.35 Find a function $y(t)$ that satisfies $y(0) = 0$ and

$$2\int_0^t (t-u)^2\, y(u)\, du + y'(t) = (t-1)^2 \quad \text{for } t > 0.$$

3.36 Find a function $f(t)$ for $t \geq 0$, that satisfies

$$f(0) = 1, \qquad f'(t) + 3f(t) + \int_0^t f(u) e^{u-t}\, du = \begin{cases} 0, & 0 \leq t < 2, \\ 1, & t > 2 \end{cases}.$$

3.37 Find a solution f of the integral-differential equation

$$5e^{-t} \int_0^t e^y \cos 2(t-y)\, f(y)\, dy = f'(t) + f(t) - e^{-t}, \quad f(0) = 0.$$

3.38 Prove the following result: if $f \in \mathcal{E}_{k_1}$ and $g \in \mathcal{E}_{k_2}$, then $f * g \in \mathcal{E}_k$ for all $k > \max\{k_1, k_2\}$.

3.5 *Laplace transforms of distributions

Laplace transforms can be used in the study of physical phenomena that take place in a time interval that starts at a certain moment, at which the clock is set to $t = 0$. It is possible to allow the functions to include instantaneous pulses and even more far-reaching generalizations of the classical notion of a function – i.e., to allow so-called distributions into the game. When we do so, it will normally be a good thing to allow such things to happen also at the very moment $t = 0$, so we modify slightly the definition of the Laplace transform into the following formula:

$$\widetilde{f}(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt = \lim_{\varepsilon \searrow 0} \int_{-\varepsilon}^{\infty} f(t)e^{-st}\,dt.$$

If f is an ordinary function, the modified definition agrees with the former one. But if f is a distribution, something new may occur.

As an example, let $\delta_a(t)$ be the Dirac pulse at the point a, where $a \geq 0$. Then

$$\widetilde{\delta_a}(s) = \int_{0-}^{\infty} \delta_a(t)e^{-st}\,dt = e^{-as}.$$

In particular, if $a = 0$, we get $\widetilde{\delta}(s) = 1$. We see that the rule that a Laplace transform must tend to zero as $s \to \infty$ no longer need hold for transforms of distributions.

The formula for the transform of a derivative must also be slightly modified. Indeed, integration by parts gives

$$\widetilde{f'}(s) = \int_{0-}^{\infty} f'(t)e^{-st}\,dt = \Big[f(t)e^{-st}\Big]_{0-}^{\infty} + s\int_{0-}^{\infty} f(t)e^{-st}\,dt = s\widetilde{f}(s) - f(0-),$$

where $f(0-)$ is the left-hand limit of $f(t)$ at 0. This may cause some confusion when dealing with functions that are considered to be zero for negative t but nonzero for positive t. In this case it may now happen that f' includes a multiple of δ, which explains the different appearance of the formula. In this situation, it is preferable to be very explicit in supplying the factor $H(t)$ in the description of functions.

Example 3.18. Solve the initial value problem

$$y'' + 4y' + 13y = \delta'(t), \quad y(0-) = y'(0-) = 0.$$

Solution. Transformation gives

$$(s^2+4s+13)\widetilde{y} = s \iff \widetilde{y} = \frac{s}{(s+2)^2+9} = \frac{s+2}{(s+2)^2+9} - \frac{2}{3}\cdot\frac{3}{(s+2)^2+9}\,.$$

The solution is found to be

$$y(t) = e^{-2t}(\cos 3t - \tfrac{2}{3}\sin 3t)H(t).$$

We check it by differentiating:

$$\begin{aligned} y'(t) &= e^{-2t}(-2\cos 3t + \tfrac{4}{3}\sin 3t - 3\sin 3t - 2\cos 3t)H(t) + \delta(t) \\ &= e^{-2t}(-4\cos 3t - \tfrac{5}{3}\sin 3t)H(t) + \delta(t), \\ y''(t) &= e^{-2t}(8\cos 3t + \tfrac{10}{3}\sin 3t + 12\sin 3t - 5\cos 3t)H(t) - 4\delta(t) + \delta'(t) \\ &= e^{-2t}(3\cos 3t + \tfrac{46}{3}\sin 3t)H(t) - 4\delta(t) + \delta'(t). \end{aligned}$$

Substituting this into the left-hand member of the equation, one sees that it indeed solves the problem. □

Example 3.19. Find the *general* solution of the differential equation $y'' + 3y' + 2y = \delta$.

Solution. It should be wellknown that the solution can be written as the sum of the general solution y_H of the corresponding *homogeneous* equation $y'' + 3y' + 2y = 0$, and one particular solution y_P of the given equation. We easily find $y_H = C_1 e^{-t} + C_2 e^{-2t}$, and proceed to look for y_P. In doing this we assume that $y_P(0-) = y_P'(0-) = 0$, which gives the simplest Laplace transforms. Indeed, $\widetilde{y_P'} = s\widetilde{y_P}$ and $\widetilde{y_P''} = s^2\widetilde{y_P}$, so that

$$s^2\widetilde{y_P} + 3s\widetilde{y_P} + 2\widetilde{y_P} = 1 \iff \widetilde{y_P} = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}.$$

Thus it turns out that

$$y_P = \left(e^{-t} - e^{-2t}\right)H(t).$$

This means that the solution of the given problem is

$$\begin{aligned} y &= C_1 e^{-t} + C_2 e^{-2t} + \left(e^{-t} - e^{-2t}\right)H(t) \\ &= (C_1 + H(t))e^{-t} + (C_2 - H(t))e^{-2t} \\ &= \begin{cases} C_1 e^{-t} + C_2 e^{-2t}, & t < 0, \\ (C_1 + 1)e^{-t} + (C_2 - 1)e^{-2t}, & t > 0. \end{cases} \end{aligned}$$

We can see that in each of the intervals $t < 0$ and $t > 0$ these expressions are solutions of the homogeneous equation, which is in accordance with the fact that $\delta = 0$ in the intervals. What happens at $t = 0$ is that the constants change value in such a way that the first derivative has a jump discontinuity and the second derivative contains a δ pulse (draw pictures!). □

The particular solution y_P found in the preceding problem is called a *fundamental solution* of the equation. Let us now denote it by E; thus,

$$E(t) = \left(e^{-t} - e^{-2t}\right)H(t).$$

It is useful in the following situation. Let f be any function, continuous for $t \geq 0$. We want to find a solution of the problem $y'' + 3y' + 2y = f$. If we assume $y(0-) = y'(0-) = 0$, we get

$$\widetilde{y} = \frac{\widetilde{f}(s)}{s^2 + 3s + 2} = \widetilde{f}(s) \cdot \frac{1}{s^2 + 3s + 2} = \widetilde{f}(s)\widetilde{E}(s).$$

This means that y can be found as the convolution of f and E:

$$y(t) = f * E(t) = \int_0^t f(t-u)\bigl(e^{-u} - e^{-2u}\bigr)\, du.$$

The fundamental solution thus provides a means for finding a particular solution for any inhomogeneous equation with the given left-hand side.

This idea can be applied to any linear differential equation with constant coefficients. The left-hand member of such an equation can be written in the form $P(D)y$, where D is the differentiation operator and $P(\cdot)$ is a polynomial. For example, if $P(r) = r^2 + 3r + 2$, then

$$P(D)y = (D^2 + 3D + 2)y = y'' + 3y' + 2y.$$

The fundamental solution E is, in the general case, the function such that

$$\widetilde{E}(s) = \frac{1}{P(s)}, \quad E(t) = 0 \text{ for } t < 0.$$

Exercises

3.39 Find a solution of the differential equation $y''' + 3y'' + 3y' + y = H(t-1) + \delta(t-2)$, that satisfies $y(0) = y'(0) = y''(0) = 0$.

3.40 Solve the differential equation $y'' + 4y' + 5y = \delta(t)$, $y(t) = 0$ for $t < 0$. Then deduce a formula for a particular solution of the equation $y'' + 4y' + 5y = f(t)$, where f is any continuous function such that $f(t) = 0$ for $t < 0$.

3.41 Find fundamental solutions for the following equations: (a) $y'' + 4y = \delta$, (b) $y'' + 4y' + 8y = \delta$, (c) $y''' + 3y'' + 3y' + y = \delta$.

3.42 Find a function y such that $y(t) = 0$ for $t \leq 0$ and

$$y'(t) + 3y(t) + 2\int_0^t y(u)\, du = 2\bigl(H(t-1) - H(t-2)\bigr) \text{ for } t > 0.$$

3.43 Find a function $f(t)$ such that $f(t) = 0$ for $t < 0$ and

$$e^{-t} \int_{0-}^{t+} f(p)\, e^p\, dp - f(t) + f'(t) = \delta(t) - t\, e^{-t}\, H(t), \quad -\infty < t < \infty.$$

3.6 The Z transform

In this section we sketch the theory of a *discrete* analogue of the Laplace transform. We have so far been considering functions $t \mapsto f(t)$, where t is a real variable (mostly thought of as representing time). Now, we shall think of t as a variable that only assumes the values $0, 1, 2, \ldots$, i.e., non-negative integer values. In applications, this is sometimes more realistic than considering a continuous variable; it corresponds to taking measurements at equidistant points in time.

A function of an integer variable is mostly written as a sequence of numbers. This will be the way we do it, at least at the beginning of the section.

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of numbers. We form the infinite series

$$A(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n} = \sum_{n=0}^{\infty} a_n z^{-n}\,.$$

If the series is convergent for some z, then it converges absolutely outside of some circle in the complex plane. More precisely, the domain of convergence is a set of the type $|z| > \sigma$, where $0 \le \sigma \le \infty$. (It may also happen that the series converges at certain points on the circle $|z| = \sigma$, but this is rarely of any importance.) Power series of this kind, that may encompass both positive and negative powers of z, are called LAURENT *series.* (A particular case is Taylor series that do not contain any negative powers of z; in the present situation we are considering a reversed case, with no positive powers.) A necessary and sufficient condition for the series to converge at all is that there exist constants M and R such that $|a_n| \le MR^n$ for all n. This condition is analogous to the condition of exponential growth for functions to have a Laplace transform.

The function $A(z)$ is called the Z *transform* of the sequence $\{a_n\}_{n=0}^{\infty}$. It can be employed to solve certain problems concerning sequences, in a manner that is largely analogous to the way that Laplace transforms can be used for solving problems for ordinary functions. Important applications occur in the theory of electronics, systems engineering, and automatic control.

When working with the Z transformation, one should be familiar with the geometric series. Recall that this is the series

$$\sum_{n=0}^{\infty} w^n,$$

where w is a real or complex number. It is convergent precisely if $|w| < 1$, and its sum is then $1/(1-w)$. This fact is used "in both directions," as the following example shows.

Example 3.20. If $a_n = 1$ for all $n \geq 0$, the Z transform is

$$\sum_{n=0}^{\infty} \frac{1}{z^n} = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n = \frac{1}{1 - \dfrac{1}{z}} = \frac{z}{z-1}\,,$$

which is convergent for all z such that $|z| > 1$. On the other hand, if λ is a nonzero complex number, we can rewrite the function $B(z) = z/(z-\lambda)$ in this way:

$$B(z) = \frac{z}{z-\lambda} = \frac{1}{1 - \dfrac{\lambda}{z}} = \sum_{n=0}^{\infty} \left(\frac{\lambda}{z}\right)^n = \sum_{n=0}^{\infty} \frac{\lambda^n}{z^n}\,, \quad |z| > |\lambda|,$$

which shows that $B(z)$ is the transform of the sequence $b_n = \lambda^n$ $(n \geq 0)$. (Here we actually use the fact that Laurent expansions are unique, which implies that two different sequences cannot have the same transform.) $\square$

We next present a simple, but typical, problem where the transform can be used.

Example 3.21. If we know that $a_0 = 1$, $a_1 = 2$ and

$$a_{n+2} = 3a_{n+1} - 2a_n\,, \qquad n = 0, 1, 2, \ldots, \tag{3.9}$$

find a formula for a_n.

An equation of the type (3.9) is often called a *difference equation.* In many respects, it is analogous to a differential equation: if differential equations are used for the description of processes taking place in "continuous time," difference equations can do the corresponding thing in "discrete time."

To solve the problem in Example 3.21, we multiply the formula (3.9) by z^{-n} and add up for $n = 0, 1, 2, \ldots$:

$$\sum_{n=0}^{\infty} a_{n+2} z^{-n} = 3 \sum_{n=0}^{\infty} a_{n+1} z^{-n} - 2 \sum_{n=0}^{\infty} a_n z^{-n}. \tag{3.10}$$

Now we introduce the Z transform of the sequence $\{a_n\}_{n=0}^{\infty}$:

$$A(z) = \sum_{n=0}^{\infty} a_n z^{-n} = 1 + \frac{2}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \cdots. \tag{3.11}$$

We notice that, firstly,

$$\begin{aligned}\sum_{n=0}^{\infty} a_{n+1} z^{-n} &= \sum_{k=1}^{\infty} a_k z^{-(k-1)} = z \left(\sum_{k=1}^{\infty} a_k z^{-k} \right) = z \left(\sum_{n=0}^{\infty} a_n z^{-n} - a_0 \right) \\ &= z(A(z) - 1),\end{aligned}$$

and, secondly,

$$\sum_{n=0}^{\infty} a_{n+2} z^{-n} = \sum_{k=2}^{\infty} a_k z^{-(k-2)} = z^2 \left(\sum_{k=2}^{\infty} a_k z^{-k} \right)$$
$$= z^2 \left(\sum_{n=0}^{\infty} a_n z^{-n} - a_0 - \frac{a_1}{z} \right) = z^2 \left(A(z) - 1 - \frac{2}{z} \right).$$

Thus, the equation (3.10) can be written as

$$z^2 \left(A(z) - 1 - \frac{2}{z} \right) = 3z(A(z) - 1) - 2A(z),$$

from which $A(z)$ can be solved. After simplification we have

$$A(z) = \frac{z}{z-2}.$$

We saw in the preceding example that this is the Z transform of the sequence

$$a_n = 2^n, \quad n = 0, 1, 2, \ldots.$$

We can check the result by returning to the statement of the problem: $a_0 = 1$ and $a_1 = 2$ are all right; and if $a_n = 2^n$ and $a_{n+1} = 2^{n+1}$, then

$$3a_{n+1} - 2a_n = 3 \cdot 2^{n+1} - 2 \cdot 2^n = 3 \cdot 2^{n+1} - 2^{n+1} = 2 \cdot 2^{n+1} = 2^{n+2},$$

which is also right. □

In the example, it is obvious from the beginning that the solution is unique. If a_0 and a_1 are given, the formula (3.9) produces the subsequent values of the a_n in an unequivocal way. In general, problems about number sequences are often uniquely determined in the same manner. However, just as for the Laplace transform, the Z transform cannot be expected to give solutions if these are very fast-growing sequences.

We take a closer look at the correspondence between sequences $\{a_n\}_{n=0}^{\infty}$ and their Z transforms $A(z)$. In order to have an efficient notation we write $a = \{a_n\}_{n=0}^{\infty}$ and $A = \mathcal{Z}[a]$. Thus, $\mathcal{Z}$ denotes a mapping from (a subset of) the set of number sequences to the set of Laurent series convergent outside of some circle.

Example 3.22. We have already seen that if $a = \{\lambda^n\}_0^{\infty}$, then

$$\mathcal{Z}[a](z) = \sum_{n=0}^{\infty} \lambda^n z^{-n} = \frac{z}{z-\lambda}, \quad |z| > |\lambda|.$$

□

Example 3.23. If $a = \{1/n!\}_0^{\infty}$, then $\mathcal{Z}[a](z) = \displaystyle\sum_{n=0}^{\infty} \frac{z^{-n}}{n!} = e^{1/z}$, $|z| > 0$.

□

Example 3.24. The sequence $a = \{n!\}_0^\infty$ has no Z transform, because the series $\displaystyle\sum_{n=0}^{\infty} n!\, z^{-n}$ diverges for all z. □

As stated at the beginning of this section, a sufficient (and actually necessary) condition for $A(z)$ to exist is that the numbers a_n grow at most exponentially: $|a_n| \leq MR^n$ for some numbers M and R. It is easy to see that this condition implies the convergence of the series for all z with $|z| > R$.

Some computational rules for the transformation $\mathcal{Z}$ have been collected in the following theorem. In the interest of brevity we introduce some notation for operations on number sequences (which can be viewed as functions $\mathbf{N} \to \mathbf{C}$). If we let $a = \{a_n\}_{n=0}^\infty$ and $b = \{b_n\}_{n=0}^\infty$, we write $a + b = \{a_n + b_n\}_{n=0}^\infty$; and if furthermore λ is a complex number, we put $\lambda a = \{\lambda a_n\}_{n=0}^\infty$. We also agree to write

$$A = \mathcal{Z}[a], \quad B = \mathcal{Z}[b].$$

The "radius of convergence" of the Z transform of a is denoted by σ_a: this means that the series is convergent for $|z| > \sigma_a$ (and divergent for $|z| < \sigma_a$).

Theorem 3.7 (i) *The transformation $\mathcal{Z}$ is linear, i.e.,*

$$\begin{aligned}\mathcal{Z}[\lambda a](z) &= \lambda \mathcal{Z}[a](z), & |z| &> \sigma_a,\\ \mathcal{Z}[a+b](z) &= \mathcal{Z}[a](z) + \mathcal{Z}[b](z), & |z| &> \max(\sigma_a, \sigma_b).\end{aligned}$$

(ii) *If λ is a complex number and $b_n = \lambda^n a_n$, $n = 0, 1, 2, \ldots$, then*

$$B(z) = A(z/\lambda), \quad |z| > \lambda \sigma_a.$$

(iii) *If k is a fixed integer > 0 and $b_n = a_{n+k}$, $n = 0, 1, 2, \ldots$, then*

$$\begin{aligned}B(z) &= z^k \left(A(z) - a_0 - \frac{a_1}{z} - \cdots - \frac{a_{k-1}}{z^{k-1}} \right)\\ &= z^k A(z) - a_0 z^k - a_1 z^{k-1} - \cdots - a_{k-1} z, \qquad |z| > \sigma_a.\end{aligned}$$

(iv) *Conversely, if k is a positive integer and $b_n = a_{n-k}$ for $n \geq k$ and $b_n = 0$ for $n < k$, then $B(z) = z^{-k} A(z)$.*

(v) *If $b_n = n a_n$, $n = 0, 1, 2, \ldots$, then*

$$B(z) = -z\, A'(z), \quad |z| > \sigma_a.$$

Proof. The assertions follow rather immediately from the definitions. We saw a couple of cases of (iii) in Example 3.21 above. We content ourselves by sketching the proofs of (ii) and (v). For (ii) we find

$$B(z) = \sum_{n=0}^{\infty} b_n z^{-n} = \sum_{n=0}^{\infty} \lambda^n a_n z^{-n} = \sum_{n=0}^{\infty} a_n \left(\frac{z}{\lambda}\right)^{-n} = A(z/\lambda).$$

And as for (v), the right-hand side is

$$-z \cdot \frac{d}{dz} \sum_{n=0}^{\infty} a_n z^{-n} = -z \sum_{n=0}^{\infty} (-n) a_n z^{-n-1} = \sum_{n=0}^{\infty} n a_n z^{-n} = \text{ left-hand side.}$$

□

Example 3.25. Example 3.23 and rule (ii) give us the transform of the sequence $\{\lambda^n/n!\}_0^\infty$, viz.,

$$\Lambda(z) = e^{1/(z/\lambda)} = e^{\lambda/z}.$$

□

When solving problems concerning the Z transform, you should have a table at hand, containing rules of computation as well as actual transforms. Such a table is included at the end of this book (p. 250).

Example 3.26. Find a formula for the so-called FIBONACCI numbers, which are defined by $f_0 = f_1 = 1$, $f_{n+2} = f_{n+1} + f_n$ for $n \geq 0$.

Solution. Let $F = \mathcal{Z}[f]$. If we Z-transform the recursion formula, using (iii) from the theorem, we get

$$z^2 F(z) - z^2 - z = (zF(z) - z) + F(z),$$

whence $(z^2 - z - 1)F(z) = z^2$ and

$$F(z) = \frac{z^2}{z^2 - z - 1} = z \cdot \frac{z}{z^2 - z - 1}.$$

In order to recover f_n, a good idea would be to expand into partial fractions, in the hope that simple expressions could be looked up in the table on page 250. A closer look at this table reveals, however, that it would be a good thing to have a z in the numerator of the partial fractions, instead of just a constant. Thus, here we have peeled off a factor z from $F(z)$ and proceed to expand the remaining expression:

$$\frac{F(z)}{z} = \frac{z}{z^2 - z - 1} = \frac{A}{z - \alpha} + \frac{B}{z - \beta},$$

where

$$\alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2}, \quad A = \frac{\sqrt{5} + 1}{2\sqrt{5}}, \quad B = \frac{\sqrt{5} - 1}{2\sqrt{5}}.$$

This gives

$$F(z) = \frac{Az}{z - \alpha} + \frac{Bz}{z - \beta}$$

and from the table we conclude that

$$f_n = A\alpha^n + B\beta^n = \frac{\sqrt{5}+1}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{\sqrt{5}-1}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

This can be rewritten as

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right], \quad n = 0, 1, 2, \ldots.$$

(In spite of all the appearances of $\sqrt{5}$ in the expression, it is an integer for all $n \geq 0$.) □

As you can see in this example, the method of expanding rational functions into partial fractions can be useful in dealing with Z transforms, provided one starts out by securing an extra factor z to be reintroduced in the numerators after the expansion.

If a and b are two number sequences, we can form a third sequence, c, called the *convolution* of a and b, by writing

$$c_n = \sum_{k=0}^{n} a_{n-k} b_k = \sum_{k=0}^{n} a_k b_{n-k}, \quad n = 0, 1, 2, \ldots.$$

One writes $c = a * b$, and we also permit ourselves to write things like $c_n = (a * b)_n$. We determine the Z transform $C = \mathcal{Z}[c]$:

$$\begin{aligned} C(z) &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{n-k} b_k \, z^{-n} = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} a_{n-k} b_k \, z^{-n} \\ &= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} a_{n-k} z^{-(n-k)} \, b_k z^{-k} = \sum_{k=0}^{\infty} b_k z^{-k} \sum_{n=k}^{\infty} a_{n-k} z^{-(n-k)} \\ &= \sum_{k=0}^{\infty} b_k z^{-k} \sum_{m=0}^{\infty} a_m z^{-m} = A(z)B(z). \end{aligned}$$

The manipulations of the double series are permitted for $|z| > \max(\sigma_a, \sigma_b)$, because in that region everything converges absolutely.

This notion of convolution appears in, e.g., control theory, if a system is considered in discrete time (see Appendix A).

Example 3.27. Find $x(t)$, $t = 0, 1, 2, \ldots$, from the equation

$$\sum_{k=0}^{t} 3^{-k} \, x(t-k) = 2^{-t}, \quad t = 0, 1, 2, \ldots.$$

Solution. The left-hand side is the convolution of x and the function $t \mapsto (1/3)^t$, so that taking Z transforms of both members gives

$$\frac{z}{z-\frac{1}{3}} \cdot X(z) = \frac{z}{z-\frac{1}{2}} \,.$$

(We have used the result of Example 3.22.) We get

$$z) = \frac{z-\frac{1}{3}}{z-\frac{1}{2}} = \frac{z}{z-\frac{1}{2}} - \frac{1}{3} \cdot \frac{1}{z-\frac{1}{2}},$$

and, using Example 3.22 and rule (iv) of Theorem 3.7, we see that

$$x(t) = \begin{cases} 1 & \text{for } t = 0, \\ \left(\frac{1}{2}\right)^t - \frac{1}{3} \cdot \left(\frac{1}{2}\right)^{t-1} & \text{for } t \geq 1. \end{cases}$$

The final expression can be rewritten as

$$x(t) = \left(\tfrac{1}{2} - \tfrac{1}{3}\right) \cdot \left(\tfrac{1}{2}\right)^{t-1} = \tfrac{1}{6} \cdot 2^{1-t} = \tfrac{1}{3} \cdot 2^{-t}, \quad t \geq 1.$$

□

In a final example, we indicate a way of viewing the Z transform as a particular case of the Laplace transform. Here we use translates of the Dirac delta "function," as in Sec. 3.5.

Example 3.28. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence having a Z transform $A(z)$, and define a function f by

$$f(t) = \sum_{n=0}^{\infty} a_n \delta_n(t) = \sum_{n=0}^{\infty} a_n \delta(t-n).$$

The convergence of this series is no problem, because for any particular t at most one of the terms is different from zero. Its Laplace transform must be

$$\widetilde{f}(s) = \sum_{n=0}^{\infty} \int_{0-}^{\infty} e^{-st} a_n \delta(t-n)\, dt = \sum_{n=0}^{\infty} a_n e^{-ns} = \sum_{n=0}^{\infty} a_n \left(e^s\right)^{-n} = A(e^s).$$

Thus, via a change of variable $z = e^s$, the two transforms are more or less the same thing. □

Exercises

3.44 Determine the Z transforms of the following sequences $\{a_n\}_{n=0}^{\infty}$:

(a) $a_n = \dfrac{1}{2^n}$ (b) $a_n = n \cdot 3^n$ (c) $a_n = n^2 \cdot 2^n$

(d) $a_n = \dbinom{n}{p} = \dfrac{n(n-1)\cdots(n-p+1)}{p!}$ for $n \geq p$, $= 0$ for $0 \leq n \leq p$ (p is a fixed integer).

3.45 Determine the sequence $a = \{a_n\}_{n=0}^{\infty}$, if its Z transform is (a) $A(z) = \dfrac{z}{3z-2}$, (b) $A(z) = \dfrac{1}{z}$.

3.46 Determine the numbers a_n and b_n, $n = 0, 1, 2, \ldots$, if $a_0 = 0$, $b_0 = 1$ and

$$\begin{cases} a_{n+1} + b_n = -2n, \\ a_n + b_{n+1} = 1, \end{cases} \qquad n = 0, 1, 2, \ldots.$$

3.47 Find the numbers a_n and b_n, $n = 0, 1, 2, \ldots$, if $a_0 = 0$, $b_0 = 1$ and

$$\begin{cases} a_{n+1} + b_n = 2, \\ a_n - b_{n+1} = 0, \end{cases} \qquad n = 0, 1, 2, \ldots.$$

3.48 Find a_n, $n = 0, 1, 2, \ldots$, such that $a_0 = a_1 = 0$ and $a_{n+2} - 3a_{n+1} + 2a_n = 1 - 2n$ for $n = 0, 1, 2, \ldots$.

3.49 Find a_n, $n = 0, 1, 2, \ldots$, if $a_0 = a_1 = 0$ and

$$a_{n+2} + 2a_{n+1} + a_n = (-1)^n n, \quad n = 0, 1, 2, \ldots.$$

3.50 Find a_n, $n = 0, 1, 2, \ldots$, if $a_0 = 1$, $a_1 = 3$ and $a_{n+2} + a_n = 2n + 4$ when $n \geq 0$.

3.51 Determine the numbers $y(t)$ for $t = 0, 1, 2, \ldots$, so that

$$\sum_{k=0}^{t} (t-k)\, 3^{t-k}\, y(k) = \begin{cases} 0, & t = 0, \\ 1, & t = 1, 2, 3, \ldots. \end{cases}$$

3.52 Find a_n for $n \geq 0$, if $a_0 = 0$ and $\displaystyle\sum_{k=0}^{n} k a_{n-k} - a_{n+1} = 2^n$ for $n \geq 0$.

3.53 Determine $x(n)$ for $n = 0, 1, 2, \ldots$, so that

$$x(n) + 2\sum_{k=0}^{n} (n-k)\, x(k) = 2^n, \quad n = 0, 1, 2, \ldots.$$

3.7 Applications in control theory

We return to the "black box" of Sec. 3.4 (p. 53). Such a box can often be described by a differential equation of the type $P(D)y(t) = x(t)$, where x is the input and y the output. If $x(t)$ is taken to be a unit pulse, $x(t) = \delta(t)$, the solution $y(t)$ with $y(t) = 0$ for $t < 0$ is called the *pulse response*, or *impulse response*, of the black box. The pulse response is the same thing as the fundamental solution. In the general case, Laplace transformation will give $P(s)\widetilde{y}(s) = 1$ and thus $\widetilde{y}(s) = 1/P(s)$. The function

$$G(s) = \frac{1}{P(s)}$$

is called the *transfer function* of the box. When solving the general problem

$$P(D)y(t) = x(t), \quad y(t) = 0 \text{ for } t < 0,$$

Laplace transformation will now result in

$$P(s)\widetilde{y}(s) = \widetilde{x}(s)$$

or

$$\widetilde{y}(s) = G(s)\widetilde{x}(s).$$

This formula is actually the Laplace transform of the convolution formula (3.8) of page 53. It provides a quick way of finding the outsignal y to any insignal x. The function g in the convolution is actually the impulse response.

In control theory, great importance is attached to the notion of *stability*. A black box is *stable*, if its impulse response is *transient*, i.e., $g(t)$ tends to zero as time goes by. This means that disturbances in the input will affect the output only for a short time and will not accumulate. If $P(s)$ is a polynomial, the impulse response will be transient if and only if all its zeroes have negative real parts.

Example 3.29. The polynomial $P_1(s) = s^2 + 2s + 2$ has zeroes $s = -1 \pm i$. Both have real part -1, so that the device described by the equation $y'' + 2y' + 2y = x(t)$ is stable. In contrast, the polynomial $P_2(s) = s^2 + 2s - 1$ has zeroes $s = -1 \pm \sqrt{2}$. One of these is positive, which implies that the corresponding black box is unstable. Finally, the polynomial $P_3(s) = s^2 + 1$ has zeroes $s = \pm i$. These have real part zero; the impulse reponse is $g(t) = \sin t$, which is not transient. The situation is considered as unstable. (It is unstable also inasmuch as a small disturbance of the coefficients of $P_3(s)$ can cause the zeroes to move into the right half-plane, which gives rise to exponentially growing solutions.) □

So far, we have assumed that the black box is described in continuous time. In the real world, it is often more realistic to assume that time is discrete, i.e., that input and output are *sampled* at equidistant points in time. For simplicity, we assume that the sampling is done at $t = 0, 1, 2, \ldots$, and that the input signal $x(t)$ and the output $y(t)$ are both zero for $t < 0$. Then, of course, the Z transform is the adequate tool.

A black box is often described by a difference equation of the type

$$y(t+k) + a_{k-1}y(t+k-1) + \cdots + a_2y(t+2) + a_1y(t+1) + a_0y(t) = x(t), \; t \in \mathbf{Z}. \tag{3.12}$$

We introduce the *characteristic polynomial*

$$P(z) = z^k + a_{k-1}z^{k-1} + \cdots + a_2z^2 + a_1z + a_0.$$

We assumed that $x(t)$ and $y(t)$ were both zero for negative t. Putting $t = -k$ in (3.12), we find that

$$y(0) = x(-k) - a_{k-1}y(-1) - \cdots - a_1y(-k+1) - a_0y(-k),$$

which implies that also $y(0) = 0$. Consequently, putting $t = -k + 1$, also $y(1) = 0$, and so on. Not until we have an $x(t)$ that is different from zero do we find a $y(t + k)$ different from zero. Thus we have initial values $y(0) = \cdots = y(k - 1) = 0$. By the rules for the Z transform, we can then easily transform the equation (3.12). With obvious notation we get

$$P(z)Y(z) = X(z).$$

Thus,

$$Y(z) = \frac{X(z)}{P(z)} = G(z)X(z),$$

where $G(z) = 1/P(z)$ is the *transfer function.* Just as in the previous situation, it is also the *impulse response,* because it is the output resulting from inputting the signal

$$\delta(t) = 1 \text{ for } t = 0, \quad \delta(t) = 0 \text{ otherwise.}$$

The *stability* of equation (3.12) hinges on the localization of the zeroes of the polynomial $P(z)$. As can be seen from a table of Z transforms, a zero a of $P(z)$ implies that the solution contains terms involving a^t. Thus we have stability precisely if *all the zeroes of $P(z)$ are in the interior of the unit disc $|z| < 1$.*

Example 3.30. The difference equation $y(t+2)+\frac{1}{2}y(t+1)+\frac{1}{4}y(t) = x(t)$ has $P(z) = z^2+\frac{1}{2}z+\frac{1}{4}$ with zeroes $z = -\frac{1}{4}\pm\frac{\sqrt{3}}{4}i$. These satisfy $|z| = \frac{1}{2} < 1$, so that the equation is stable. The equation

$$y(t + 3) + 2y(t + 2) - y(t + 1) + 2y(t) = x(t)$$

is unstable. This can be seen from the constant term ($= 2$) of the characteristic polynomial; as is well known, this term is (plus or minus) the product of the zeroes, which implies that these cannot all be of modulus less than one. □

More sophisticated methods for localizing the zeroes of polynomials can be found in the literature on complex analysis and in books dealing with these applications.

Exercises

3.54 Investigate the stability of the following equations:
(a) $y''+2y'+3y = x(t)$, (b) $y'''+3y''+3y'+y = x(t)$, (c) $y''+4y = x(t)$.

3.55 Are these difference equations stable or unstable?
(a) $2y(t + 2) - 2y(t + 1) + y(t) = x(t)$,
(b) $y(t + 2) - y(t + 1) + y(t) = x(t)$,
(c) $2y(t + 3) - y(t + 2) + 3y(t + 1) + 3y(t) = x(t)$.

Summary of Chapter 3

To provide an overview of the results of this chapter, we collect the main definitions and theorems here. The precise details of the conditions for the validity of the results are sometimes indicated rather sketchily. Thus, this summary should serve as a memory refresher. Details should be looked up in the core of the text. Facts that rather belong in a table of transforms, such as rules of computation, are not included here, but can be found at the end of the book (p. 247 ff).

Definition
If $f(t)$ is defined for $t \in \mathbf{R}$ and $f(t) = 0$ for $t < 0$, its *Laplace transform* is defined by

$$\widetilde{f}(s) = \int_0^\infty f(t)e^{-st}\,dt,$$

provided the integral is abolutely convergent for some value of s.

Theorem
For $\widetilde{f}$ to exist it is sufficient that f grows at most exponentially, i.e., that $|f(t)| \le Me^{kt}$ for some constants M and k.

Theorem
If $\widetilde{f}(s) = \widetilde{g}(s)$ for all (sufficiently large) s, then $f(t) = g(t)$ for all t where both f and g are continuous.

Theorem
If we define the *convolution* $h = f * g$ by

$$h(t) = f * g(t) = \int_0^t f(t-u)g(u)\,du = \int_0^t f(u)g(t-u)\,du,$$

then its Laplace transform is $\widetilde{h} = \widetilde{f}\widetilde{g}$.

Definition
If $\{a_n\}_{n=0}^\infty$ is a sequence of numbers, its *zeta transform* is defined by

$$A(z) = \sum_{n=0}^\infty a_n z^{-n},$$

provided the series is convergent for some value of z. This holds if $|a_n| \le MR^n$ for some constants M and R.

Historical notes

The Laplace transform is, not surprisingly, found in the works of Pierre Simon de Laplace, notably his *Théorie analytique des probabilités* of 1812. In this book, he made free use of Laplace transforms and also generating functions (which are related to the Z transform) in a way that baffled his contemporaries. During the nineteenth century, the technique was developed further, and also influenced by similar ideas such as the "operational calculus" of Oliver Heaviside (British physicist and applied mathematician, 1850–1925). With the development of modern technology in computing and control theory, the importance of these methods has grown enormously.

Problems for Chapter 3

3.56 Solve the system $y' - 2z = (1-t)e^{-t}$, $z' + 2y = 2te^{-t}$, $t > 0$, with initial conditions $y(0) = 0$, $z(0) = 1$.

3.57 Solve the problem $y'' + 2y' + 2y = 5e^t$, $t > 0$; $y(0) = 1$, $y'(0) = 0$.

3.58 Solve the problem $y''' + y'' + y' - 3y = 1$, $t > 0$, when $y(0) = y'(0) = 0$, $y''(0) = 1$.

3.59 Solve the problem $y'' + 4y = f(t)$, $t > 0$; $y(0) = 0$, $y'(0) = 1$, where

$$f(t) = \begin{cases} (t-1)^2, & t \geq 1 \\ 0, & 0 < t < 1. \end{cases}$$

3.60 Find $y = y(t)$ for $t > 0$ that solves $y'' - 4y' + 5y = \varphi(t)$, $y(0) = 2$, $y'(0) = 0$, where $\varphi(t) = 0$ for $t < 2$, $\varphi(t) = 5$ for $t > 2$.

3.61 Find $f(t)$ for $t \geq 0$, such that $f(0) = 1$ and

$$8 \int_0^t f(t-u)\, e^{-u}\, du + f'(t) - 3f(t) + 2e^{-t} = 0, \quad t > 0.$$

3.62 Let f be the function described by

$$f(t) = 0,\ t \leq 0; \quad f(t) = t,\ 0 < t \leq 1; \quad f(t) = 1,\ t > 1.$$

Solve the differential equation $y''(t) + y(t) = f(t)$ with initial values $y(0) = 0$, $y'(0) = 1$.

3.63 Solve $y''' + y' = t - 1$, $y(0) = 2$, $y'(0) = y''(0) = 0$.

3.64 Solve $y''' + 3y'' + 3y' + y = t + 3$, $t > 0$; $y(0) = 0$, $y'(0) = 1$, $y''(0) = 2$.

3.65 Solve the initial value problem

$$\begin{cases} z'' - y' = e^{-t}, \\ y'' + y' + z' + z = 0, \end{cases} \quad t > 0; \quad \begin{matrix} y(0) = 0, & y'(0) = 1; \\ z(0) = 0, & z'(0) = -1. \end{matrix}$$

3.66 Find f such that $f(0) = 1$ and

$$2e^{-t} \int_0^t (t-u)\, e^u\, f(u)\, du + f'(t) + 2t^2 e^{-t} = 0.$$

3.67 Solve the problem

$$\begin{cases} y''(t) + 2z'(t) - y(t) = 4e^t, \\ z''(t) - 2y'(t) - z(t) = 0, \end{cases} \quad t > 0; \quad \begin{matrix} y(0) = 0, \quad y'(0) = 2, \\ z(0) = z'(0) = 0. \end{matrix}$$

3.68 Solve $y''' + y'' + y' + y = 4e^{-t}$, $t > 0$; $y(0) = 0$, $y'(0) = 3$, $y''(0) = -6$.

3.69 Find f that solves

$$\int_0^t f(u)(t-u)\sin(t-u)\,du - 2f'(t) = 12e^{-t},\ t > 0; \quad f(0) = 6.$$

3.70 Solve $y''' + 3y'' + y' - 5y = 0$, $t > 0$; $y(0) = 1$, $y'(0) = -2$, $y''(0) = 3$.

3.71 Solve $y'''(t) + y''(t) + 4y'(t) + 4y(t) = 8t + 4$, $t > 0$, with initial values $y(0) = -1$, $y'(0) = 4$, $y''(0) = 0$.

3.72 Solve $y''' + y'' + y' + y = 2e^{-t}$, $t > 0$; $y(0) = 0$, $y'(0) = 2$, $y''(0) = -2$.

3.73 Find a solution $y = y(t)$ for $t > 0$ to the initial value problem $y'' + 2ty' - 4y = 1$, $y(0) = y'(0) = 0$.

3.74 Find a solution of the partial differential equation $u_{tt} + 2u_t + xu_x + u = xt$ for $x > 0$, $t > 0$, such that $u(x,0) = u_t(x,0) = 0$ for $x > 0$ and $u(0,t) = 0$ for $t > 0$.

3.75 Use Laplace transformation to find a solution of

$$y''(t) - ty'(t) + y(t) = 5, \quad t > 0; \qquad y(0) = 5, \quad y'(0) = 3.$$

3.76 Find f such that $f(t) = 0$ for $t < 0$ and

$$5e^{-t}\int_0^t e^y \cos 2(t-y)\, f(y)\,dy = f'(t) + f(t) - e^{-t}, \quad t > 0.$$

3.77 Solve the integral equation $y(t) + \int_0^t (t-u)\,y(u)\,du = 3\sin 2t$.

3.78 Solve the difference equation $a_{n+2} - 2a_{n+1} + a_n = b_n$ for $n \geq 0$ with initial values $a_0 = a_1 = 0$ and right-hand member (a) $b_n = 1$, (b) $b_n = e^n$, (c) $b_0 = 1$, $b_n = 0$ for $n > 0$.

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