

4

Fourier series

4.1 Definitions

We are going to solve, as far as we can, the approximation problem that was presented in Sec. 1.4. The strategy will perhaps appear somewhat surprising: starting from a function f , we shall define a certain series, and in due time we shall find that the function can be recovered from this series in various ways.

All functions that we consider will have period 2π . The whole theory could just as well be carried through for functions having some other period. This is equivalent to the standard case that we treat, via a simple linear transformation of the independent variable. The formulae that hold in the general case are collected in Sec. 4.5.

A function defined on \mathbf{R} with period 2π can alternatively be thought of as defined on the unit circle \mathbf{T} , the variable being the polar coordinate. We shall frequently take this point of view. For example, the integral of f over an interval of one period can be written $\int_{\mathbf{T}} f(t) dt$. When we want to compute this integral, we can choose any convenient period interval for the actual calculations:

$$\int_{\mathbf{T}} = \int_{-\pi}^{\pi} = \int_0^{2\pi} = \int_a^{a+2\pi}, \quad a \in \mathbf{R}.$$

(If \mathbf{T} is viewed as a circle, the integral $\int_{\mathbf{T}} f(t) dt$ is *not* to be considered as a line integral of the sort used to calculate amounts of work in mechanics, or

that appears in complex analysis. Instead, it is a line integral *with respect to arc length*.)

One must be careful when working on \mathbf{T} and speaking of notions such as *continuity*. The statement that $f \in C(\mathbf{T})$ must mean that f is continuous at all points of the circle. If we switch to viewing f as a 2π -periodic function, this function must also be continuous. The formula $f(t) = t$ for $-\pi < t < \pi$, for instance, defines a function that *cannot* be made continuous on \mathbf{T} : at the point on \mathbf{T} that corresponds to $t = \pm\pi$, the limits of $f(t)$ from different directions are different.

Similar care must be taken when speaking of functions belonging to $C^k(\mathbf{T})$, i.e., having continuous derivatives of orders up to and including k . As an example, the definition $g(t) = t^2$, $|t| \leq \pi$, describes a function that is in $C(\mathbf{T})$, but not in $C^1(\mathbf{T})$. The first derivative does not exist at $t = \pm\pi$. This can be seen graphically by drawing the periodic continuation, which has corners at these points (sketch a picture!).

Let us now do a preparatory maneuver. Suppose that a function f is the sum of a series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int} = \sum_{n \in \mathbf{Z}} c_n e^{int}. \quad (4.1)$$

We assume that the coefficients c_n are complex numbers such that

$$\sum_{n \in \mathbf{Z}} |c_n| < \infty.$$

By the Weierstrass M -test, the series actually converges absolutely and uniformly, since $|e^{int}|$ is always equal to 1. Each term of the series is continuous and has period 2π , and the sum function f inherits both these properties.

Now let m be any integer (positive, negative, or zero), and multiply the series by e^{-imt} . It will still converge uniformly, and it can be integrated term by term over a period, such as the interval $(-\pi, \pi)$:

$$\int_{-\pi}^{\pi} f(t) e^{-imt} dt = \int_{-\pi}^{\pi} \sum_{n \in \mathbf{Z}} c_n e^{i(n-m)t} dt = \sum_{n \in \mathbf{Z}} c_n \int_{-\pi}^{\pi} e^{i(n-m)t} dt.$$

But it is readily seen that

$$\int_{-\pi}^{\pi} e^{ikt} dt = \begin{cases} 2\pi, & k = 0, \\ 0, & k \neq 0. \end{cases}$$

It follows that all the terms in the sum vanish, except the one where $n-m = 0$, which is the same thing as $n = m$, and the result is that

$$\int_{-\pi}^{\pi} f(t) e^{-imt} dt = 2\pi c_m.$$

Thus, for an absolutely convergent series of the form (4.1), the coefficients can be computed from the sum function using this formula. This fact can be taken as a motivation for the following definition.

Definition 4.1 *Let f be a function with period 2π that is absolutely Riemann-integrable over a period. Define the numbers c_n , $n \in \mathbf{Z}$, by*

$$c_n = \frac{1}{2\pi} \int_{\mathbf{T}} f(t) e^{-int} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt.$$

These numbers are called the Fourier coefficients of f , and the Fourier series of f is the series

$$\sum_{n \in \mathbf{Z}} c_n e^{int}.$$

Notice that the definition does not state anything about the convergence of the series, even less what its sum might be if it happens to converge. It is the main task of this chapter to investigate these questions.

When dealing simultaneously with several functions and their Fourier coefficients it is convenient to indicate to what function the coefficients belong by writing things like $c_n(f)$. Another commonly used way of denoting the Fourier coefficients of f is $\hat{f}(n)$.

When we want to state, as a formula, that f has a certain Fourier series, we write

$$f(t) \sim \sum_{n \in \mathbf{Z}} c_n e^{int}.$$

This means nothing more or less than the fact that the numbers c_n are computable from f using certain integrals.

There are a number of alternative ways of writing the terms in a Fourier series. For instance, when dealing with real-valued functions, the complex-valued functions e^{int} are often felt to be rather “unnatural.” One can then write $e^{int} = \cos nt + i \sin nt$ and reshape the two terms corresponding to $\pm n$ like this:

$$\begin{aligned} c_n e^{int} + c_{-n} e^{-int} &= c_n (\cos nt + i \sin nt) + c_{-n} (\cos nt - i \sin nt) \\ &= (c_n + c_{-n}) \cos nt + i(c_n - c_{-n}) \sin nt = a_n \cos nt + b_n \sin nt, \\ &\quad n = 1, 2, \dots \end{aligned}$$

In the special case $n = 0$ we have only one term, c_0 . This gives a series of the form

$$c_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt).$$

The coefficients in this series are given by new integral formulae:

$$a_n = c_n + c_{-n} = \frac{1}{2\pi} \int_{\mathbf{T}} f(t) e^{-int} dt + \frac{1}{2\pi} \int_{\mathbf{T}} f(t) e^{int} dt$$

$$= \frac{1}{\pi} \int_{\mathbf{T}} f(t) \frac{1}{2} (e^{int} + e^{-int}) dt = \frac{1}{\pi} \int_{\mathbf{T}} f(t) \cos nt dt, \quad n = 1, 2, 3, \dots,$$

and similarly one shows that

$$b_n = \frac{1}{\pi} \int_{\mathbf{T}} f(t) \sin nt dt, \quad n = 1, 2, 3, \dots$$

If we extend the validity of the formula for a_n to $n = 0$, we find that $a_0 = 2c_0$. For this reason the Fourier series is commonly written

$$f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt). \quad (4.2)$$

This is sometimes called the “real” or *trigonometric* version of the Fourier series for f . It should be stressed that this is nothing but a different way of writing the series — it is really the same series as in the definition.

The terms in the series (4.2) can be interpreted as vibrations of different frequencies. The constant term $\frac{1}{2}a_0$ is a “DC component,” the term $a_1 \cos t + b_1 \sin t$ has period 2π , the term with $n = 2$ has half the period length, for $n = 3$ the period is one-third of 2π , etc. These terms can be written in yet another way, that emphasizes this physical interpretation. The reader should be familiar with the fact that the sum of a cosine and a sine with the same period can always be rewritten as a single cosine (or sine) function with a phase angle:

$$\begin{aligned} a \cos nt + b \sin nt &= \sqrt{a^2 + b^2} \left(\frac{a}{\sqrt{a^2 + b^2}} \cos nt + \frac{b}{\sqrt{a^2 + b^2}} \sin nt \right) \\ &= \sqrt{a^2 + b^2} (\cos \alpha \cos nt + \sin \alpha \sin nt) = \sqrt{a^2 + b^2} \cos(nt - \alpha), \end{aligned}$$

where the *phase angle* α is a number such that $\cos \alpha = a/\sqrt{a^2 + b^2}$, $\sin \alpha = b/\sqrt{a^2 + b^2}$. This means that (4.2) can be written in the form

$$\sum_{n=0}^{\infty} A_n \cos(nt - \alpha_n). \quad (4.3)$$

This is sometimes called the *physical* version of the Fourier series. In this formula one can immediately see the *amplitude* A_n of each partial frequency. In this text, however, we shall not work with this form of the series, since it is slightly unwieldy from a mathematical point of view.

When asked to compute the Fourier series of a specific function, it is normally up to the reader to choose what version to work with. This is illustrated by the following examples.

Example 4.1. Define f by saying that $f(t) = e^t$ for $-\pi < t < \pi$ and $f(t + 2\pi) = f(t)$ for all t . (This leaves $f(t)$ undefined for $t = (2n + 1)\pi$,

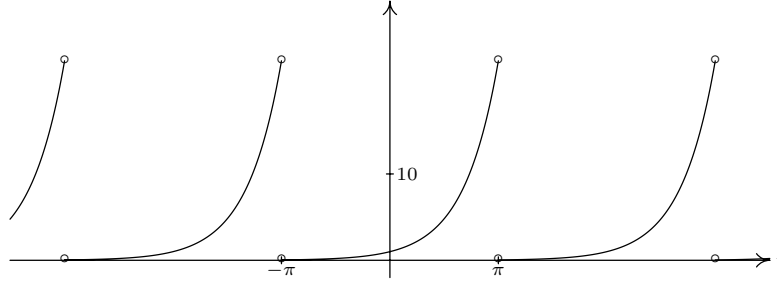


FIGURE 4.1.

but this does not matter. The value of a function at one point or another does not affect the values of its Fourier coefficients!) We get a function with period 2π (see Figure 4.1). Its Fourier coefficients are

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^t e^{-int} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(1-in)t} dt = \frac{1}{2\pi} \left[\frac{e^{(1-in)t}}{1-in} \right]_{t=-\pi}^{\pi} \\ &= \frac{e^{\pi-in\pi} - e^{-\pi+in\pi}}{2\pi(1-in)} = \frac{(-1)^n(e^{\pi} - e^{-\pi})}{2\pi(1-in)} = \frac{(-1)^n \sinh \pi}{\pi(1-in)}. \end{aligned}$$

Here we used the fact that $e^{\pm in\pi} = (-1)^n$. Now we can write

$$f(t) \sim \frac{1}{\pi} \sum_{n \in \mathbf{Z}} \frac{(-1)^n \sinh \pi}{1-in} e^{int} = \frac{\sinh \pi}{\pi} \sum_{n \in \mathbf{Z}} \frac{(-1)^n}{1-in} e^{int}.$$

□

We remind the reader of a couple of notions of symmetry that turn out to be useful in connection with Fourier series. A function f defined on \mathbf{R} is said to be *even*, if $f(-t) = f(t)$ for all $t \in \mathbf{R}$. A function f is *odd*, if $f(-t) = -f(t)$. (The terms should bring to mind the special function $f(t) = t^n$, which is even if n is an even integer, odd if n is an odd integer.) An odd function f on a symmetric interval $(-a, a)$ has the property that the integral over $(-a, a)$ is equal to zero. This has useful consequences for the so-called real Fourier coefficients a_n and b_n . If f is even and has period 2π , the sine coefficients b_n will be zero, and furthermore the cosine coefficients will be given by the formula

$$f \text{ even} \quad \Rightarrow \quad a_n = \frac{2}{\pi} \int_0^{\pi} f(t) \cos nt \, dt.$$

In an analogous way, an odd function has all cosine coefficients equal to zero, and its sine coefficients are given by

$$f \text{ odd} \quad \Rightarrow \quad b_n = \frac{2}{\pi} \int_0^{\pi} f(t) \sin nt \, dt.$$

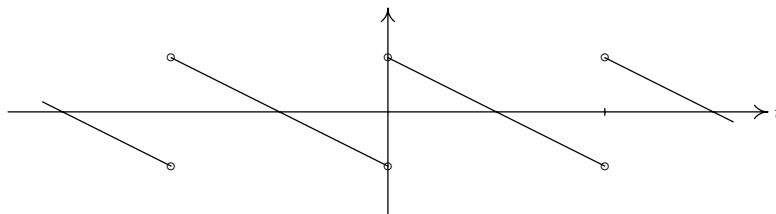


FIGURE 4.2.

When computing the Fourier series for an even or odd function these facts are often useful.

Example 4.2. Let f be an odd function with period 2π , that satisfies $f(t) = (\pi - t)/2$ for $0 < t < \pi$. Find its Fourier series! (See Figure 4.2.)

Solution. Notice that the description as given actually determines the function completely (except for its value at one point in each period, which does not matter). Because the function is odd we have $a_n = 0$ and

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^\pi \frac{\pi - t}{2} \sin nt \, dt \\ &= \frac{1}{\pi} \left[(\pi - t) \frac{-\cos nt}{n} \right]_{t=0}^\pi + \frac{1}{n\pi} \int_0^\pi (-1) \cos nt \, dt \\ &= \frac{1}{n} - \frac{1}{n^2\pi} \left[\sin nt \right]_{t=0}^\pi = \frac{1}{n}. \end{aligned}$$

Thus,

$$f(t) \sim \sum_{n=1}^{\infty} \frac{\sin nt}{n}.$$

□

Example 4.3. Let $f(t) = t^2$ for $|t| \leq \pi$ and define f outside of this interval by proclaiming it to have period 2π (draw a picture!). Find the Fourier series of this function.

Solution. Now the function is even, and so $b_n = 0$ and

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^\pi t^2 \cos nt \, dt \stackrel{n \neq 0}{=} \frac{2}{\pi} \left[t^2 \frac{\sin nt}{n} \right]_0^\pi - \frac{2}{n\pi} \int_0^\pi 2t \sin nt \, dt \\ &= -\frac{4}{n\pi} \left[t \frac{-\cos nt}{n} \right]_0^\pi - \frac{4}{n^2\pi} \int_0^\pi 1 \cdot \cos nt \, dt = \frac{4\pi \cos n\pi}{n^2\pi} - 0 = \frac{4(-1)^n}{n^2}. \end{aligned}$$

For $n = 0$ we must do a separate calculation:

$$a_0 = \frac{2}{\pi} \int_0^\pi t^2 \, dt = \frac{2}{\pi} \cdot \frac{\pi^3}{3} = \frac{2\pi^2}{3}.$$

Collecting the results we get

$$f(t) \sim \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nt.$$

□

The series obtained in Example 4.3 is clearly convergent; indeed it even converges uniformly, by Weierstrass. At this stage we cannot tell what its sum is. The goal of the next few sections is to investigate this. For the moment, we can notice two facts about Fourier coefficients:

Lemma 4.1 *Suppose that f is as in the definition of Fourier series. Then*

1. *The sequence of Fourier coefficients is bounded; more precisely,*

$$|c_n| \leq \frac{1}{2\pi} \int_{\mathbf{T}} |f(t)| dt \quad \text{for all } n.$$

2. *The Fourier coefficients tend to zero as $|n| \rightarrow \infty$.*

Proof. For the c_n we have

$$|c_n| = \frac{1}{2\pi} \left| \int_{\mathbf{T}} f(t) e^{-int} dt \right| \leq \frac{1}{2\pi} \int_{\mathbf{T}} |f(t)| |e^{-int}| dt = \frac{1}{2\pi} \int_{\mathbf{T}} |f(t)| dt = M,$$

where M is a fixed number that does not depend on n . (In just the same way one can estimate a_n and b_n .) The second assertion of the lemma is just a case of Riemann–Lebesgue’s lemma. □

The constant term in a Fourier series is of particular interest:

$$c_0 = \frac{a_0}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt.$$

This can be interpreted as the *mean value* of the function f over one period (or over \mathbf{T}). This can often be useful in problem-solving. It is also intuitively reasonable in that all the other terms of the series have mean value 0 over any period (think of the graph of, say, $\sin nt$).

Exercises

- 4.1 Prove the formulae $c_n = \frac{1}{2}(a_n - ib_n)$ and $c_{-n} = \frac{1}{2}(a_n + ib_n)$ for $n \geq 0$ (where $b_0 = 0$).
- 4.2 Assume that f and g are odd functions and h is even. Find out which of the following functions are odd or even: $f + g$, fg , fh , f^2 , $f + h$.
- 4.3 Show that an arbitrary function f on a symmetric interval $(-a, a)$ can be decomposed as $f_E + f_O$, where f_E is even and f_O is odd. Also show that this decomposition is unique. Hint: put $f_E(t) = (f(t) + f(-t))/2$.

- 4.4 Determine the Fourier series of the 2π -periodic function described by $f(t) = t + 1$ for $|t| < \pi$.
- 4.5 Prove the following relations for a (continuous) function f and its “complex” Fourier coefficients c_n :
- (a) If f is even, then $c_n = c_{-n}$ for all n .
 - (b) If f is odd, then $c_n = -c_{-n}$ for all n .
 - (c) If f is real-valued, then $\overline{c_n} = c_{-n}$ for all n (where $\overline{}$ denotes complex conjugation).
- 4.6 Find the Fourier series (in the “real” version) of the functions (a) $f(t) = \cos 2t$, (b) $g(t) = \cos^2 t$, (c) $h(t) = \sin^3 t$. *Sens moral?*
- 4.7 Let f have the Fourier coefficients $\{c_n\}$. Prove the following rules for Fourier coefficients (F.c.’s):
- (a) Let $a \in \mathbf{Z}$. Then the function $t \mapsto e^{iat} f(t)$ has F.c.’s $\{c_{n-a}\}$.
 - (b) Let $b \in \mathbf{R}$. Then the function $t \mapsto f(t-b)$ has F.c.’s $\{e^{-inb} c_n\}$.
- 4.8 Find the Fourier series of $h(t) = e^{3it} f(t-4)$, when f has period 2π and satisfies $f(t) = 1$ for $|t| < 2$, $f(t) = 0$ for $2 < |t| < \pi$.
- 4.9 Compute the Fourier series of f , where $f(t) = e^{-|t|}$, $|t| < \pi$, $f(t+2\pi) = f(t)$, $t \in \mathbf{R}$.
- 4.10 Let f and g be defined on \mathbf{T} with Fourier coefficients $c_n(f)$ resp. $c_n(g)$. Define the function h by

$$h(t) = \frac{1}{2\pi} \int_{\mathbf{T}} f(t-u) g(u) du.$$

Show that h is welldefined on \mathbf{T} (i.e., h has also period 2π), and prove that $c_n(h) = c_n(f) c_n(g)$. (The function h is called the *convolution* of f and g .)

4.2 Dirichlet’s and Fejér’s kernels; uniqueness

It is a regrettable fact that a Fourier series need not be convergent. For example, it is possible to construct a continuous function such that its Fourier series diverges at a specified point (see, for example, the book by THOMAS KÖRNER mentioned in the bibliography). We shall see, in due time, that if we impose somewhat harder requirements on the function, such as differentiability, the results are more positive.

It is, however, true that the Fourier series of a continuous function is Cesàro summable to the values of the function, and this is the main result of this section.

We start by establishing a closed formula for the partial sums of a Fourier series. To this end we shall use the following formula:

Lemma 4.2

$$D_N(u) := \frac{1}{\pi} \left(\frac{1}{2} + \sum_{n=1}^N \cos nu \right) = \frac{1}{2\pi} \sum_{n=-N}^N e^{inu} = \frac{\sin(N + \frac{1}{2})u}{2\pi \sin \frac{1}{2}u}.$$

Proof. The equality of the two sums follows easily from Euler's formulae. Let us then start from the “complex” version of the sum and compute it as a finite geometric sum:

$$\begin{aligned}
 2\pi D_N(u) &= \sum_{n=-N}^N e^{inu} = e^{-iNu} \sum_{n=0}^{2N} e^{inu} = e^{-iNu} \cdot \frac{1 - e^{i(2N+1)u}}{1 - e^{iu}} \\
 &= e^{-iNu} \cdot \frac{e^{i(N+\frac{1}{2})u} (e^{-i(N+\frac{1}{2})u} - e^{i(N+\frac{1}{2})u})}{e^{iu/2} (e^{-iu/2} - e^{iu/2})} \\
 &= \frac{e^{-iNu+i(N+\frac{1}{2})u}}{e^{iu/2}} \cdot \frac{-2i \sin(N + \frac{1}{2})u}{-2i \sin \frac{1}{2}u} = \frac{\sin(N + \frac{1}{2})u}{\sin \frac{1}{2}u}.
 \end{aligned}$$

□

The function D_N is called the DIRICHLET kernel. Its graph is shown in Figure 4.3 on page 87.

When discussing the convergence of Fourier series, the natural partial sums are those containing all frequencies up to a certain value. Thus we define the partial sum $s_N(t)$ to be

$$s_N(t) := \frac{1}{2} a_0 + \sum_{n=1}^N (a_n \cos nt + b_n \sin nt) = \sum_{n=-N}^N c_n e^{int}.$$

Using the Dirichlet kernel we can obtain an integral formula for this sum, assuming the c_n to be the Fourier coefficients of a function f :

$$\begin{aligned}
 s_N(t) &= \sum_{n=-N}^N c_n e^{int} = \sum_{n=-N}^N \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u) e^{-inu} du \cdot e^{int} \\
 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(u) \cdot \frac{1}{2} \sum_{n=-N}^N e^{in(t-u)} du = \int_{-\pi}^{\pi} f(u) D_N(t-u) du \\
 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-u) \frac{\sin(N + \frac{1}{2})u}{\sin \frac{1}{2}u} du.
 \end{aligned}$$

In the last step we change the variable ($t-u$ is replaced by u) and make use of the periodicity of the integrand. We shall presently take another step and form the arithmetic means of the $N+1$ first partial sums. To achieve this we need a formula for the mean of the corresponding Dirichlet kernels:

Lemma 4.3

$$F_N(u) := \frac{1}{N+1} \sum_{n=0}^N D_n(u) = \frac{1}{2\pi(N+1)} \left(\frac{\sin \frac{1}{2}(N+1)u}{\sin \frac{1}{2}u} \right)^2.$$

The proof can be done in a way similar to Lemma 4.2 (or in some other way). It is left as an exercise. The function $F_N(t)$ is called the FEJÉR kernel.

Now we can form the mean of the partial sums:

$$\begin{aligned}\sigma_N(t) &= \frac{s_0(t) + s_1(t) + \cdots + s_N(t)}{N+1} = \frac{1}{N+1} \sum_{n=0}^N \int_{-\pi}^{\pi} f(t-u) D_n(u) du \\ &= \int_{-\pi}^{\pi} f(t-u) \cdot \frac{1}{N+1} \sum_{n=0}^N D_n(u) du = \int_{-\pi}^{\pi} f(t-u) F_N(u) du.\end{aligned}$$

Lemma 4.4 *The Fejér kernel $F_N(u)$ has the following properties:*

1. F_N is an even function, and $F_N(u) \geq 0$.
2. $\int_{-\pi}^{\pi} F_N(u) du = 1$.
3. If $\delta > 0$, then $\lim_{N \rightarrow \infty} \int_{\delta}^{\pi} F_N(u) du = 0$.

Proof. Property 1 is obvious. Number 2 follows from

$$\int_{-\pi}^{\pi} D_n(u) du = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{1}{2} + \cos u + \cdots + \cos nu \right) du = 1, \quad n = 0, 1, 2, \dots, N,$$

and the fact that F_N is the mean of these Dirichlet kernels. Finally, property 3 can be proved thus:

$$\begin{aligned}0 &\leq \int_{\delta}^{\pi} F_N(u) du = \frac{1}{2\pi(N+1)} \int_{\delta}^{\pi} \frac{\sin^2 \frac{1}{2}(N+1)u}{\sin^2 \frac{1}{2}u} du \\ &\leq \frac{1}{2\pi(N+1)} \int_{\delta}^{\pi} \frac{1}{\sin^2 \frac{1}{2}\delta} du = \frac{1}{2\pi(N+1)} \frac{\pi - \delta}{\sin^2 \frac{1}{2}\delta} = \frac{C_{\delta}}{N+1} \rightarrow 0\end{aligned}$$

as $N \rightarrow \infty$. □

The lemma implies that $\{F_N\}_{N=1}^{\infty}$ is a positive summation kernel such as the ones studied in Sec. 2.4. Applying Corollary 2.1 we then have the result on Cesàro sums of Fourier series.

Theorem 4.1 (Fejér's theorem) *If f is piecewise continuous on \mathbf{T} and continuous at the point t , then $\lim_{N \rightarrow \infty} \sigma_N(t) = f(t)$.*

Remark. Using the remark following Corollary 2.1, we can sharpen the result of the theorem a bit. If f is continuous in an interval $I_0 =]a_0, b_0[$, and $I = [a, b]$ is a compact subinterval of I_0 , then $\sigma_N(t)$ will converge to $f(t)$ *uniformly on I* . □

If a series is convergent in the traditional sense, then its sum coincides with the Cesàro limit. This means that if a *continuous* function happens to have a Fourier series, which is *seen* to be convergent, in one way or another, then it actually converges to the function it comes from. In particular we have the following theorem.

Theorem 4.2 *If f is continuous on \mathbf{T} and its Fourier coefficients c_n are such that $\sum |c_n|$ is convergent, then the Fourier series is convergent with sum $f(t)$ for all $t \in \mathbf{T}$, and the convergence is even uniform on \mathbf{T} .*

The uniform convergence follows using the Weierstrass M -test just as at the beginning of this chapter.

This result can be applied to Example 4.3 of the previous section, where we computed the Fourier series of $f(t) = t^2$ ($|t| \leq \pi$). Applying the usual comparison test, the series obtained is easily seen to be convergent, and now we know that its sum is also equal to $f(t)$. We now have this formula:

$$t^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nt, \quad -\pi \leq t \leq \pi. \quad (4.4)$$

(Why does this formula hold even for $t = \pm\pi$?) In particular, we can amuse ourselves by inserting various values of t just to see what we get. For $t = 0$ the result is

$$0 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

From this we can conclude that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}.$$

If $t = \pi$ is substituted into (4.4), we have

$$\pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} (-1)^n = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which enables us to state that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Thus, Fourier series provide a means of computing the sums of numerical series. Regrettably, it can hardly be called a “method”: if one faces a more-or-less randomly chosen series, there is no general method to find a function whose Fourier expansion will help us to sum it. As an illustration we mention that it is rather easy to find nice expressions for the values of

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $s = 2, 4, 6, \dots$, but no one has so far found such an expression for, say, $\zeta(3)$.

The following uniqueness result is also a consequence of Theorem 4.2.

Theorem 4.3 *Suppose that f is piecewise continuous and that all its Fourier coefficients are 0. Then $f(t) = 0$ at all points where f is continuous.*

In fact, all the partial sums are zero and the series is trivially convergent, and by Theorem 4.2 it must then converge to the function from which it is formed.

Corollary 4.1 *If two continuous functions f and g have the same Fourier coefficients, then $f = g$.*

Proof. Apply Theorem 4.3 to the function $h = f - g$. □

Exercises

- 4.11 Prove the formula for the Fejér kernel (i.e., Lemma 4.3).
 4.12 Study the function $f(t) = t^4 - 2\pi^2 t^2$, $|t| < \pi$, and compute the value of $\zeta(4)$.
 4.13 Determine the Fourier series of $f(t) = |\cos t|$. Prove that the series converges uniformly to f and find the value of

$$s = \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}.$$

- 4.14 Prove converse statements to the assertions in Exercise 4.5; i.e., show that if f is continuous (say), we can say that
 (a) if $c_n = c_{-n}$ for all n , then f is even;
 (b) If $c_n = -c_{-n}$ for all n , then f is odd;
 (c) If $\bar{c}_n = c_{-n}$ for all n , then f is real-valued.

4.3 Differentiable functions

Suppose that $f \in C^1(\mathbf{T})$, which means that both f and its derivative f' are continuous on \mathbf{T} . We compute the Fourier coefficients of the derivative:

$$\begin{aligned} c_n(f') &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(t) e^{-int} dt = \frac{1}{2\pi} [f(t) e^{-int}]_{-\pi}^{\pi} - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) (-in) e^{-int} dt \\ &= \frac{1}{2\pi} (f(\pi)(-1)^n - f(-\pi)(-1)^n) + in c_n(f) = in c_n(f). \end{aligned}$$

(The fact that f is continuous on \mathbf{T} implies that $f(-\pi) = f(\pi)$.) This means that if f has the Fourier series $\sum c_n e^{int}$, then f' has the series $\sum in c_n e^{int}$. This indeed means that the Fourier series can be differentiated termwise (even if we have no information at all concerning the convergence of either of the two series).

If $f \in C^2(\mathbf{T})$, the argument can be repeated, and we find that the Fourier series of the second derivative is $\sum (-n^2)c_n e^{int}$. Since the Fourier coefficients of f'' are bounded, by Lemma 4.1, we conclude that $|-n^2c_n| \leq M$ for some constant M , which implies that $|c_n| \leq M/n^2$ for $n \neq 0$. But then we can use Theorem 4.2 to conclude that the Fourier series of f converges to $f(t)$ for all t . Here we have a first, simple, sufficient condition on the function f itself that ensures a nice behavior of its Fourier series.

In the next section, we shall see that C^2 can be improved to C^1 and indeed even less demanding conditions.

By iteration of the argument above, the following general result follows.

Theorem 4.4 *If $f \in C^k(\mathbf{T})$, then $|c_n| \leq M/|n|^k$ for some constant M .*

The smoother the function, the smaller the Fourier coefficients: a function with high differentiability contains small high-frequency components.

The assertion of the theorem is really rather weak. Indeed, one can say more, which is exemplified in Exercises 4.15 and 4.17.

The situation concerning integration of Fourier series is extremely favorable. It turns out that termwise integration is *always* possible, both when talking about antiderivatives and integrals over an interval. There is one complication: if the constant term in the series is not zero, the formally integrated series is no longer a Fourier series. However, we postpone the treatment of these matters until later on, when it will be easier to carry through. (Sec. 5.4, Theorem 5.9 on p. 122.)

The fact that termwise differentiation is possible can be used when looking for periodic solutions of differential equations and similar problems. We give an example of this.

Example 4.4. Find a solution $y(t)$ with period 2π of the differential-difference equation $y'(t) + 2y(t - \pi) = \sin t$, $-\infty < t < \infty$.

Solution. Assume the solution to be the sum of a “complex” Fourier series (a “real” series could also be used):

$$y(t) = \sum_{n \in \mathbf{Z}} c_n e^{int}.$$

If we differentiate termwise and substitute into the given equation, we get

$$y'(t) + 2y(t - \pi) = \sum in c_n e^{int} + 2 \sum c_n e^{int - in\pi} = \sum (in + 2(-1)^n) c_n e^{int}. \quad (4.5)$$

This should be equal to $\sin t = (e^{it} - e^{-it})/(2i) = \frac{1}{2} i e^{-it} - \frac{1}{2} i e^{it}$. The equality must imply that the coefficients in the last series of (4.5) are zeroes for all $n \neq \pm 1$, and furthermore

$$(i - 2)c_1 = -\frac{i}{2}, \quad (-i - 2)c_{-1} = \frac{i}{2}.$$

From this we solve $c_1 = \frac{1}{10}(2i - 1)$, $c_{-1} = \frac{1}{10}(-2i - 1)$ (and $c_n = 0$ for all other n), which gives

$$\begin{aligned} y(t) &= c_1 e^{it} + c_{-1} e^{-it} = -\frac{1}{10}(e^{it} + e^{-it}) + \frac{2}{10}i(e^{it} - e^{-it}) \\ &= -\frac{1}{5}\cos t + \frac{1}{5}i \cdot 2i \sin t = \frac{1}{5}(-\cos t - 2\sin t). \end{aligned}$$

Check the solution by substituting into the original equation! \square

Exercises

- 4.15 Prove the following improvement on Theorem 4.4: If $f \in C^k(\mathbf{T})$, then $\lim_{n \rightarrow \pm\infty} n^k c_n = 0$.
- 4.16 Find the values of the constant a for which the problem $y''(t) + ay(t) = y(t + \pi)$, $t \in \mathbf{R}$, has a solution with period 2π which is not identically zero. Also, determine all such solutions.
- 4.17 Try to prove the following partial improvements on Theorem 4.4:
- (a) If f' is continuous and differentiable on \mathbf{T} except possibly for a finite number of jump discontinuities, then $|c_n| \leq M/|n|$ for some constant M .
 - (b) If f is continuous on \mathbf{T} and has a second derivative everywhere except possibly for a finite number of points, where there are “corners” (i.e., the left-hand and right-hand first derivatives exist but are different from each other), then $|c_n| \leq M/n^2$ for some constant M .

4.4 Pointwise convergence

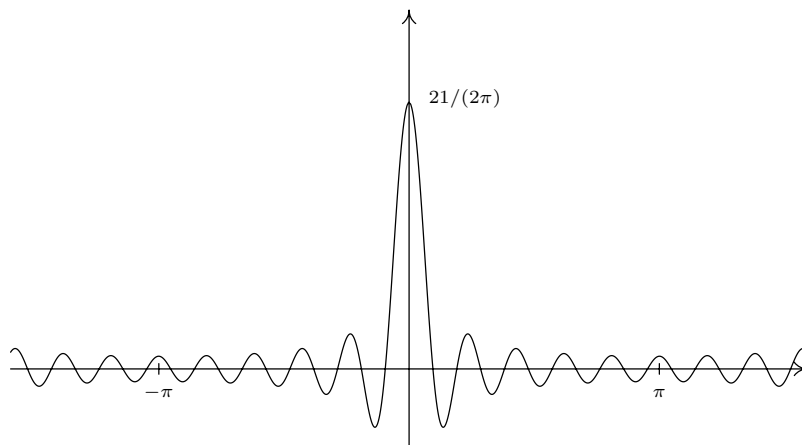
Time is now ripe for the formulation and proof of our most general theorem on the pointwise convergence of Fourier series. We have already mentioned that continuity of the function involved is not sufficient. Now let us assume that f is defined on \mathbf{T} and continuous except possibly for a finite number of finite jumps. This means that f is permitted to be discontinuous at a finite number of points in each period, but at these points we assume that both the one-sided limits exist and are finite. For convenience, we introduce this notation for these limits:

$$f(t_0-) = \lim_{t \nearrow t_0} f(t), \quad f(t_0+) = \lim_{t \searrow t_0} f(t).$$

In addition, we assume that the “generalized left-hand derivative” $f'_L(t_0)$ exists:

$$f'_L(t_0) = \lim_{h \nearrow 0} \frac{f(t_0 + h) - f(t_0-)}{h} = \lim_{u \searrow 0} \frac{f(t_0 - u) - f(t_0-)}{-u}.$$

If f happens to be continuous at t_0 , this coincides with the usual left-hand derivative; if f has a discontinuity at t_0 , we take care to use the left-hand limit instead of just writing $f(t_0)$.

FIGURE 4.3. The graph of D_{10}

Symmetrically, we shall also assume that the “generalized right-hand derivative” exists:

$$f'_R(t_0) = \lim_{h \searrow 0+} \frac{f(t_0 + h) - f(t_0+)}{h}.$$

Intuitively, the existence of these generalized derivatives amounts to the fact that at a jump discontinuity, the graphs of the two parts of the function on either side of the jump have each an end-point tangent direction.

In Sec. 4.2 we proved the following formula for the partial sums of the Fourier series of f :

$$s_N(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-u) \frac{\sin(N + \frac{1}{2})u}{\sin \frac{1}{2}u} du. \quad (4.6)$$

What complicates matters is that the Dirichlet kernel occurring in the integral is not a positive summation kernel. On the contrary, it takes a lot of negative values, which causes a proof along the lines of Theorem 2.1 to fail completely (see Figure 4.3).

We shall make use of the following formula:

$$\frac{1}{\pi} \int_0^{\pi} \frac{\sin(N + \frac{1}{2})u}{\sin \frac{1}{2}u} du = 1. \quad (4.7)$$

This follows directly from the fact that the integrated function is $2\pi D_N(u) = 1 + 2 \sum_{n=1}^N \cos nu$, where all the cosine terms have integral zero over $[0, \pi]$.

We split the integral (4.6) in two parts, each covering half of the interval of integration, and begin by taking care of the right-hand half:

Lemma 4.5

$$\lim_{N \rightarrow \infty} \frac{1}{\pi} \int_0^\pi f(t_0 - u) \frac{\sin(N + \frac{1}{2})u}{\sin \frac{1}{2}u} du = f(t_0-).$$

Proof. Rewrite the difference between the integral on the left and the number on the right, using (4.7):

$$\begin{aligned} & \frac{1}{\pi} \int_0^\pi f(t_0 - u) \frac{\sin(N + \frac{1}{2})u}{\sin \frac{1}{2}u} du - f(t_0-) \\ &= \frac{1}{\pi} \int_0^\pi (f(t_0 - u) - f(t_0-)) \frac{\sin(N + \frac{1}{2})u}{\sin \frac{1}{2}u} du \\ &= \frac{1}{\pi} \int_0^\pi \frac{f(t_0 - u) - f(t_0-)}{-u} \cdot \frac{-u}{\sin \frac{1}{2}u} \cdot \sin(N + \frac{1}{2})u du. \end{aligned}$$

The last integrand consists of three factors: The first one is continuous (except for jumps), and it has a finite limit as $u \rightarrow 0+$, namely, $f'_L(t_0)$. The second factor is continuous and bounded. The product of the two first factors is thus a function $g(u)$ which is clearly Riemann-integrable on the interval $[0, \pi]$. By the Riemann–Lebesgue lemma we can then conclude that the whole integral tends to zero as N goes to infinity, which proves the lemma. \square

In just the same way one can prove that if f has a generalized right-hand derivative at t_0 , then

$$\lim_{N \rightarrow \infty} \frac{1}{\pi} \int_{-\pi}^0 f(t_0 - u) \frac{\sin(N + \frac{1}{2})u}{\sin \frac{1}{2}u} du = f(t_0+).$$

Taking the arithmetic mean of the two formulae, we have proved the convergence theorem:

Theorem 4.5 *Suppose that f has period 2π , and suppose that t_0 is a point where f has one-sided limiting values and (generalized) one-sided derivatives. Then the Fourier series of f converges for $t = t_0$ to the mean value $\frac{1}{2}(f(t_0+) + f(t_0-))$. In particular, if f is continuous at t_0 , the sum of the series equals $f(t_0)$.*

We emphasize that if f is continuous at t_0 , the sum of the series is simply $f(t_0)$. At a point where the function has a jump discontinuity, the sum is instead the mean value of the right-hand and left-hand limits.

It is important to realize that the convergence of a Fourier series at a particular point is really dependent only on the local behavior of the function in the neighborhood of that point. This is sometimes called the Riemann localization principle.

Example 4.5. Let us return to Example 4.2 on page 78. Now we finally know that the series

$$\sum_{n=1}^{\infty} \frac{\sin nt}{n}$$

is indeed convergent for all t . If, for example, $t = \pi/2$, we have $\sin nt$ equal to zero for all even values of n , while $\sin(2k+1)t = (-1)^k$. Since f is continuous and has a derivative at $t = \pi/2$, and $f(\pi/2) = \pi/4$, we obtain

$$\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

(In theory, this formula could be used to compute numerical approximations to π , but the series converges so extremely slowly that it is of no practical use whatever.) \square

The most comprehensive theorem concerning pointwise convergence of Fourier series of continuous functions was proved in 1966 by Lennart CARLESON. In order to formulate it we first introduce the notion of a *zero set*: a set $E \subset \mathbf{T}$ is called a zero set if, for every $\varepsilon > 0$, it is possible to construct a sequence of intervals $\{\omega_n\}_{n=1}^{\infty}$ on the circle, that together cover the set E and whose total length is less than ε .

Theorem 4.6 (Carleson's theorem) *If f is continuous on \mathbf{T} , then its Fourier series converges at all points of \mathbf{T} except possibly for a zero set.*

In fact, it is not even necessary that f be continuous; it is sufficient that $f \in L^2(\mathbf{T})$, which will be explained in Chapter 5. The proof is very complicated.

Carleson's theorem is "best possible" in the following sense:

Theorem 4.7 (Kahane and Katznelson) *If E is a zero set on \mathbf{T} , then there exists a continuous function such that its Fourier series diverges precisely for all $t \in E$.*

Exercises

- 4.18 Define f by letting $f(t) = t \sin t$ for $|t| < \pi$ and $f(t + 2\pi) = f(t)$ for all t . Determine the Fourier series of f and investigate for which values of t it converges to $f(t)$.
- 4.19 If $f(t) = (t + 1) \cos t$ for $-\pi < t < \pi$, what is the sum of the Fourier series of f for $t = 3\pi$? (Note that you do not have to compute the series itself!)
- 4.20 The function f has period 2π and satisfies

$$f(t) = \begin{cases} t + \pi, & -\pi < t < 0, \\ 0, & 0 \leq t \leq \pi. \end{cases}$$

(a) Find the Fourier series of f and sketch the sum of the series on the interval $[-3\pi, 3\pi]$.

(b) Sum the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

4.21 Let $f(x)$ be defined for $-\pi < x < \pi$ by $f(x) = \cos \frac{3}{2}x$ and for other values of x by $f(x) = f(x + 2\pi)$. Determine the Fourier series of f . For all real x , investigate whether the series is convergent. Find its sum for $x = n \cdot \pi/2$, $n = 1, 2, 3$.

4.22 Let α be a complex number but not an integer. Determine the Fourier series of $\cos \alpha t$ ($|t| \leq \pi$). Use the result to prove the formula

$$\pi \cot \pi z = \lim_{N \rightarrow \infty} \sum_{n=-N}^N \frac{1}{z-n} \quad (z \notin \mathbf{Z})$$

(“expansion into partial fractions of the cotangent”).

4.5 Formulae for other periods

Here we have collected the formulae for Fourier series of functions with a period different from 2π . It is convenient to have a notation for the *half-period*, so we assume that the period is $2P$, where $P > 0$:

$$f(t + 2P) = f(t) \text{ for all } t \in \mathbf{R}.$$

Put $\Omega = \pi/P$. The number Ω could be called the *fundamental angular frequency*. A linear change of variable in the usual formulae results in the following set of formulae:

$$f(t) \sim \sum_{n \in \mathbf{Z}} c_n e^{in\Omega t}, \quad \text{where } c_n = \frac{1}{2P} \int_{-P}^P f(t) e^{-in\Omega t} dt,$$

and, alternatively,

$$f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\Omega t + b_n \sin n\Omega t), \quad \text{where } \begin{matrix} a_n \\ b_n \end{matrix} = \frac{1}{P} \int_{-P}^P f(t) \begin{matrix} \cos \\ \sin \end{matrix} n\Omega t dt.$$

In all cases, the intervals of integration can be changed from $(-P, P)$ to an arbitrary interval of length $2P$. If f is even or odd, we have the special cases

$$\begin{aligned} f \text{ even} &\Rightarrow b_n = 0, \quad a_n = \frac{2}{P} \int_0^P f(t) \cos n\Omega t dt, \\ f \text{ odd} &\Rightarrow a_n = 0, \quad b_n = \frac{2}{P} \int_0^P f(t) \sin n\Omega t dt. \end{aligned}$$

All results concerning summability, convergence, differentiability, etc., that we have proved in the preceding sections, will of course hold equally well for any period length.

Exercises

- 4.23 (a) Determine the Fourier series for the even function f with period 2 that satisfies $f(t) = t$ for $0 < t < 1$.
 (b) Determine the Fourier series for the odd function f with period 2 that satisfies $f(t) = t$ for $0 < t < 1$.
 (c) Compare the convergence properties of the series obtained in (a) and (b). Illuminate by drawing pictures!
- 4.24 Find, in the guise of a “complex” Fourier series, a periodic solution with a continuous first derivative on \mathbf{R} of the differential equation $y'' + y' + y = g$, where g has period 4π and $g(t) = 1$ for $|t| < \pi$, $g(t) = 0$ for $\pi < |t| < 2\pi$.
- 4.25 Determine a solution with period 2 of the differential-difference equation $y'(t) + y(t - 1) = \cos^2 \pi t$.
- 4.26 Compute the Fourier series of the odd function f with period 2 that satisfies $f(x) = x - x^2$ for $0 < x < 1$. Use the result to find the sum of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}.$$

4.6 Some worked examples

In this section we give a few more examples of the computational work that may occur in calculating the Fourier coefficients of a function.

Example 4.6. Take $f(t) = t \cos 2t$ for $-\pi < t < \pi$, and assume f to have period 2π . First of all, we try to see if f is even or odd — indeed, it is odd. This means that it should be a good idea to compute the Fourier series in the “real” version; because all a_n will be zero, and b_n is given by the half-range integral

$$b_n = \frac{2}{\pi} \int_0^{\pi} t \cos 2t \sin nt \, dt.$$

The computation is now greatly simplified by using the product formula

$$\sin x \cos y = \frac{1}{2} (\sin(x+y) + \sin(x-y)).$$

Integrating by parts, we get

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_0^{\pi} t (\sin(n+2)t + \sin(n-2)t) \, dt \quad (n \neq 2) \\ &= \frac{1}{\pi} \left[t \left(-\frac{\cos(n+2)t}{n+2} - \frac{\cos(n-2)t}{n-2} \right) \right]_0^{\pi} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\pi} \int_0^\pi \left(\frac{\cos(n+2)t}{n+2} + \frac{\cos(n-2)t}{n-2} \right) dt \\
& = -\frac{1}{\pi} \cdot \pi \left(\frac{\cos(n+2)\pi}{n+2} + \frac{\cos(n-2)\pi}{n-2} \right) + 0 = -\left(\frac{(-1)^n}{n+2} + \frac{(-1)^n}{n-2} \right) \\
& = -\frac{2n(-1)^n}{n^2-4}.
\end{aligned}$$

This computation fails for $n = 2$. For this n we get instead

$$\begin{aligned}
b_2 &= \frac{1}{\pi} \int_0^\pi t(\sin 4t + 0) dt = \frac{1}{\pi} \left[t \frac{-\cos 4t}{4} \right] + \frac{1}{4\pi} \int_0^\pi \cos 4t dt \\
&= -\frac{1}{4} + 0 = -\frac{1}{4}.
\end{aligned}$$

Noting that $b_1 = -\frac{2}{3}$, we can conveniently describe the Fourier series as

$$f(t) \sim -\frac{2}{3} \sin t - \frac{1}{4} \sin 2t - 2 \sum_{n=3}^{\infty} \frac{n(-1)^n}{n^2-4} \sin nt.$$

□

Example 4.7. Find the Fourier series of the odd function of period 2 that is described by $f(t) = t(1-t)$ for $0 \leq t \leq 1$. Using the result, find the value of the sum

$$s_1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}.$$

Solution. Since the function is odd, we compute a sine series. The coefficients are

$$b_n = 2 \int_0^1 t(1-t) \sin n\pi t dt = (\text{integrations by parts}) = \frac{4(1 - (-1)^n)}{n^3\pi^3},$$

which is zero for all even values of n . Writing $n = 2k+1$ when n is odd, we get the series

$$f(t) \sim \frac{8}{\pi^3} \sum_{k=0}^{\infty} \frac{\sin(2k+1)\pi t}{(2k+1)^3}.$$

A sketch of the function shows that f is everywhere continuous and has both right- and left-hand derivatives everywhere, which permits us to replace the sign \sim by $=$. In particular we note that if $t = \frac{1}{2}$, then $\sin(2k+1)\pi t = \sin(k + \frac{1}{2})\pi = (-1)^k$, so that

$$\frac{1}{4} = f\left(\frac{1}{2}\right) = \frac{8}{\pi^3} \cdot s_1 \quad \Rightarrow \quad s_1 = \frac{\pi^3}{32}.$$

□

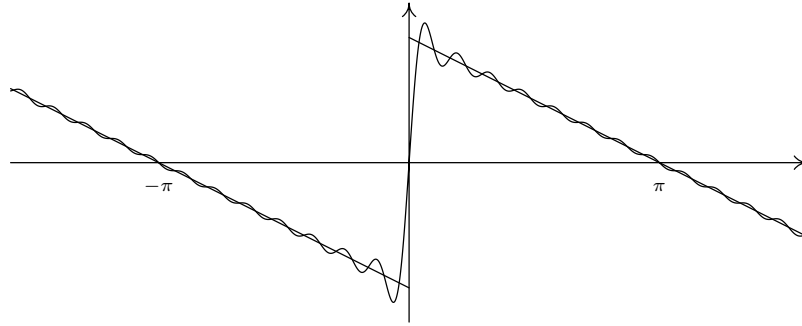


FIGURE 4.4.

Exercises

- 4.27 Find the Fourier series of f with period 1, when $f(x) = x$ for $1 < x < 2$. Indicate the sum of the series for $x = 0$ and $x = \frac{1}{2}$. Explain your answer!
- 4.28 Develop into Fourier series the function f given by

$$f(x) = \sin \frac{x}{2}, \quad -\pi < x \leq \pi; \quad f(x + 2\pi) = f(x), \quad x \in \mathbf{R}.$$

- 4.29 Compute the Fourier series of period 2π for the function $f(x) = (|x| - \pi)^2$, $|x| \leq \pi$, and use it to find the sums

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

4.7 The Gibbs phenomenon

Let f be a function that satisfies the conditions for pointwise convergence of the Fourier series (Theorem 4.5) and that has a jump discontinuity at a certain point t_0 . If we draw a graph of a partial sum of the series, we discover a peculiar behavior: When t approaches t_0 , for example, from the left, the graph of $s_n(t)$ somehow grows restless; you might say that it prepares to take off for the jump; and when the jump is accomplished, it overshoots the mark somewhat and then calms down again. Figure 4.4 shows a typical case.

This sort of behavior had already been observed during the nineteenth century by experimental physicists, and it was then believed to be due to imperfection in the measuring apparatuses. The fact that this is not so, but that we are dealing with an actual mathematical phenomenon, was proved by J. W. GIBBS, after whom the behavior has also been named.

The behavior is fundamentally due to the fact that the Dirichlet kernel $D_n(t)$ is restless near $t = 0$. We are going to analyze the matter in detail in one special case and then, using a simple maneuver, show that the same sort of thing occurs in the general case.

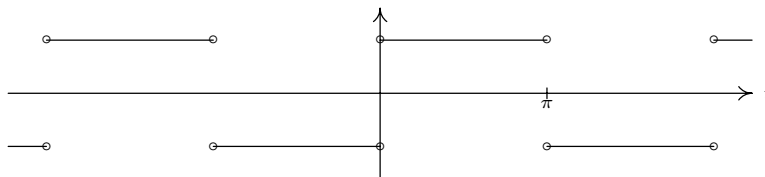


FIGURE 4.5.

Let $f(t)$ be a so-called square-wave function with period 2π , described by $f(t) = 1$ for $0 < t < \pi$, $f(t) = -1$ for $-\pi < t < 0$ (see Figure 4.5). Since f is odd, it has a sine series, with coefficients

$$b_n = \frac{2}{\pi} \int_0^\pi \sin nt \, dt = \frac{2}{\pi} \left[-\frac{\cos nt}{n} \right]_0^\pi = \frac{2}{n\pi} (1 - (-1)^n),$$

which is zero if n is even. Thus,

$$f(t) \sim \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin(2k+1)t}{2k+1} = \frac{4}{\pi} \left(\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \cdots \right). \quad (4.8)$$

Because of symmetry we can restrict our study to the interval $(0, \pi/2)$. For a while we dump the factor $4/\pi$ and consider the partial sums of the series in the brackets:

$$S_n(t) = \sin t + \frac{1}{3} \sin 3t + \frac{1}{5} \sin 5t + \cdots + \frac{1}{2n+1} \sin(2n+1)t.$$

By differentiation we find

$$\begin{aligned} S'_n(t) &= \cos t + \cos 3t + \cdots + \cos(2n+1)t = \frac{1}{2} \sum_{k=0}^n (e^{i(2k+1)t} + e^{-i(2k+1)t}) \\ &= \frac{1}{2} e^{-i(2n+1)t} \sum_{k=0}^{2n+1} e^{i2kt} = \frac{1}{2} e^{-i(2n+1)t} \frac{1 - e^{i2(2n+2)t}}{1 - e^{i2t}} = \frac{\sin 2(n+1)t}{2 \sin t} \end{aligned}$$

(compare the method that we used to sum $D_n(t)$). The last formula does not hold for $t = 0$, but it does hold in the half-open interval $0 < t \leq \pi/2$. The derivative has zeroes in this interval; they are easily found to be where $2(n+1)t = k\pi$ or $t = \tau_k = (k\pi)/((2(n+1)))$, $k = 1, 2, \dots, n$. Considering the sign of the derivative between the zeroes one realizes that these points are alternatingly maxima and minima of S_n . More precisely, since $S_n(0) = 0$, integration gives

$$S_n(t) = \int_0^t \frac{\sin 2(n+1)u}{2 \sin u} \, du,$$

where the numerator of the integrand oscillates in a smooth fashion between the successive τ_k , while the denominator increases throughout the interval.

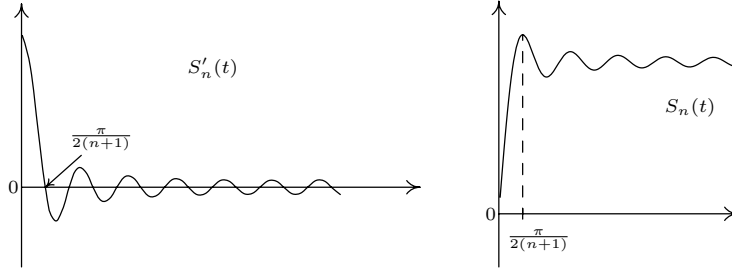


FIGURE 4.6.

This means that the first maximum value, for $t = \tau_1$, is also the largest, and the oscillations in S_n then quiet down as t increases (see Figure 4.6).

It follows that the maximal value of $S_n(t)$ on $]0, \pi/2]$ is given by

$$A_n = S_n(\tau_1) = S_n\left(\frac{\pi}{2(n+1)}\right) = \sum_{k=0}^n \frac{1}{2k+1} \sin \frac{(2k+1)\pi}{2(n+1)}.$$

We can interpret the last sum as a Riemann sum for a certain integral: Let $t_k = k\pi/(n+1)$ and $\xi_k = \frac{1}{2}(t_k + t_{k+1})$. Then the points $0 = t_0, t_1, \dots, t_{n+1} = \pi$ describe a subdivision of the interval $(0, \pi)$, the point ξ_k lies in the subinterval (x_k, x_{k+1}) and, in addition, $\xi_k = (2k+1)\pi/(2(n+1))$. Thus we have

$$A_n = \frac{1}{2} \sum_{k=0}^n \frac{\sin \xi_k}{\xi_k} \Delta x_k \rightarrow \frac{1}{2} \int_0^\pi \frac{\sin x}{x} dx \quad \text{as } n \rightarrow \infty.$$

A more detailed scrutiny of the limit process would show that the numbers A_n decrease toward the limit.

Now we reintroduce the factor $4/\pi$. We have then established that the partial sums of the Fourier series (4.8) have maximum values that tend to the limit

$$\frac{2}{\pi} \int_0^\pi \frac{\sin t}{t} dt \approx 1.1789797,$$

and the maximal value of $S_n(t)$ is taken at $t = \pi/(2(n+1))$. On the right-hand side of the maximum, the partial sums oscillate around the value 1 with a decreasing amplitude, up to the point $t = \pi/2$. Because of symmetry, the behavior to the left will be analogous. What we want to stress is the fact that the maximal oscillation does not tend to zero when more terms of the series are added; on the contrary, it stabilizes toward a value that is approximately 9 percent of the total size of the jump. The point where the maximum oscillation takes place moves indefinitely closer to the point of the jump. It is even possible to prove that the Fourier series is actually uniformly convergent to 1 on intervals of the form $[a, \pi - a]$, where $a > 0$.

Now let g be any function with a jump discontinuity at t_0 with the size of the jump equal to $\delta = g(t_0+) - g(t_0-)$, and assume that g satisfies the

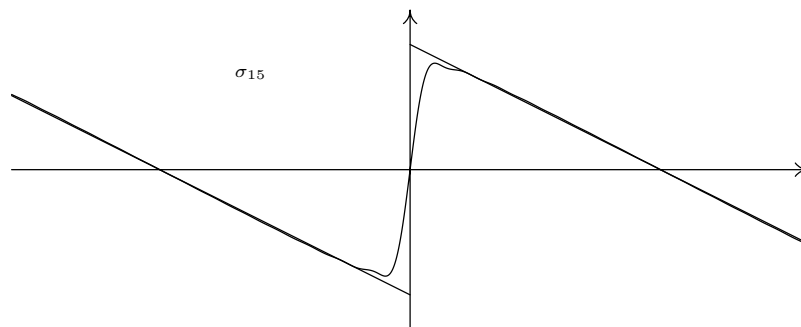


FIGURE 4.7.

conditions of Theorem 4.5 for convergence of the Fourier series in some neighborhood of t_0 . Form the function $h(t) = g(t) - \frac{1}{2}\delta f(t - t_0)$, where f is the square-wave function just investigated. Then, $h(t_0+) = h(t_0-)$, so that h is actually continuous at t_0 if one defines $h(t_0)$ in the proper way. Furthermore, h has left- and right-hand derivatives at t_0 , and so the Fourier series of h will converge nicely to h in a neighborhood of $t = t_0$. The Fourier series of g can be written as the series of h plus some multiple of a translate of the series of f ; the former series is calm near t_0 , but the latter oscillates in the manner demonstrated above. It follows that the series of g exhibits on the whole the same restlessness when we approach t_0 , as does the series of f when we approach 0. The size of the maximum oscillation is also approximately 9 percent of the size of the whole jump.

If a Fourier series is summed according to Cesàro (Theorem 4.1) or Poisson–Abel (see Sec. 6.3), the Gibbs phenomenon disappears completely. Compare the graphs of $s_{15}(t)$ in Figure 4.4 and $\sigma_{15}(t)$ (for the same f) in Figure 4.7.

4.8 *Fourier series for distributions

We shall here consider the generalized functions of Sec. 2.6 and 2.7 and their Fourier series. Since the present chapter deals with objects defined on \mathbf{T} , or, equivalently, periodic phenomena, we begin by considering periodic distributions as such.

In this context, the Heaviside function H is not really interesting. But we can still think of the object $\delta_a(t)$ as a “unit pulse” located at a point $a \in \mathbf{T}$, having the property

$$\int_{\mathbf{T}} \varphi(t) \delta_a(t) dt = \varphi(a) \quad \text{if } \varphi \text{ is continuous at } a.$$

The periodic description of the same object consists of a so-called *pulse train* consisting of unit pulses at all the points $a + n \cdot 2\pi$, $n \in \mathbf{Z}$. As an

object defined on \mathbf{R} , this pulse train could be described by

$$\sum_{n=-\infty}^{\infty} \delta_{a+2\pi n}(t) = \sum_{n=-\infty}^{\infty} \delta(t - a - 2\pi n).$$

The convergence of this series is uncontroversial, because at any individual point t at most one of the terms is different from zero.

The derivatives of δ_a can be described using integration by parts, just as in Sec. 2.6, but now the integrals are taken over \mathbf{T} (i.e., over one period). Because everything is periodic, the contributions at the ends of the interval will cancel:

$$\int_{\mathbf{T}} \varphi(t) \delta'_a(t) dt = \left[\varphi(t) \delta_a(t) \right]_b^{b+2\pi} - \int_{\mathbf{T}} \varphi'(t) \delta_a(t) dt = -\varphi'(a).$$

What would be the Fourier series of these distributions? Let us first consider δ_a . The natural approach is to define Fourier coefficients by the formula

$$c_n = \frac{1}{2\pi} \int_{\mathbf{T}} \delta_a(t) e^{-int} dt = \frac{1}{2\pi} \cdot e^{-ina}.$$

The series then looks like this:

$$\delta_a(t) \sim \frac{1}{2\pi} \sum_{n \in \mathbf{Z}} e^{-ina} \cdot e^{int}.$$

In particular, when $a = 0$, the Fourier coefficients are all equal to $1/(2\pi)$, and the series is

$$\delta(t) \sim \frac{1}{2\pi} \sum_{n \in \mathbf{Z}} e^{int}.$$

By pairing terms with the same values of $|n|$, we can formally rewrite this as

$$\delta(t) \sim \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \cos nt.$$

Compare the Dirichlet kernel! We might say that δ is the limit of D_N as $N \rightarrow \infty$.

These series cannot be convergent in the usual sense, since their terms do not tend to zero. But for certain values of t they can be summed according to Cesàro. Indeed, we can use the result of Exercise 2.16 on page 22. The series for $2\pi\delta_a$ can be written (with $z = e^{i(t-a)}$)

$$\begin{aligned} \sum_{n \in \mathbf{Z}} e^{in(t-a)} &= \sum_{n=0}^{\infty} \left(e^{i(t-a)} \right)^n + \sum_{n=-\infty}^{-1} \left(e^{i(t-a)} \right)^n \\ &= \sum_{n=0}^{\infty} z^n + \sum_{n=1}^{\infty} \left(e^{-i(t-a)} \right)^n = \sum_{n=0}^{\infty} z^n + \bar{z} \sum_{n=0}^{\infty} \bar{z}^n \end{aligned}$$

According to Exercise 2.16, both the series in the last expression can be summed $(C, 1)$ if $|z| = 1$ but $z \neq 1$, which is the case if $t \neq a$, and the result will be

$$\frac{1}{1-z} + \frac{\bar{z}}{1-\bar{z}} = \frac{1-\bar{z} + \bar{z}(1-z)}{|1-z|^2} = \frac{1-z\bar{z}}{|1-z|^2} = \frac{1-|z|^2}{|1-z|^2} = 0.$$

If $t = a$, all the terms are ones, and the series diverges to infinity.

Thus the series behaves in a way that is most satisfactory, as it enhances our intuitive image of what δ_a looks like.

Next we find the Fourier series of δ'_a . The coefficients are

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta'_a(t) e^{-int} dt = -\frac{1}{2\pi} \cdot \frac{d}{dt} e^{-int} \Big|_{t=a} \\ &= -\frac{1}{2\pi} (-ine^{-int}) \Big|_{t=a} = \frac{in}{2\pi} e^{-ina}. \end{aligned}$$

We recognize that the rule in Sec. 4.3 for the Fourier coefficients of a derivative holds true. The summation of the series

$$\delta'_a(t) = \frac{i}{2\pi} \sum_{n \in \mathbf{Z}} ne^{-ian} e^{int} = \frac{i}{2\pi} \sum_{n \in \mathbf{Z}} ne^{in(t-a)}$$

is tougher than that of δ_a itself, because the terms now have moduli that even tend to infinity as $|n| \rightarrow \infty$. It can be shown, however, that for $t \neq a$ the series is summable $(C, 2)$ to 0.

We give a couple of examples to illustrate the use of these series.

Example 4.8. Consider the function of Example 4.2 on page 78. Its Fourier series can be written

$$f(t) \sim \sum_{n=1}^{\infty} \frac{\sin nt}{n} = \sum_{n \neq 0} \frac{1}{2in} e^{int}.$$

(Notice that the last version is correct — the minus sign in the Euler formula for \sin is incorporated in the sign of the n in the coefficient.)

The derivative of f consists of an “ordinary” term $-\frac{1}{2}$, which takes care of the slope between the jumps, and a pulse train that on \mathbf{T} is identified with $\pi \cdot \delta(t)$. This would mean that the Fourier series of the derivative is given by

$$\begin{aligned} f'(t) &= -\frac{1}{2} + \pi \delta(t) \sim -\frac{1}{2} + \pi \cdot \frac{1}{2\pi} \sum_{n \in \mathbf{Z}} e^{int} \\ &= -\frac{1}{2} + \frac{1}{2} \sum_{n \in \mathbf{Z}} e^{int} = \frac{1}{2} \sum_{n=1}^{\infty} (e^{int} + e^{-int}) = \sum_{n=1}^{\infty} \cos nt. \end{aligned}$$

Notice that this is precisely what a formal differentiation of the original series would yield. \square

Example 4.9. Find a 2π -periodic solution of the differential equation $y' + y = 1 + \delta(t)$ ($-\pi < t < \pi$).

Solution. We try a solution of the form $y = \sum c_n e^{int}$. Differentiating this and expanding the right-hand member in Fourier series, we get

$$\sum_{n \in \mathbf{Z}} in c_n e^{int} + \sum_{n \in \mathbf{Z}} c_n e^{int} = 1 + \sum_{n \in \mathbf{Z}} \frac{1}{2\pi} e^{int},$$

or

$$c_0 + \sum_{n \neq 0} (in + 1) c_n e^{int} = \left(1 + \frac{1}{2\pi}\right) + \sum_{n \neq 0} \frac{1}{2\pi} e^{int}.$$

Identification of coefficients yields $c_0 = 1 + 1/(2\pi)$ and, for $n \neq 0$, $c_n = 1/(2\pi(1 + in))$. A solution should thus be given by

$$y(t) \sim \left(1 + \frac{1}{2\pi}\right) + \frac{1}{2\pi} \sum_{n \neq 0} \frac{e^{int}}{1 + in}.$$

By a stroke of luck, it happens that this series has been almost encountered before in the text: in Example 4.1 on page 76 f. we found that

$$f(u) \sim \frac{\sinh \pi}{\pi} \left(1 + \sum_{n \neq 0} \frac{(-1)^n}{1 - in} e^{inu}\right),$$

where $f(u) = e^u$ for $-\pi < u < \pi$ and f has period 2π . From this we can find that

$$\sum_{n \neq 0} \frac{(-1)^n}{1 - in} e^{inu} = \frac{\pi}{\sinh \pi} f(u) - 1.$$

On the other hand, the series on the left of this equation can be rewritten, using $(-1)^n = e^{in\pi}$ and letting $t = \pi - u$:

$$\sum_{n \neq 0} \frac{(-1)^n}{1 - in} e^{inu} = \sum_{n \neq 0} \frac{(-1)^n}{1 + in} e^{-inu} = \sum_{n \neq 0} \frac{e^{in(\pi-u)}}{1 + in} = \sum_{n \neq 0} \frac{e^{int}}{1 + in}.$$

This means that our solution can be expressed in the following way:

$$y(t) \sim \left(1 + \frac{1}{2\pi}\right) + \frac{1}{2\pi} \left(\frac{\pi}{\sinh \pi} f(u) - 1\right) = 1 + \frac{1}{2 \sinh \pi} f(u) = 1 + \frac{f(\pi - t)}{2 \sinh \pi}.$$

In particular,

$$y(t) = 1 + \frac{e^{\pi-t}}{2 \sinh \pi} = 1 + \frac{e^\pi}{2 \sinh \pi} e^{-t}, \quad 0 < t < 2\pi,$$

since this condition on t is equivalent to $-\pi < \pi - t < \pi$. At the points $t = n \cdot 2\pi$, $y(t)$ has an upward jump of size 1 (check this!).

Let us check the solution by substitution into the equation. Differentiating, we find that $y'(t)$ contains the pulse $\delta(t)$ at the origin, and between jumps one has $y'(t) = -(y(t) - 1)$. This proves that we have indeed found a solution. \square

Exercises

4.30 Let f be the even function with period 2π that satisfies $f(t) = \pi - t$ for $0 \leq t \leq \pi$. Determine f' and f'' , and use the result to find the Fourier series of f .

4.31 Let f have period 2π and satisfy

$$f(t) = \begin{cases} e^t, & |t| < \pi/2, \\ 0, & \pi/2 < |t| < \pi. \end{cases}$$

Compute $f' - f$, and then determine the Fourier series of f .

Summary of Chapter 4

Definition

If f is a sufficiently nice function defined on \mathbf{T} , we define its *Fourier coefficients* by

$$c_n = \frac{1}{2\pi} \int_{\mathbf{T}} f(t) e^{-int} dt \quad \text{or} \quad \begin{matrix} a_n \\ b_n \end{matrix} = \frac{1}{\pi} \int_{\mathbf{T}} f(t) \begin{matrix} \cos \\ \sin \end{matrix} nt dt.$$

The *Fourier series* of f is the series

$$\sum_{n \in \mathbf{Z}} c_n e^{int}, \quad \text{resp.} \quad \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt).$$

If f has a period other than 2π , the formulae have to be adjusted accordingly. If f is even or odd, the formulae for a_n and b_n can be simplified.

Theorem

If two continuous functions f and g have the same Fourier coefficients, then $f = g$.

Theorem

If f is piecewise continuous on \mathbf{T} and continuous at the point t , then, for this value of t , its Fourier series is summable $(C, 1)$ to the value $f(t)$.

Theorem

If f is continuous on \mathbf{T} and its Fourier coefficients satisfy $\sum |c_n| < \infty$, then its Fourier series converges absolutely and uniformly to $f(t)$ on all of \mathbf{T} .

Theorem

If f is differentiable on \mathbf{T} , then the Fourier series of the derivative f' can be found by termwise differentiation.

Theorem

If $f \in C^k(\mathbf{T})$, then its Fourier coefficients satisfy $|c_n| \leq M/|n|^k$.

Theorem

If f is continuous except for jump discontinuities, and if it has (generalized) one-sided derivatives at a point t , then its Fourier series for this value of t converges with the sum $\frac{1}{2}(f(t+) + f(t-))$.

Formulae for Fourier series are found on page 251.

Historical notes

JOSEPH FOURIER was not the first person to consider trigonometric series of the kind that came to bear his name. Around 1750, both Daniel Bernoulli and Leonhard Euler were busy investigating these series, but the standard of rigor in mathematics then was not sufficient for a real understanding of them. Part of the problem was the fact that the notion of a function had not been made precise, and different people had different opinions on this matter. For example, a graph pieced together as in Figure 4.2 on page 78 was not considered to represent one function but several. It was not until the times of BERNHARD RIEMANN and KARL WEIERSTRASS that something similar to the modern concept of a function was born. In 1822, when Fourier's great treatise appeared, it was generally regarded as absurd that a series with terms that were smooth and nice trigonometric functions should be able to represent functions that were not everywhere differentiable, or even worse—discontinuous!

The convergence theorem (Theorem 4.5) as stated in the text is a weaker version of a result by the German mathematician J. PETER LEJEUNE-DIRICHLET (1805–59). At the age of 19, the Hungarian LIPÓT FEJÉR (1880–1959) had the bright idea of applying Cesàro summation to Fourier series.

In the twentieth century the really hard questions concerning the convergence of Fourier series were finally resolved, when LENNART CARLESON (1928–) proved his famous Theorem 4.6. The author of this book, then a graduate student, attended the series of seminars in the fall of 1965 when Carleson step by step conquered the obstacles in his way. The final proof consists of 23 packed pages in one of the world's most famous mathematical journals, the *Acta Mathematica*.

Problems for Chapter 4

4.32 Determine the Fourier series of the following functions. Also state what is the sum of the series for all t .

(a) $f(t) = 2 + 7 \cos 3t - 4 \sin 2t$, $-\pi < t < \pi$.

(b) $f(t) = |\sin t|$, $-\pi < t < \pi$.

(c) $f(t) = (\pi - t)(\pi + t), \quad -\pi < t < \pi.$

(d) $f(t) = e^{|t|}, \quad -\pi < t < \pi.$

4.33 Find the cosine series of $f(t) = \sin t, 0 < t < \pi.$

4.34 Find the sine series of $f(t) = \cos t, 0 < t < \pi.$ Use this series to show that

$$\frac{\pi\sqrt{2}}{16} = \frac{1}{2^2 - 1} - \frac{3}{6^2 - 1} + \frac{5}{10^2 - 1} - \frac{7}{14^2 - 1} + \cdots.$$

4.35 Let f be the 2π -periodic continuation of the function $H(t - a) - H(t - b)$, where $-\pi < a < b < \pi$. Find the Fourier series of f . For what values of t does it converge? Indicate its sum for all such $t \in [-\pi, \pi]$.

4.36 Let f be given by $f(x) = -1$ for $-1 < x < 0$, $f(x) = x$ for $0 \leq x \leq 1$ and $f(x + 2) = f(x)$ for all x . Compute the Fourier series of f . State the sum of this series for $x = 10$, $x = 10.5$, and $x = 11$.

4.37 Develop $f(t) = t(t - 1), 0 < t < 1$, period 1, in a Fourier series. Quote some criterion that implies that the series converges to $f(t)$ for all values of t .

4.38 The function f is defined by $f(t) = t^2$ for $0 \leq t \leq 1$, $f(t) = 0$ for $1 < t < 2$ and by the statement that it has period 2.

(a) Develop f in a Fourier series with period 2 and indicate the sum of the series in the interval $[0, 5]$.

(b) Compute the value of the sum $s = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$

4.39 Suppose that f is integrable, has period T , and Fourier series

$$f(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t / T}.$$

Determine the Fourier series of the so-called *autocorrelation function* r of f , which is defined by

$$r(t) = \frac{1}{T} \int_0^T f(t + u) \overline{f(u)} du.$$

4.40 An application to sound waves: Suppose the variation in pressure, p , that causes a sound has period $\frac{1}{262}$ s (seconds), and satisfies

$$p(t) = 1, \quad 0 < t < \frac{1}{1048}, \quad p(t) = -\frac{7}{8}, \quad \frac{1}{1048} < t < \frac{1}{524},$$

$$p(t) = \frac{7}{8}, \quad \frac{1}{524} < t < \frac{3}{1048}, \quad p(t) = -1, \quad \frac{3}{1048} < t < \frac{1}{262}.$$

What frequencies can be heard in this sound? Which is the dominant frequency?

4.41 Compute the Fourier series of f , given by

$$f(x) = \left| \sin \frac{x}{2} \right|, \quad -\pi < x \leq \pi; \quad f(x + 2\pi) = f(x), \quad x \in \mathbf{R}.$$

Then find the values of the sums

$$s_1 = \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \quad \text{and} \quad s_2 = \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}.$$

- 4.42 Let f be an even function of period 2π described by $f(x) = \cos 2x$ for $0 \leq x \leq \frac{1}{2}\pi$ and $f(x) = -1$ for $\frac{1}{2}\pi < x \leq \pi$. Find its Fourier series and compute the value of the sum

$$s = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)(2k-1)(2k+3)}.$$

- 4.43 Find all solutions $y(t)$ with period 2π of the differential-difference equation

$$y'(t) + y(t - \tfrac{1}{2}\pi) - y(t - \pi) = \cos t, \quad -\infty < t < \infty.$$

- 4.44 Let f be an even function with period 4 such that $f(x) = 1 - x$ for $0 \leq x \leq 1$ and $f(x) = 0$ for $1 < x \leq 2$. Find its Fourier series and compute

$$s = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}.$$

- 4.45 Let α be a real number but not an integer. Define $f(x)$ by putting $f(x) = e^{i\alpha x}$ for $-\pi < x < \pi$ and $f(x+2\pi) = f(x)$. By studying its Fourier series, prove the following formulae:

$$\frac{\pi}{\sin \pi \alpha} = \frac{1}{\alpha} + \sum_{n=1}^{\infty} \frac{2(-1)^n \alpha}{\alpha^2 - n^2}. \quad \left(\frac{\pi}{\sin \pi \alpha} \right)^2 = \sum_{n=-\infty}^{\infty} \frac{1}{(\alpha - n)^2}.$$

- 4.46 Compute the Fourier series of the 2π -periodic function f given by $f(x) = x^3 - \pi^2 x$ for $-\pi < x < \pi$. Find the sum

$$s = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3}.$$

- 4.47 Let f be a 2π -periodic function with (“complex”) Fourier coefficients c_n ($n \in \mathbf{Z}$). Assume that for an integer $k > 0$ it holds that

$$\sum_{n \in \mathbf{Z}} |n|^k |c_n| < \infty.$$

Prove that f is of class C^k , i.e., that the k th derivative of f is continuous.

- 4.48 Find the Fourier series of f with period 2 which is given for $|x| < 1$ by $f(x) = 2x^2 - x^4$. The result can be used to find the value of the sum

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4}.$$



<http://www.springer.com/978-0-387-00836-3>

Fourier Analysis and Its Applications

Vretblad, A.

2003, XII, 272 p., Hardcover

ISBN: 978-0-387-00836-3